STRATIFIED SPACES TWIGS

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1. Introduction

These are the notes from the TWIGS[1] talk on stratified spaces. The theory of these spaces is the work of many people, including Whitney, Thom, Mather, Hardt, Hironaka, and Łojasiewicz. The best places to read about the theory i know about are are [1, 4, 5]. Another reference is the first chapter of [6], but I'm not as familiar with this.

2. The basic idea

Most people are familiar with manifolds: a *manifold* is a topological space that locally looks like $\mathbb{R}^n$ (plus other stuff such as countable basis, Hausdorff.) Roughly speaking, a stratified space is a topological space that is built from manifolds in a nice way.

Definition 2.1. Let $X \subset \mathbb{R}^n$ be a closed subset. Then X is said to be *stratified* if there is a poset I and a locally finite collection of disjoint locally closed subsets $S_i, i \in I$ such that

(1) $Z = \bigcup S_i$
(2) $S_i \cap \overline{S}_j$ is nonempty if and only if $S_i \subset \overline{S}_j$, and this happens if and only if $i = j$ or $i < j$ (this is called the axiom of the frontier)
(3) Each S_i is a locally closed smooth submanifold of $\mathbb{R}^n$

The pieces S_i are called *strata*. The set of strata is itself a poset, with the relation induced from inclusion.

Note: *locally closed* means each subset is the intersection of a closed and open subset. For example, the open 2D disc of radius 1 embedded is $\mathbb{R}^3$ is locally closed. Any open set is locally closed. Also, since X is closed, the axiom of the frontier implies that the closure of each stratum is a union of strata.

The decomposition above is called a *stratification*. We caution that there are actually several different definitions of stratification floating around the literature, in which one requires more or less than the above (see the last section for an example or requiring more). For example, sometimes the axiom of the frontier is omitted

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[1] "The What is Graduate Seminar", initiated by F. Hajir. I thank him for inviting me to speak.

from the definition, and a stratification satisfying the axiom of the frontier is called a *primary stratification*. However, the above definition seems to require the least while still being useful, and so we will use it.

3. Examples

There are many examples of such spaces, and in fact this is the main point about them: they arise naturally in many situations where manifolds just won't do.

Example 3.1. A manifold with boundary M is a stratified space. One stratum is the boundary ∂M, and the other stratum is the complement $M \smallsetminus \partial M$. Note that strata need not be connected. All the above conditions are obviously met.

Example 3.2. More generally we can take M to be a manifold with corners. This is a space that is locally modelled not on $\mathbb{R}^n$, or even H_n (the halfspace $\{x \in \mathbb{R}^n \mid x_1 \geq 0\}$), but rather the "corner spaces" $H_{n,k} = \{x \in \mathbb{R}^n \mid x_1, \ldots, x_k \geq 0\}$, where $k = 0, \ldots, n$. A manifold with corners is homeomorphic to a manifold with boundary (simply "round off" the corners into H_n's), but it is useful in many applications to have the additional structure of the corners.

Example 3.3. Affine algebraic varieties over $\mathbb{C}$ are stratified spaces. Recall that an affine algebraic variety is the zero set of a set of polynomials $f_1, \ldots, f_k \subset \mathbb{C}[x_1, \ldots, x_n]$. For example, the variety $C = \{x^2 + y^2 - z^2 = 0\} \subset \mathbb{C}^3$ is a double cone (as usual it's easiest to picture the real points of such a space). This fails to be a manifold at the origin, and we can stratify it by taking the strata to be $C \smallsetminus \{0\}$ and $\{0\}$.

Another example is provided by the variety $P \subset \mathbb{C}^3$ defined by the vanishing of the polynomial $f = x^2 + y^3 + z^5$. The origin O is a singular point; observe that the gradient of f vanishes there. Otherwise P is a manifold.

In fact the hypersurface P is a very interesting example. Take a small sphere S^5_ε of radius ε centered at the origin, and consider the intersection $L = P \cap S^5_\varepsilon$. This is called the *link* of the isolated singular point O. One can show that L is uniquely determined if ε is sufficiently small. If P were a manifold at O, then this intersection would be a 3-sphere, since P has real dimension 4. But one can show that actually L is diffeomorphic to the *Poincare homology 3-sphere*. Thus L is a 3-manifold whose homology coincides with that of S^3, yet L is not diffeomorphic to S^3. In fact, L isn't simply-connected; its fundamental group is isomorphic to the *binary icosahedral group*, which has the presentation $\langle a, b, c \mid a^2 = b^3 = c^5 = abc\rangle$. For more about this beautiful manifold see the extremely entertaining [3].

