On the cohomology of congruence subgroups of $\mathrm{SL}_4(\mathbb{Z})$

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References

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Basic problem

Let $G = \mathrm{SL}_4(\mathbb{R})$, $K = \mathrm{SO}(4)$, $X = G/K$.
Let $\Gamma = \Gamma_0(N) \subset \mathrm{SL}_4(\mathbb{Z})$ be the subgroup with bottom row congruent to $(0, 0, 0, *)$ mod N.
Our goal is to compute the cohomology

$$H^5(\Gamma; \mathbb{C}) = H^5(\Gamma\backslash X; \mathbb{C}),$$

and to understand the action of the Hecke operators on this space.

Why?

The cohomology of any arithmetic group is built out of certain automorphic forms, yet can be computed using topological tools.

- Gives a concrete way to compute automorphic forms that complements other approaches (e.g., theta series).
- Gives explicit examples of various constructions in automorphic forms (e.g, functorial liftings).
- Gives examples of automorphic forms that should be related to arithmetic objects (e.g., Galois representations). Gives way to test various "motivic $\Leftrightarrow$ automorphic" conjectures.

Arithmetic groups and automorphic forms

$\mathbf{G}$ semisimple connected algebraic group $/\mathbb{Q}$

$G = \mathbf{G}(\mathbb{R})$ group of real points (Lie group)

$K \subset G$ maximal compact subgroup

$X = G/K$ global symmetric space

$\Gamma \subset \mathbf{G}(\mathbb{Q})$ arithmetic subgroup

E finite-dimensional rational complex representation of $\mathbf{G}(\mathbb{Q})$

Arithmetic groups and automorphic forms

If Γ is torsion-free, then $\Gamma\backslash X$ is an Eilenberg–Mac Lane space. We have

$$H^*(\Gamma; E) = H^*(\Gamma\backslash X; \mathscr{E}),$$

where $\mathscr{E}$ is the local coefficient system attached to E.
True even if Γ has torsion, since we're using complex representations.

$(\mathfrak{g}, K)$-cohomology

We can get automorphic forms into the picture via the de Rham theorem. Let $\Omega^p = \Omega^p(X, E)$ be the space of E-valued p-forms on X. Let $\Omega^p(X, E)^\Gamma$ be the subspace of Γ-invariant forms. We have a differential $d\colon \Omega^p \to \Omega^{p+1}$ and have an isomorphism

$$H^*(\Gamma; E) = H^*(\Omega^*(X, E)^\Gamma).$$

$(\mathfrak{g}, K)$-cohomology

We can identify

$$\Omega^p(\Gamma\backslash X, \mathbb{C}) = \mathrm{Hom}_K(\wedge^p(\mathfrak{g}/\mathfrak{k}), C^\infty(\Gamma\backslash G))$$

or more generally

$$\Omega^p(\Gamma\backslash X, E) = \mathrm{Hom}_K(\wedge^p(\mathfrak{g}/\mathfrak{k}), C^\infty(\Gamma\backslash G) \otimes E)$$

RHS inherits a differential. The cohomology is denoted

$$H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G) \otimes E)$$

and is called $(\mathfrak{g}, K)$-*cohomology*.

Cuspidal cohomology

We have

$$H^*(\Gamma; E) = H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G) \otimes E).$$

We can use this to identify important subspaces of the cohomology. For instance the inclusion

$$L^2_{\text{cusp}}(\Gamma\backslash G)^\infty \hookrightarrow C^\infty(\Gamma\backslash G)$$

induces an injective map

$$H^*(\mathfrak{g}, K; L^2_{\text{cusp}}(\Gamma\backslash G)^\infty \otimes E) \to H^*(\mathfrak{g}, K; C^\infty(\Gamma\backslash G) \otimes E).$$

The image $H^*_{\text{cusp}}(\Gamma; E) \subset H^*(\Gamma; E)$ is called the *cuspidal cohomology*.

Borel conjecture

We also have the subspace of automorphic forms

$$A(\Gamma, G) \subset C^\infty(\Gamma \backslash G)$$

(subspace of functions that are right K-finite, left $Z(\mathfrak{g})$-finite, and of moderate growth). '

Theorem (Franke)

The inclusion $A(\Gamma, G) \to C^\infty(\Gamma \backslash G)$ induces an isomorphism

$$H^*(\mathfrak{g}, K; A(\Gamma, G) \otimes E) \to H^*(\mathfrak{g}, K; C^\infty(\Gamma \backslash G) \otimes E)$$

Thus we can think of $H^*(\Gamma; E)$ as being a concrete realization of certain automorphic forms, namely those with nonvanishing $(\mathfrak{g}, K)$-cohomology. These were classified by Vogan–Zuckermann.

