TORIC VARIETIES AND MODULAR FORMS

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ABSTRACT. Let $N \subset \mathbb{R}^r$ be a lattice, and let $\deg\colon N \to \mathbb{C}$ be a piecewise-linear function that is linear on the cones of a complete rational polyhedral fan. Under certain conditions on deg, the data $(N, \deg)$ determines a function $f\colon \mathfrak{H} \to \mathbb{C}$ that is a holomorphic modular form of weight r for the congruence subgroup $\Gamma_1(l)$.

Moreover, by considering all possible pairs $(N, \deg)$, we obtain a natural subring $\mathscr{T}(l)$ of modular forms with respect to $\Gamma_1(l)$. We construct an explicit set of generators for $\mathscr{T}(l)$, and show that $\mathscr{T}(l)$ is stable under the action of the Hecke operators. Finally, we relate $\mathscr{T}(l)$ to the Hirzebruch elliptic genera that are modular with respect to $\Gamma_1(l)$.

1. Introduction

1.1. The construction of arithmetically distinguished automorphic forms from theta series has a long and rich history. An early spectacular manifestation is the computation of the number of representations of an integer by a sum of four squares, in which theta series are compared with Eisenstein series (cf. [12]). Another is Hecke's basis problem, which asks if all modular forms can be expressed in terms of theta functions of quaternary quadratic forms associated to orders in definite rational quaternion algebras [4, 5, 7]. In [14], Waldspurger has made a deep study of the generation of modular forms by theta series.

In this paper, we construct a subspace $\mathscr{T}(l)$ of the level l holomorphic modular forms with character using toric geometry and the combinatorics of rational polyhedral fans. We show that our modular forms are related to products of logarithmic derivatives of theta functions with characteristic. Our main results say that $\mathscr{T}(l)$ is a natural subspace:

- It is a finitely generated ring over $\mathbb{C}$.
- It is stable under the action of the Hecke operators.
- It is stable under the Fricke involution.
- It is stable under Atkin-Lehner lifting.

For $l \geq 5$, the generators of $\mathscr{T}(l)$ are the *Hecke-Eisenstein* forms (see Remark 5.10).

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1.2. To describe our construction in more detail, we must fix notation. Let l be a positive integer, and let $\Gamma_1(l) \subset SL_2(\mathbb{Z})$ be congruence subgroup of matrices satisfying

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \bmod l.$$

For any positive integer r, let $\mathscr{M}_r(\Gamma_1(l))$ be the $\mathbb{C}$-vector space of weight r holomorphic modular forms for $\Gamma_1(l)$.

Let X be a complete, possibly singular, toric variety of dimension r, and let $\Sigma \subset \mathbb{R}^r$ be the corresponding fan, which we assume to be rational with respect to a lattice N. Let $\deg\colon N \to \mathbb{C}$ be a piecewise-linear function that is linear on the cones of Σ. Under certain conditions on deg, the data $(N, \deg)$ determines a function $f_{N,\deg}\colon \mathfrak{H} \to \mathbb{C}$ that lies in $\mathscr{M}_r(\Gamma_1(l))$. We call $f_{N,\deg}$ a *toric form.*

Essentially, $f_{N,\deg}$ is constructed as the alternating sum over the cones of Σ of a collection of q-expansions

$$\sum_{m\in M} \sum_{n\in C\cap N} q^{m\cdot n} \mathrm{e}^{2\pi \mathrm{i} \deg(n)}, \tag{1}$$

where M is the dual lattice of N, and $m \cdot n$ is the natural pairing. However, we emphasize that (1) is a formal expression, and some effort is required to see that (1), and the sum over the cones of Σ, has meaning.

1.3. In a sense, our q-expansions differ dramatically from those constructed from quadratic forms, in that they are "piecewise-linear" theta series: instead of simply counting the number of lattice points on an ellipsoid, we record an underlying convex-geometric structure by twisting by roots of unity through the function deg. The simple nature of (1) might make one expect that toric forms have little arithmetic content, or at best that their span contains only Eisenstein series. However, this is definitely not the case. For weight 2 and prime level $l < 37$, for example, we have $\mathscr{T}_2(l) = \mathscr{M}_2(l)$. In general we do not know how much of $\mathscr{M}(\Gamma_1(l))$ is captured by $\mathscr{T}(l)$, nor can we provide an arithmetic characterization of the toric forms (however, see Remark 4.14).