To see how affine varieties can be stratified, let $X \subset \mathbb{C}^n$ be defined by the vanishing of the polynomials $f_1, \ldots, f_k$, where $k \leq n$. Assume for simplicity that the codimension of X in $\mathbb{C}^n$ is k. The f_i determine a $k \times n$ matrix J, the *jacobian matrix*, in which the (i, j)th entry is the polynomial $\partial f_i/\partial x_j$. Let $\Sigma(X) \subset X$ be the subvariety on which this matrix has rank less than k. Then $\Sigma(X)$ is called the *singular* *locus*. We have that $\Sigma(X)$ is a proper subvariety of X (possibly empty), and that $Y_0 := X \smallsetminus \Sigma(X)$ is a manifold of complex codimension k.([2]) This is the first step in the stratification. To get the full stratification, we inductively let $\Sigma^j(X) = \Sigma(\Sigma^{j-1}(X))$ and put $Y_j = \Sigma^j(X) \smallsetminus \Sigma^{j+1}(X)$. Then the Y_j are the strata of our stratification.

In the examples above, the varieties are only defined by one equation (i.e. they are *hypersurfaces*). Thus the jacobian is just the gradient, and the singular locus is given by all points on the varieties where the gradient vanishes. In both cases this is just the origin.

For another example, consider the *Whitney cusp*, which is the variety X defined by $x^3 + z^2x^2 - y^2 = 0$. A picture of its real points is shown in Figure 1. Computing the gradient shows that $\Sigma(X)$ is defined by $x = y = 0$, which is the z-axis. $\Sigma(X)$ is clearly a submanifold, so there are only two strata in the stratification.

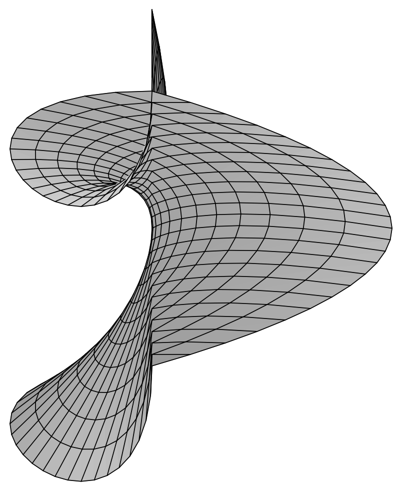

FIGURE 1. The Whitney cusp

Example 3.4. What's the largest interesting class of spaces that can be stratified? One answer is the class of *semianalytic* spaces. By definition, a *real semianalytic set* is a subset X of $\mathbb{R}^n$ if locally at each point it is defined by a finite collection of equalities $f_1 = \cdots = f_k = 0$ and inequalities $g_1 > 0, \ldots, g_l > 0$ in which the functions f_i, g_j are analytic. This includes the case where the functions are polynomials, in which case X is called *semialgebraic*. There is also a complex version, in which $\mathbb{R}^n$ is replaced by $\mathbb{C}^n$, and the inequalities $g_j > 0$ are replaced with $g_j \neq 0$. Thus this includes the affine algebraic variety example above (and the real version includes the picture). That such spaces admit stratifications is due to Łojasiewicz. Apparently

[2]It's a subvariety because the condition that J have rank $< k$ can be expressed by requiring that all $k \times k$ minors of J vanish. Thus $\Sigma(X)$ is defined by finitely many polynomial equations.

the original source for this result is a set of unpublished lecture notes from IHES in 1965, but one can also look at [4, p. 1583] and the references given there.

Example 3.5. The examples above were stratifications of singular spaces, namely spaces that aren't themselves manifolds. In fact we were really trying to decompose the singular spaces into manifolds. However, sometimes it can be useful to decompose manifolds into strata to reveal some additional structure.

As an example, recall that the (real) grassmannian $G(k, n)$ is the manifold parametrizing all k-dimensional subspaces of $\mathbb{R}^n$. We get a stratification here by fixing a *flag* $0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{n-1} \subsetneq \mathbb{R}^n$, where F_i is a subspace of dimension i. Each k-dimensional subspace $V \subset \mathbb{R}^n$ determines a sequence $\mathbf{m}(V)$ of nondecreasing integers $m_1, \ldots, m_{n-1}$, where m_i is the dimension of $V \cap F_i$. Fix such a sequence $\mathbf{m}$, and let $S_{\mathbf{m}} \subset G(k, n)$ be the subvariety of all V such that $\mathbf{m}(V) = \mathbf{m}$. Note that $S_{\mathbf{m}}$ may very well be empty, depending on $\mathbf{m}$. There are only finitely many $S_{\mathbf{m}}$, and they form a stratification of $G(k, n)$ called the *Schubert stratification.* (The varieties $S_{\mathbf{m}}$ are called *Schubert cells.*) For example, there are 6 Schubert cells in the stratification of $G(2, 4)$. Representative planes for the 6 cells are shown in Figure 2 (actually, we cheat and draw the pictures using lines in $\mathbb{P}^3(\mathbb{R})$ instead of 2-planes in $\mathbb{R}^4$).