Example: SL_2 and modular forms

If $\mathbf{G} = \mathrm{SL}_2$, then $G = \mathrm{SL}_2(\mathbb{R})$, $K = \mathrm{SO}(2)$, and X is the upper halfplane.
Let $\Gamma = \Gamma_0(N) \subset \mathrm{SL}_2(\mathbb{Z})$, the subgroup of matrices upper triangular modulo N.
Let E_k be the k-dimensional complex representation of G, say on the vector space of degree $k-1$ homogeneous complex polynomials in two variables.
We have

$$H^1(\Gamma; E_k) \simeq S_{k+1}(\Gamma) \oplus \overline{S}_{k+1}(\Gamma) \oplus \mathrm{Eis}_{k+1}(\Gamma),$$

where S_{k+1} is the space of holomorphic weight $k+1$ modular forms, and Eis_{k+1} is the space of weight $k+1$ Eisenstein series.

Virtual cohomological dimension

Let $q = q(\mathbf{G})$ be the $\mathbb{Q}$-rank of $\mathbf{G}$.

Theorem (Borel–Serre)

For all Γ *and* E *as above, we have* $H^i(\Gamma; E) = 0$ *if* $i > \dim X - q$.

The number $\nu = \dim X - q$ is called the *virtual cohomological dimension*.

Cuspidal range

The cuspidal cohomology doesn't appear in every cohomological degree. In fact, one can show that $H^i_{\mathrm{cusp}}(\Gamma; E) = 0$ unless the degree i lies in a small interval about $(\dim X)/2$ (Li–Schwermer, Saper).

n	2	3	4	5	6	7	8	9
$\dim X$	2	5	9	14	20	27	35	44
$\nu(\Gamma)$	1	3	6	10	15	21	28	36
top degree of H^*_{cusp}	1	3	5	8	11	15	19	24
bottom degree of H^*_{cusp}	1	2	4	6	9	12	16	20

Table: The virtual cohomological dimension and the cuspidal range for subgroups of $\mathrm{SL}_n(\mathbb{Z})$

Connection with arithmetic geometry

The groups $H^*(\Gamma; E)$ have an action of the *Hecke operators*, which are endomorphisms of the cohomology associated to certain finite index subgroups of Γ.

We expect eigenclasses of these operators to reveal arithmetic information in the cohomology.

Galois representations and eigenclasses

$\mathbf{G} = \mathrm{SL}_n/\mathbb{Q}, \quad \Gamma = \Gamma_0(N)$
$\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ absolute Galois group of $\mathbb{Q}$
$\rho\colon \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to GL_n(\mathbb{Q}_p)$ continuous semisimple Galois representation unramified outside pN.
Frob_l Frobenius congugacy class over l.
We can consider the characteristic polynomial

$$\det(1 - \rho(\mathrm{Frob}_l)\,T),$$

On the cohomology side, for each prime l not dividing N we have Hecke operators $T(l,k)$, $k = 1, \ldots, n-1$. These operators generalize the classical operator T_l on modular forms.
If ξ is a Hecke eigenclass, define the *Hecke polynomial*

$$H(\xi) = \sum_k (-1)^k l^{k(k-1)/2} a(l,k) T^k \in \mathbb{C}[T].$$

where $a(l,k)$ is the eigenvalue of $T(l,k)$.

Fix an isomorphism $\bar{\mathbb{Q}}_p \simeq \mathbb{C}$.

Conjecture (Ash)

For any Hecke eigenclass ξ of level N, there is a Galois representation $\rho\colon Gal(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_n(\mathbb{Q}_p)$ unramified outside pN such that for every prime l not dividing pN, we have

$$H(\xi) = \det(1 - \rho(Frob_l)T).$$

This is the conjecture we're ultimately testing. Note that as stated the conjecture is primarily of interest in the case of *nonselfdual* eigenclasses, since one knows how to attach Galois representations to selfdual classes (Clozel). (Selfdual classes have palindromic and real eigenvalues.)
Note that $\Gamma \backslash X$ is *not* an algebraic variety, so can't use etale cohomology to look for Galois action.
Also of interest: consider torsion coefficients (will be the subject of later work).

Our goals

- Compute $H^5(\Gamma_0(N); \mathbb{C})$ for as big a range of levels N as possible. The degree 5 is chosen because it's in the cuspidal range, and is as close to the vcd $\nu(\Gamma)$ as possible (more below).
- Compute the action of the Hecke operators on this space.
- Identify Galois representations attached to the cohomology
- Try to understand whatever we can about this cohomology space.

