QUADRATIC WEYL MULTIPLE DIRICHLET SERIES

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Abstract. We construct multiple Dirichlet series in several complex variables whose coefficients involve quadratic residue symbols. The series are shown to have an analytic continuation and satisfy a group of functional equations isomorphic to a Weyl group. This is joint work with Gautam Chinta; for full details we refer to [2].

1. Multiple Dirichlet series

We begin with an overview of multiple Dirichlet series, with some hints as to what this talk is about. We will be intentionally vague.

A (usual, single-variable) Dirichlet series is a sum of the form

$$Z(s) = \sum_{m} \frac{a(m)}{m^s}, \tag{1.1}$$

where s is a complex variable, and each $a(m)$ is a complex number. The ms can range over the positive integers, or perhaps some interesting subset of the positive integers. Or sometimes the ms range over integral ideals in a number field, perhaps prime to some fixed ideal; in this case we replace m^{-s} with $|\,\mathrm{N}(m)|^{-s}$ in (1.1). Or maybe the ms range over a subset of the integers in a number field, in which case we probably want to mod out by the action of a unit group, ... There is basically no end to what number theorists are willing to consider in the basic framework of (1.1).

Typically (but not always) the $a(m)$s come from some arithmetic situation. One hopes to gain insight into their arithmetic nature by packaging them together into a complex-valued function Z, and then studying properties of the latter. This is an old idea with lots of applications.

If one is really lucky, then Z will have lots of nice properties. For instance Z should converge as long as the real part of s is big enough. One hopes that Z extends to be a meromorphic function on the whole complex plane. After multiplying by a simple function of s (*Gamma factors*), one hopes that Z satisfies a functional equation of the shape

$$s \to 1 - s. \tag{1.2}$$

These features can't always be realized, even when the coefficients in (1.1) come from nice arithmetic objects (e.g. partial zeta functions don't satisfy a functional equation).

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But there are plenty of times when they can be, and if they can then $Z(s)$ is certainly telling us something interesting about the $a(m)$s.

At this level of imprecision, it's clear what a multiple Dirichlet series should be. We should have a sum of the shape

$$Z(s_1,\dots,s_r)=\sum_{m_1,\dots,m_r}\frac{a(m_1,\dots,m_r)}{m_1^{s_1}\cdots m_r^{s_r}},\quad a(m_1,\dots,m_r)\in\mathbb{C}. \tag{1.3}$$

We hope that (1.3) has good analytic properties, for example convergence if the real part of the vector $(s_1,\dots,s_r)$ lies in some positive r-cone. We also want functional equations, not just involutions like $s_i\to 1-s_i$, but a whole *group* of functional equations intermixing all the variables. For example, one of the two variable series $Z(s,w)$ that we'll consider has two basic involutions of its variables:

$$\sigma_1\colon (s,w)\to(1-s,s+w-1/2),\qquad \sigma_2\colon (s,w)\to(s+w-1/2,1-w). \tag{1.4}$$

It's not hard to see that the subgroup of the affine transformations of $\mathbb{C}^2$ generated by σ_1,σ_2 is isomorphic to the symmetric group S_3, with these generators satisfying the standard relation $(\sigma_1\sigma_2)^3=1$. Moreover $(1/2,1/2)$ is the unique fixed point. Figure 1 shows that the action is the same as the usual reflection representation of S_3 on a 2-dimensional vector space; the point $(s,w)=(1/2,1/2)$ is at the center. In the figure, we represent elements of S_3 as words in σ_1,σ_2, showing only the ordered list of subscripts to save space. Thus 121 is the longest word $\sigma_1\sigma_2\sigma_1$; e is the identity. A word w over a region R means that $R=wC$, where C is the red cone in the upper right of the figure.