1.4. Here is an overview of this paper. In §2, we define toric forms, and show that they are well-defined by relating them to a construction in homological algebra. In §3, we describe toric forms in terms of theta functions by interpreting the former as the graded Euler characteristic of a certain infinite-dimensional bundle over X. In §4, we show that toric forms are modular, and give a set of generators for $\mathscr{T}(l)$. In §5, we prove our main results about the compatibility of $\mathscr{T}(l)$ with Hecke theory and lifting. Finally, in an appendix (§6), we present background on toric varieties for readers not familiar with them.

2. Degree functions and functions on the upper halfplane

2.1. Let N be a lattice of rank r, and let $\deg\colon N \to \mathbb{C}$ be a piecewise-linear function. Then deg is a *degree function* if there exists a complete rational polyhedral fan Σ such that deg is linear on all cones of Σ.

Let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice, and let $m \cdot n$ be the natural pairing. Let $M_{\mathbb{C}} = M \otimes \mathbb{C}$. For every cone $C \in \Sigma$, we can define a map

$$f_C\colon \mathfrak{H} \times M_{\mathbb{C}} \to \mathbb{C}$$

as follows. For $\tau \in \mathfrak{H}$, let $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$. If $m \in M$ satisfies

$$m \cdot (C \smallsetminus \{0\}) > 0, \tag{2}$$

then we set

$$f_C(q, m) := \sum_{n \in C} q^{m\cdot n} \mathrm{e}^{2\pi\mathrm{i}\deg(n)},$$

for all τ with sufficiently large imaginary part. For all other m and q, we define f_C by analytic continuation. It is easy to see that f_C is a rational function of $q^{m\cdot n_i}$, where $\{n_i\}$ is any basis of N. To emphasize the origin of f_C as a sum, we will usually write

$$\mathrm{a.c.}\Big(\sum_{n\in C} q^{m\cdot n}\mathrm{e}^{2\pi\mathrm{i}\deg(n)}\Big)$$

instead of $f_C(q, m)$.

Definition 2.2. Let $\deg\colon N \to \mathbb{C}$ be a degree function with respect to the fan Σ. Assume that $\deg(d) \notin \mathbb{Z}$ for the generator of any one-dimensional cone of Σ. Then the *toric form* associated to $(N, \deg)$ is the function $f_{N,\deg}\colon \mathfrak{H} \to \mathbb{C}$ defined by

$$f_{N,\deg}(q) := \sum_{m\in M}\Big(\sum_{C\in\Sigma}(-1)^{\operatorname{codim} C}\mathrm{a.c.}\Big(\sum_{n\in C} q^{m\cdot n}\mathrm{e}^{2\pi\mathrm{i}\deg(n)}\Big)\Big).$$

We denote by $\mathscr{T}$ the $\mathbb{C}$-vector space generated by the toric forms, and by $\mathscr{T}_r \subset \mathscr{T}$ the subspace generated by those $f_{N,\deg}$ with $\operatorname{rank} N = r$.

Example 2.3. Suppose that $\deg(d) = 1/2$ for all generators d of one-dimensional cones of Σ, and suppose that the toric variety X associated to Σ is nonsingular. Then the function $f_{N,\deg}$ is a normalized elliptic genus of X [2]. This example was our major motivation and the starting point for this paper.

Example 2.4. Let $N = \mathbb{Z}^2 \subset \mathbb{R}^2$, and let Σ be the fan in Figure 1. Then the corresponding toric variety is the projective plane $\mathbb{P}^2$. Assume that deg takes the indicated values on the generators of the one-dimensional cones. Then after simplifying, one sees that the toric form associated to this data is

$$f_{N,\deg}(q) = \sum_{a,b\in\mathbb{Z}} \frac{1 - \mathrm{e}^{2\pi\mathrm{i}\alpha}\mathrm{e}^{2\pi\mathrm{i}\beta}\mathrm{e}^{2\pi\mathrm{i}\gamma}}{(1-\mathrm{e}^{2\pi\mathrm{i}\alpha}q^a)(1-\mathrm{e}^{2\pi\mathrm{i}\beta}q^b)(1-\mathrm{e}^{2\pi\mathrm{i}\gamma}q^{-a-b})}.$$

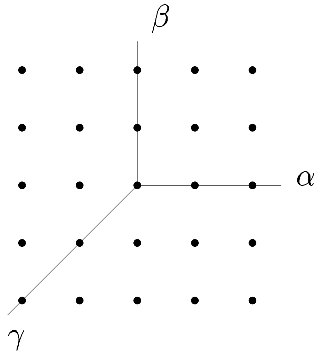

FIGURE 1. A toric form associated to $\mathbb{P}^2$.