Tools to compute the cohomology

For modular forms, i.e. the cohomology of subgroups of $\mathrm{SL}_2(\mathbb{Z})$, we can use *modular symbols* to perform computations.
Recall that X is the upper halfplane. Let $X^* = X \cup \mathbb{Q} \cup \{i\infty\}$. Given two cusps $q_1, q_2 \in X^* \smallsetminus X$, we can form the geodesic from q_1 to q_2 and can look at the image in $\Gamma\backslash X^*$.
This gives a relative homology class

$$[q_1, q_2] \in H_1(\Gamma\backslash X^*, \text{cusps}; \mathbb{C}) \simeq H^1(\Gamma\backslash X; \mathbb{C}).$$

One knows that the vector space generated by the symbols $[q_1, q_2]$ maps surjectively onto the cohomology, and can determine the relations:

- $[q_1, q_2] = -[q_2, q_1]$
- $[q_1, q_2] + [q_2, q_3] + [q_3, q_1] = 0.$
- $[\gamma q_1, \gamma q_2] = [q_1, q_2]$, $\gamma \in \Gamma$.

Hecke action

The Hecke operators act on the modular symbols. Given T_l, we can find finitely many matrices $\{\gamma_i\}$ such that, if $\xi = [q_1, q_2]$, then

$$T_l \xi = \sum_i [\gamma_i q_1, \gamma_i q_2]$$

Moreover, we can identify a special set of modular symbols—the *unimodular symbols*—that

- is finite modulo Γ and
- spans $H^1(\Gamma)$.

These are the $\mathrm{SL}_2(\mathbb{Z})$-translates of $[0, i\infty]$. The Hecke operators do not preserve the subspace of unimodular symbols, but there is an algorithm ("Manin's trick") to write any modular symbol as a linear combination of unimodular symbols.

$n > 2$

The situation for $n > 2$ is more complicated:

- There is an analogue of the modular and unimodular symbols, and they provide a model for $H^\nu(\Gamma; \mathbb{C})$. One takes n-tuples of cusps $[q_1, \ldots, q_n]$ modulo relations, where now cusp means a minimal boundary component in a certain Satake compactification X^*.
- One can describe an analogue of Manin's trick (Ash–Rudolph)
- BUT: usually $H^\nu_{\text{cusp}} = 0$, since $\nu(\Gamma)$ usually falls outside the cuspidal range.

Remark: for $n = 3$, one can use modular symbols to compute cuspidal cohomology (Ash–Grayson–Green).

$n = 4$

Nevertheless, we can overcome these problems, at least for $n = 4$ (the first interesting case):

- We can define an explicit complex computing $H^*(\Gamma; \mathbb{C})$, the *sharbly complex* (named in honor of Sczarba and Lee).
- We can identify a finite subcomplex mod Γ, using the well-rounded retract of Ash–Soule–Lannes (equivalently, Voronoi's work on perfect quadratic forms).
- We can formulate an analogue of the Manin trick to compute the Hecke action on $H^{\nu-1}(\Gamma; \mathbb{C})$. (G)

The first two can be done for any n. The last one has only been tested in dimension 4.

Remark about other cases

This setup is also useful in other cases, such as

- $R_{F/\mathbb{Q}}(\mathrm{SL}_2)$, where F is real quadratic (Hilbert modular case) or F is complex quartic. (G–Yasaki)
- $R_{F/\mathbb{Q}}(\mathrm{SL}_3)$, where F is complex quadratic.

In all these cases, the cuspidal cohomology meets $H^{\nu-1}$ and not H^{ν}.

Results

We have computed $H^5(\Gamma_0(N); \mathbb{C})$ for N prime and ≤ 211, and for composite N up to 52. The biggest computation involed matrices of size 845712×3277686 ($N = 211$).

- No nonselfdual cuspidal classes were found :(
- We found Eisenstein classes (boundary cohomology) attached to weight 2 and weight 4 modular forms.
- We found Eisenstein classes attached to SL_3 cuspidal cohomology.
- Found selfdual cuspidal classes that are apparently functorial lifts of Siegel modular forms.

For N prime we believe this is a complete description of the cohomology, apart from nonselfdual classes.