2. Why study them?

One reason to study multiple Dirichlet series is that *Passing to more variables can shed light on the single variable case*. We illustrate with an example. Let d be a positive squarefree integer, and consider the Dirichlet L-function attached to the quadratic extension $\mathbb{Q}(\sqrt{d})$ of $\mathbb{Q}$. If we write the quadratic character for this extension as χ_d, then we're considering the L-series

$$L(s,\chi_d)=\sum_{m>0}\frac{\chi_d(m)}{m^s}.$$

For instance,

$$L(s,\chi_3)=1-\frac{1}{2^s}+\frac{1}{4^s}-\frac{1}{5^s}-\frac{1}{7^s}-\frac{1}{8^s}+\frac{1}{10^s}+\dots$$

Now suppose we want to study the central values $L(1/2,\chi_d)$ as a function of d. One approach is to package all the L-series into a bigger function by introducing a new variable w:

$$Z^*(s,w)=\sum_d\frac{L(s,\chi_d)}{d^w},\quad s,w\in\mathbb{C}. \tag{2.1}$$

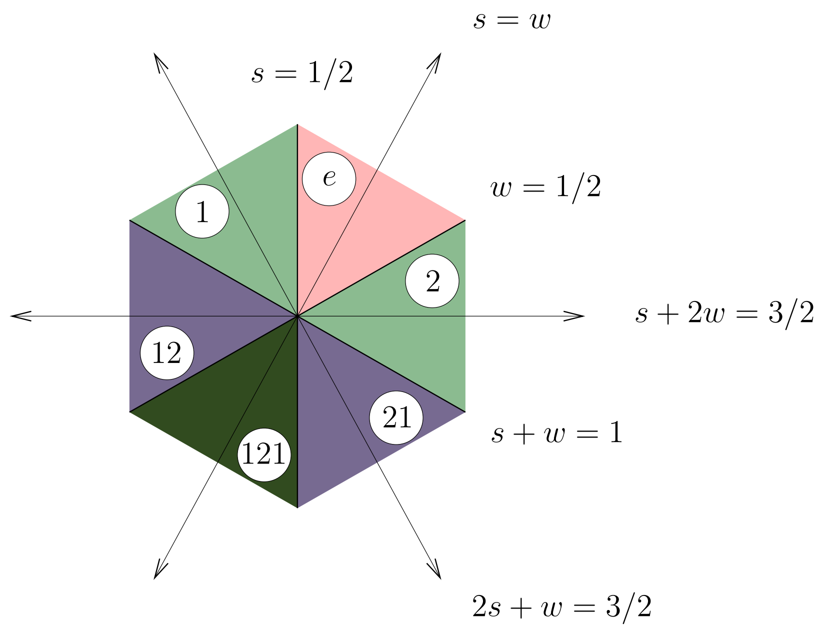

FIGURE 1.

Here we might take d to run over all discriminants of real quadratic fields. Of course it's not clear that this is a nice object, but Siegel [7] and Goldfeld and Hoffstein [6] showed that it is. More precisely, Goldfeld and Hoffstein proved

- $Z^*(1/2, w)$ is absolutely convergent if $\Re w > 1$.
- As a function of w, $Z^*(1/2, w)$ extends past $\Re w = 1$ with a double pole at $w = 1$.

From this they concluded[1]

$$\sum_{0<d<X} L(1/2, \chi_d) \sim CX \log X, \qquad X \gg 0,$$

where C is the constant[2]

$$C = \frac{3}{16} \prod_{p \neq 2} (1 - 2p^{-2} + p^{-3}) \simeq 0.1284748\ldots,$$

and the summation runs over real discriminants. This can be interpreted as the truth of the "Lindelöf hypothesis in the d aspect on average" for this family of L-functions;[3] one can see how a multiple Dirichlet series helped to obtain this result;

[1]Similar asymptotics, for sums over slightly different sets of d, were achieved earlier by Takhtadzjan–Vinogradov (1980) and Jutila (1981).

[2]The product converges very slowly. The floating point approximation is our guess, which was computed using 41000 primes past $p = 2$.

[3]The *Lindelöf hypothesis* for the Riemann zeta function asserts that for any $\varepsilon > 0$ there exists a constant C_ε such that $|\zeta(1/2 + it)| < C_\varepsilon |t|^\varepsilon$. The *Lindelöf hypothesis in the d aspect* for Dirichlet

for further details we refer to [4]. In the literature one can find more examples of multiple Dirichlet series being used to compute mean value/nonvanishing results for other families of L-functions.