This example also shows that even though the strata are assumed to be manifolds, their closures need not be. For a very rewarding exercise, try to determine the topology of all the closures of the Schubert cells in this case.

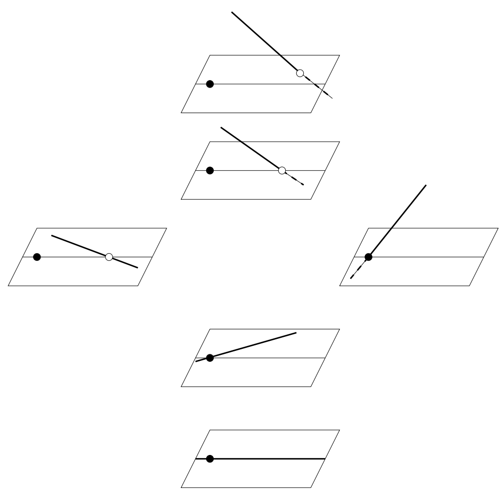

FIGURE 2. Six Schubert cells in $G(2, 4)$

Example 3.6. For a final example, suppose that a finite group G acts on a manifold M. For any subgroup $H \subset G$, let M_H be the set of all $m \in M$ whose stabilizer is equal to H. Then as H ranges over all subgroups of G, we get a stratification called the *stabilizer stratification*. Again, it's not hard to cook up examples where the closures of strata aren't manifolds.

4. Whitney stratifications

Consider the Whitney cusp X with its decomposition into two strata S_0 and S_1. A close inspection of its geometry reveals that the origin O "looks different" from the other points in the 1-dimensional stratum S_1. Here's one way to see what I mean by this. Take a point $p \in S_1$, take a plane P through p that is transverse to S_1, and consider the intersection of a small disc $D_\varepsilon \subset P$ with X. (Here we're working with the real points of X.) Then if ε is sufficiently small and p is not the origin, the intersection looks like a little X with center $P \cap S_1$. But if $p = O$, we get a different intersection: it's a V, not an X. This is telling us that the geometry is locally different around O.

The first point of a stratification is that it is a decomposition of a space X into simpler pieces, namely manifolds. But we can require more. We can require that the geometry of X looks the same at every point in a given stratum. I don't want to make this precise here, but it's clear that this fails for the cusp, and the obvious fix is to work with *three* strata instead of two. Namely, we take for our stratification the sets $X \smallsetminus S_1$, $S_1 \smallsetminus O$, and O.

How do we determine that O should be pulled out of S_1? This was Whitney's great insight.

Definition 4.1. Let X be a stratified space with strata S_i. We say the stratification is a *Whitney* stratification if the following conditions are satisfied for any pair $S_\alpha < S_\beta$:

Suppose $\{x_i\} \subset S_\alpha$ is a sequence converging to $y \in S_\beta$, and $\{y_i\} \subset S_\beta$ is a sequence also converging to y. Let τ be the limit of the sequence of tangent spaces $T_{y_i} S_\beta$, and let ℓ be the limiting secant line for the sequence $\ell_i = x_i y_i$. Then

(A) $T_y S_\alpha \subset \tau$ and
(B) $\ell \subset \tau$

For an exercise, you can check that the stratification of the cusp isn't Whitney unless O is taken to be a separate stratum. For a recent exposition of the construction of Whitney stratifications, one can look at [2].

References

1. Mark Goresky and Robert MacPherson, *Stratified Morse theory*, Ergebnisse der Mathematik und ihrer Grenzgebiete (3) [Results in Mathematics and Related Areas (3)], vol. 14, Springer-Verlag, Berlin, 1988.
2. V. Yu. Kaloshin, *A geometric proof of the existence of Whitney stratifications*, Mosc. Math. J. **5** (2005), no. 1, 125–133.
3. R. C. Kirby and M. G. Scharlemann, *Eight faces of the Poincaré homology* 3*-sphere*, Geometric topology (Proc. Georgia Topology Conf., Athens, Ga., 1977), Academic Press, New York, 1979, pp. 113–146.

4. Stanislas Łojasiewicz, *Sur la géométrie semi- et sous-analytique*, Ann. Inst. Fourier (Grenoble) **43** (1993), no. 5, 1575–1595.
5. John Mather, *Notes on topological stability*, available from `www.math.princeton.edu`.
6. Masahiro Shiota, *Geometry of subanalytic and semialgebraic sets*, Progress in Mathematics, vol. 150, Birkhäuser Boston Inc., Boston, MA, 1997.

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