Eisenstein cohomology

$\bar{X}$ partial bordification of X due to Borel–Serre
$\Gamma\backslash\bar{X}$ Borel–Serre compactification (orbifold with corners)
$\partial(\Gamma\backslash\bar{X}) = \Gamma\backslash\bar{X} \smallsetminus \Gamma\backslash X$.
We have

$$H^*(\Gamma\backslash\bar{X};\mathbb{C}) \simeq H^*(\Gamma\backslash X;\mathbb{C}).$$

The inclusion $\partial(\Gamma\backslash\bar{X}) \hookrightarrow \Gamma\backslash\bar{X}$ induces a restriction map

$$H^*(\Gamma\backslash\bar{X};\mathbb{C}) \to H^*(\partial(\Gamma\backslash\bar{X});\mathbb{C}),$$

and *Eisenstein classes* are those restricting nontrivially to the boundary (Harder–Schwermer–Mannkopf)

Weights 2 and 4

Each weight 2 eigenform f contributes to $H^5(\Gamma; \mathbb{C})$ in two different ways, with the Hecke polynomials

$$(1 - l^2 T)(1 - l^3 T)(1 - \alpha T + l T^2)$$

and

$$(1 - T)(1 - l T)(1 - l^2 \alpha T + l^5 T^2),$$

where $T_l f = \alpha f$.
A weight 4 eigenform g contributes with Hecke polynomial

$$(1 - l T)(1 - l^2 T)(1 - \beta T + l^3 T^2),$$

where $T_l g = \beta g$, if and only if the central special value of the L-function of g vanishes.

SL_3 cuspidal classes

These cohomology classes were originally computed by Ash–Grayson–Green.
An SL_3 cuspidal class with eigenvalues γ and γ' contributes in two different ways, with the Hecke polynomials

$$(1 - l^3 T)(1 - \gamma T + l\gamma' T^2 - l^3 T^3)$$

and

$$(1 - T)(1 - l\gamma T + l^3\gamma' T^2 - l^6 T^3).$$

Siegel modular forms

Recall the *paramodular group* of prime level

$$K(p) = \begin{pmatrix} \mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} & \mathbb{Z} & p^{-1}\mathbb{Z} \\ \mathbb{Z} & p\mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ p\mathbb{Z} & p\mathbb{Z} & p\mathbb{Z} & \mathbb{Z} \end{pmatrix} \subset \mathrm{Sp}_4(\mathbb{Q}).$$

Let $S^3(p)$ be the space of weight three paramodular forms (they are all cuspforms; there are no Eisenstein series).
This space contains the subspace $S^3_{\mathrm{G}}(p)$ of *Gritsenko lifts*, which are lifts from certain weight 3 Jacobi forms to $S^3(p)$.
Let $S^3_{\mathrm{nG}}(p)$ be the complement to $S^3_{\mathrm{G}}(p)$ in $S^3(p)$.

Siegel modular forms

The space of cuspidal paramodular forms is known pretty explicitly. First we have a dimension formula due to Ibukiyama.

Let $\kappa(a)$ be the Kronecker symbol $(\frac{a}{p})$. Define functions $f, g\colon \mathbb{Z} \to \mathbb{Q}$ by

$$f(p) = \begin{cases} 2/5 & \text{if } p \equiv 2, 3 \bmod 5, \\ 1/5 & \text{if } p = 5, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$g(p) = \begin{cases} 1/6 & \text{if } p \equiv 5 \bmod 12, \\ 0 & \text{otherwise.} \end{cases}$$

Ibukiyama's theorem

Theorem (Ibukiyama)

For p prime we have $\dim S^3(2) = \dim S^3(3) = 0$*. For $p \geq 5$, we have*

$$\begin{aligned}\dim S^3(p) = (p^2-1)/2880 \\ &+ (p+1)(1-\kappa(-1))/64 + 5(p-1)(1+\kappa(-1))/192 \\ &+ (p+1)(1-\kappa(-3))/72 + (p-1)(1+\kappa(-3))/36 \\ &+ (1-\kappa(2))/8 + f(p) + g(p) - 1.\end{aligned}$$

Using this one can easily compute the dimension of $S^3_{\mathrm{nG}}(p)$.

Hecke eigenvalues

Next, Poor and Yuen have developed a technique to compute Hecke eigenvales for forms in $S^3_{\mathrm{nG}}(p)$.
Putting these two together, we find

- For all p, the dimension of the subspace of $H^5(\Gamma_0(p); \mathbb{C})$ not accounted for by the Eisenstein classes above matches $2 \dim S^3_{\mathrm{nG}}(p)$ according to Ibukiyama.
- In cases where we have computed the Hecke action on this subspace, we find full agreement with the data produced by Poor–Yuen.

To do

- Prove that the Eisenstein classes we see actually occur for all p.
- Prove that we do indeed have a lift from Siegel modular forms to the cohomology.
- Investigate nontrivial coefficients, torsion coefficients.