Now let's play with (2.1) a bit. We work heuristically in a perfect world, in which all pesky details can be ignored. In this perfect world

- quadratic reciprocity has no signs: $\left(\frac{d}{m}\right) = \left(\frac{m}{d}\right)$;
- all integers are pairwise relatively prime; and
- all positive integers are discriminants of real quadratic fields.

If we insert the definition of $L(s, \chi_d)$ in (2.1), using our idealized assumption $\chi_d(m) = \left(\frac{d}{m}\right)$, we find

$$Z^*(s,w) = \sum_{d,m} \left(\frac{d}{m}\right) m^{-s} d^{-w}. \tag{2.2}$$

In our perfect world, (2.2) is completely symmetric in m and d. We already have a functional equation $s \to 1-s$, which comes from the original family of L-functions $L(s, \chi_d)$; interchanging the order of summation and collecting the sums over ds into Dirichlet L-functions, we get a functional equation of the form $w \to 1-w$. So we might hope heuristically to cook up something like (2.2) with good analytic properties in both s and w.

Of course all this is nonsense. Nevertheless it can be fixed. In fact Siegel essentially studied Z^* in 1956 [7], and the resulting series is an example of our main construction. We define

$$Z(s,w) = \sum\sum_{\substack{d,m \geq 1 \\ d,m \text{ odd}}} \frac{\chi_d(\hat{m})}{m^s d^w} a(d,m), \tag{2.3}$$

where $\hat{m}$ denotes the part of m relatively prime to the squarefree part of d and as before χ_d is the quadratic character associated to the extension $\mathbb{Q}(\sqrt{d})$ of $\mathbb{Q}$. The multiplicative factor $a(d,m)$ is defined by

$$a(d,m) = \prod_{\substack{p \text{ prime} \\ p^k || d, \ p^l || m}} a(p^k, p^l),$$

and

$$a(p^k, p^l) = \begin{cases} \min(p^{k/2}, p^{l/2}) & \text{if } \min(k,l) \text{ is even,} \\ 0 & \text{otherwise.} \end{cases} \tag{2.4}$$

The function a is shown in Figure 2. With this definition, $Z(s,w)$ satisfies the functional equations in (1.4).

L-functions asserts that for any $\varepsilon > 0$ there exists a constant C_ε such that $|L(1/2), \chi_d| < C_\varepsilon |d|^\varepsilon$ as $d \to \infty$, where χ_d is some Dirichlet character mod d for each d.

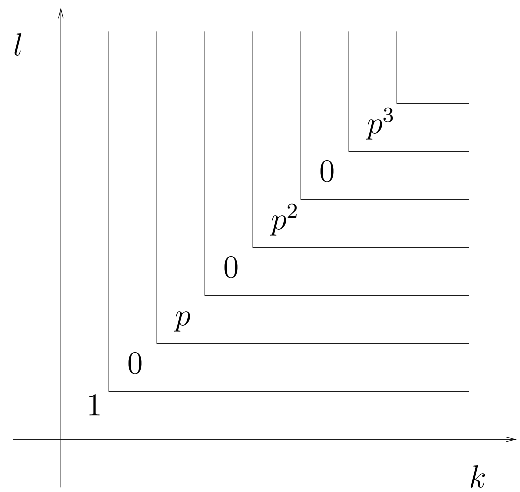

FIGURE 2. The function $a(p^k, p^l)$

3. Quadratic Weyl multiple Dirichlet series

We are now ready to define our multiple Dirichlet series. They will be generalizations of Siegel's series (2.3), in that they will involve quadratic characters, and among the examples will be r-variable series with group of functional equations isomorphic to S_{r+1}. For full details, see [2].

Let K be a number field with ring of integers $\mathscr{O}$. Let $S = S_f \cup S_\infty$ be a set of places with S_∞ all archimedian places and S_f large enough so that the ring of S-integers $\mathscr{O}_{S_f}$ has class number 1. Let $\mathscr{I}(S)$ be the group of integral ideals prime to S_f, and let $\mathscr{J}(S)$ be the group of fractional ideals coprime with S_f.

There is a quadratic residue symbol $\left(\frac{*}{*}\right) : \mathscr{J}(S) \times \mathscr{J}(S) \to \{-1, 0, 1\}$. Essentially it's set up so that $\left(\frac{a}{*}\right)$ gives the character attached to the abelian extension $K(\sqrt{a})$, but there are some technicalities that we will ignore for this talk. The full details of this symbol were worked through by Fisher–Friedberg [5].

Let Φ be an irreducible, reduced root system of rank r. We choose a subset of positive roots Φ^+ and a subset of simple roots $\alpha_1, \ldots, \alpha_r$. We will assume Φ is simply-laced, although we did this in [2] mainly for convenience. If one is interested in other root systems there is no loss of generality because we are working with quadratic characters: our main construction will give series for non-simply-laced root systems by setting certain variables equal to one another in a simply-laced construction.

Let $\mathbf{s} = (s_1, \ldots, s_r)$ be a vector of r complex variables, indexed by the simple roots in Φ, that is by the nodes of the Dynkin diagram for Φ. Let $\mathbf{I} = (I_1, \ldots, I_r)$ be a tuple of ideals from $\mathscr{I}(S)$, and let $\Psi = (\psi_1, \ldots, \psi_r)$ be a collection of r idèle class characters unramified outside of S. We denote by $\Psi(\mathbf{I})$ the product

$$\prod_i \psi_i(I_i).$$

Now we come to our main construction. We provisionally define

$$Z_S(\mathbf{s},\Psi) = \sum_{\mathbf{I}\in\mathscr{I}(S)^r} \frac{\Psi(\mathbf{I})H(\mathbf{I})}{\prod_j |I_j|^{s_j}}, \tag{3.1}$$

where $H\colon \mathscr{I}(S)^r \to \mathbb{Z}$ is a function we will specify later. In fact correctly defining H is the main part of the whole story; for type A_2 and $K=\mathbb{Q}$, this is essentially the function a shown in Figure 2. The function H will be constructed so that $Z_S(\mathbf{s},\Psi)$ will satisfy r functional equations $\sigma_1,\dots,\sigma_r$, taking $\mathbf{s}=(s_1,\dots,s_r)$ to $\sigma_{j_0}\mathbf{s}=(s_1',\dots,s_r')$, where

$$s_j' = \begin{cases} s_j + s_{j_0} - 1/2 & \text{if } j \text{ and } j_0 \text{ are adjacent,} \\ 1 - s_{j_0} & \text{if } j = j_0, \text{ and} \\ s_j & \text{otherwise.} \end{cases} \tag{3.2}$$

Here adjacent means that the variables correspond to adjacent nodes of the Dynkin diagram for Φ. It is easy to check that these involutions generate a group isomorphic to the Weyl group of Φ. Note that H is the *only* part of the definition reflecting the structure of Φ; without it (3.1) has nothing to do with Φ, except that the number of variables is the same as the rank of Φ.

What properties should H have? First, it should satisfy a *twisted multiplicativity* condition: given ideals $I_j, I_j' \in \mathscr{I}(S)$ with $(I_1I_2\cdots I_r, I_1'I_2'\cdots I_r') = 1$ we'll have

$$\frac{H(I_1I_1',\dots,I_rI_r')}{H(I_1,\dots,I_r)H(I_1',\dots,I_r')} = \prod_{\substack{i,j\text{ adj.}\\ i<j}} \left(\frac{I_i}{I_j'}\right)\left(\frac{I_i'}{I_j}\right) \tag{3.3}$$

Note that if H were actually *multiplicative*, then the right of (3.3) would be 1. Instead it is a product of symbols reflecting the structure of Φ. Twisted multiplicativity is a nice property, but it means that Z_S won't have an Euler product.

Next, the twisted multiplicativity means that to compute H we only have to know how to do it on tuples of the form $(P^{k_1},\dots,P^{k_r})$, where P is a fixed prime ideal, and where the k_i are nonnegative integers. But H can't be just anything on such tuples; otherwise we won't get the functional equations.

So how do we build H? Our main result is the following: let $\mathbf{x}=(x_1,\dots,x_r)$ be a vector of variables and let F be the function field $\mathbb{C}(\mathbf{x})$. Then there is a rational function $f(\mathbf{x})$ such that

$$f(\mathbf{x}) = \sum_{k_1,\dots,k_r\geq 0} H(P^{k_1},\dots,P^{k_r})x_1^{k_1}\cdots x_r^{k_r}. \tag{3.4}$$

For example, the function f for Siegel's series is

$$f(x,y) = \frac{1 + x + y - xy^2 - x^2y - x^2y^2}{(1-x^2)(1-y^2)(1-px^2y^2)}. \tag{3.5}$$

It is a pleasant exercise to check that (3.5) agrees with (2.4). In the next section, we explain how to construct the function f attached to Φ, which completes the definition of (3.1).

4. A Weyl group action

There is an obvious similarity between (3.5) and the Weyl character formula. The Weyl formula expresses the character of a representation as a ratio of two W-alternating polynomials; the denominator can also be written as a monomial times a product of linear factors over the positive roots. These features are clearly visible in (3.5). For instance, the demonominator clearly corresponds to a product over the positive roots. The numerator is a sum of 6 terms; each can be seen to correspond to a vertex of a weight polygon.

In fact our construction of $f(\mathbf{x})$ is by forming a certain average over the Weyl group and dividing by a standard denominator $\Delta(\mathbf{x})$. The difference is that we average rational functions (not monomials), and the final expression ends up having a slightly different denominator than $\Delta(\mathbf{x})$ (although the final denominator is still given by a product over the positive roots). There is a tantalizing similarity between our construction and the Weyl formula, but the exact relationship remains elusive.

Let us now describe the construction of $f(\mathbf{x})$. We begin by defining an action of the Weyl group W on the field $\mathbb{C}(\mathbf{x})$. This action is cooked up so that when H is given by (3.3) and (3.4), then $Z_S(\mathbf{s}, \Psi)$ will satisfy the correct group of functional equations if and only if $f(\mathbf{x})$ is invariant under the action of W.

We define the W-action in stages. Let q be an indeterminate; we formally add $\sqrt{q}$ and its inverse to $\mathbb{C}(\mathbf{x})$ (eventually q will be a prime power, but we keep it as a variable for now). First, for $\mathbf{x} = (x_1, x_2, \ldots, x_r)$ define $\sigma_i \mathbf{x} = \mathbf{x}'$, where

$$x'_j = \begin{cases} x_i x_j \sqrt{q} & \text{if } i \text{ and } j \text{ are adjacent,} \\ 1/(qx_j) & \text{if } i = j, \text{ and} \\ x_j & \text{otherwise.} \end{cases} \tag{4.1}$$

It is easy to see that

$$\begin{array}{ll} \sigma_i^2 \mathbf{x} = \mathbf{x} & \text{for all } i, \\ \sigma_i\sigma_j\sigma_i \mathbf{x} = \sigma_j\sigma_i\sigma_j \mathbf{x} & \text{if } i \text{ and } j \text{ are adjacent,} \\ \sigma_i\sigma_j \mathbf{x} = \sigma_j\sigma_i \mathbf{x} & \text{otherwise.} \end{array} \tag{4.2}$$

Next, define $\epsilon_i \mathbf{x} = \mathbf{x}'$, where

$$x'_j = \begin{cases} -x_j & \text{if } i \text{ and } j \text{ are adjacent,} \\ x_j & \text{otherwise.} \end{cases} \tag{4.3}$$

For $f \in F$ define

$$f_i^+(\mathbf{x}) = \frac{f(\mathbf{x}) + f(\epsilon_i \mathbf{x})}{2} \quad \text{and} \quad f_i^-(\mathbf{x}) = \frac{f(\mathbf{x}) - f(\epsilon_i \mathbf{x})}{2}. \tag{4.4}$$

Hence $f_i^+(\mathbf{x})$ is the "even part of f in the variables adjacent to x_i," and similarly for $f_i^-(\mathbf{x})$. Finally we can define the action of W on F for a generator $\sigma_i \in W$:

$$(f|\sigma_i)(\mathbf{x}) = -\frac{1 - qx_i}{qx_i(1 - x_i)} f_i^+(\sigma_i \mathbf{x}) + \frac{1}{x_i\sqrt{q}} f_i^-(\sigma_i \mathbf{x}) \tag{4.5}$$

This extends to an action of W on F (not obvious, but can be proved by direct computation).

We need one more ingredient to define f. Let Λ be the root lattice (the lattice generated by the simple roots). For any $\alpha \in \Lambda$, let $\mathbf{x}^\alpha$ be the Laurent monomial $x_1^{d_1} \cdots x_r^{d_r}$, where $\alpha = d_1\alpha_1 + \cdots + d_r\alpha_r$, and let $d(\alpha) = \sum d_i$ be the height of α. Then we define $\Delta(\mathbf{x})$ to be the polynomial

$$\Delta(\mathbf{x}) = \prod_{\alpha \in \Phi^+} (1 - q^{d(\alpha)} \mathbf{x}^{2\alpha}),$$

and let

$$j(w, \mathbf{x}) = \Delta(\mathbf{x})/\Delta(w\mathbf{x}).$$

One can check that j satisfies the one-cocycle relation

$$j(ww', \mathbf{x}) = j(w, w'\mathbf{x})j(w', \mathbf{x}). \tag{4.6}$$

We are now ready to construct the rational function $f(\mathbf{x})$. Define

$$f_0(\mathbf{x}) = \sum_{w \in W} j(w, \mathbf{x})(1|w)(\mathbf{x}), \tag{4.7}$$

and put

$$f(\mathbf{x}) = f_0(\mathbf{x})\Delta(\mathbf{x})^{-1}. \tag{4.8}$$

Here is our main theorem:

Theorem 4.1. [2] *If $P \in \mathscr{I}(S)$ is a prime with norm q, define $H(P^{k_1}, \ldots, P^{k_r})$ via* (3.4), *where $f(\mathbf{x})$ is given in* (4.8). *Extend H to a function on $\mathscr{I}(S)^r$ using the twisted multiplicativity* (3.3). *Then we have the following:*

- *The function $Z_S(\mathbf{s}, \Psi)$ has an analytic continuation to $\mathbb{C}^r$.*
- *The collection of these functions as Ψ ranges over r-tuples of quadratic idèle class characters unramified outside of S satisfies a group of functional equations isomorphic to W. These functional equations can be expressed in terms of a certain scattering matrix. (The exact statement is somewhat complicated; we refer to [2, Theorems 5.4–5.5] for the full statement.) The action on the variables is given by* (3.2).
- *Finally, $Z_S(\mathbf{s}, \Psi)$ is analytic outside the hyperplanes $(w\mathbf{s})_j = 1$, for $w \in W, 1 \leq j \leq r$. Here $(w\mathbf{s})_j$ denotes the jth component of $w\mathbf{s}$.*

5. Open problems and complements

- The series Z described above are one of an infinite set of similary defined series; the general series includes the choice of a dominant weight θ. For Z the weight is ρ, the sum of the fundamental weights; the general series are called *twisted Weyl group multiple Dirichlet series*. Our construction also gives rise to twisted series, if one modifies the Weyl group action to incorporate this weight.

  One can show that in the untwisted case $f(\mathbf{x})$ and hence the function H is uniquely determined by the functional equations, but that $f(\mathbf{x})$ is not uniquely determined in general. In general the space of possible solutions is one less than he number of regular dominant weights in the multiplicity diagram for θ [3].
- If one defines
$$D(\mathbf{x}) = \prod_{\alpha\in\Phi^+} (1 - q^{d(\alpha)-1}\mathbf{x}^{2\alpha}),$$
  then it turns out that $f(\mathbf{x})D(\mathbf{x})$ is actually a polynomial $N(\mathbf{x})$ in the x_i [3]. ($D(\mathbf{x})$ is the denominator appearing in (3.5).) The same is true if one considers twisted series. Moreover, for type A and all dominant weights θ, Brubaker, Bump, Friedberg, and Hoffstein have given a conjectural description for a P-part polynomial in terms of Gelfand–Tsetlin patterns [1].[4] They conjecture that the resulting multiple Dirichlet series coincides with a Whittaker coefficient of an Eisenstein series for the minimal parabolic subgroup of a 2-fold cover of GL_n. We conjecture that their P-part polynomial coincides with ours [3].

References

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[4] Actually they have given a description for the P-part polynomial for multiple Dirichlet series constructed using n-power residue symbols, not just quadratic residue symbols.

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