In the rest of this section we will show that $f_{N,\deg}$ is well-defined. Moreover, in §4, we will justify our nomenclature by showing that if $\deg(N) \subset \mathbb{Q}$, then $f_{N,\deg}$ is a holomorphic modular form of weight r.

2.5. Our first goal is to investigate the function

$$r(q,m) = \sum_{C\in\Sigma} (-1)^{\operatorname{codim} C} \text{a.c.}\Big(\sum_{n\in C} q^{m\cdot n} \mathrm{e}^{2\pi \mathrm{i} \deg(n)}\Big).$$

First of all, it is easy to see that $r(q,m)$ does not depend on Σ, in the sense that Σ can be replaced by any fan on which deg is piecewise-linear. Indeed, it suffices to show that $r(q,m)$ does not change if Σ is subdivided. This is easily seen to be true for those cones and $m \in M$ where the corresponding sums are absolutely convergent. Analytic continuation then finishes the argument. Therefore, $f_{N,\deg}$ is independent of Σ, as our notation suggests.

2.6. Now we want to give a homological interpretation of $r(q,m)$. Let $\mathbb{C}[N]^\Sigma$ be the following deformation of $\mathbb{C}[N]$. As a vector space $\mathbb{C}[N]^\Sigma = \otimes_{n\in N}\mathbb{C}y^n$, where y is a dummy multi-variable. We define multiplication in $\mathbb{C}[N]^\Sigma$ by

$$y^{n_1}y^{n_2} = \begin{cases} y^{n_1+n_2} & \text{if there exists } C\in\Sigma \text{ with } n_1,n_2\in C,\\ 0 & \text{otherwise.}\end{cases}$$

Furthermore, for each cone $C\in\Sigma$, we have the submodule $\mathbb{C}[C]\subset\mathbb{C}[N]^\Sigma$.

Let R be the polynomial ring $\mathbb{C}[y^d \mid d\in D]$, where D is the set of generators of the one-dimensional cones of Σ. There is a natural action of R on $\mathbb{C}[N]^\Sigma$, where any product y^dy^n is zero unless d and n lie in some cone of Σ. It is straightforward to see that $\mathbb{C}[N]^\Sigma$ and $\mathbb{C}[C]$ are finitely generated R-modules.

The modules $\mathbb{C}[N]^\Sigma$ and $\mathbb{C}[C]$ have a natural grading by N that is compatible with the R-action. Moreover, these modules have a $\mathbb{C}$-grading induced by deg. In particular, the spaces $\operatorname{Tor}^i_R(\mathbb{C}[N]^\Sigma,\mathbb{C})$ have a natural $N\oplus\mathbb{C}$ grading.

Proposition 2.7. *Let D be the set of generators of one-dimensional cones of Σ. Then*

$$r(q,m)=\frac{\sum_i(-1)^i\mathrm{Trace}_{\mathrm{Tor}^i_R(\mathbb{C}[N]^\Sigma,\mathbb{C})}\,q^{m\cdot\mathbf{n}}\mathrm{e}^{2\pi\mathrm{i}\mathbf{deg}}}{\prod_{d\in D}(1-\mathrm{e}^{2\pi\mathrm{i}\deg(d)}q^{m\cdot d})}.$$

Here $m\cdot\mathbf{n}$ (respectively $\mathbf{deg}$) denotes the linear operator that takes the value $m\cdot n$ (resp. $\deg$) on each graded component.

Proof. To begin, notice the long exact sequence of Tor implies that the function

$$r(A;q,m):=\frac{\sum_i(-1)^i\mathrm{Trace}_{\mathrm{Tor}^i_R(A,\mathbb{C})}\,q^{m\cdot\mathbf{n}}\mathrm{e}^{2\pi\mathrm{i}\mathbf{deg}}}{\prod_{d\in D}(1-\mathrm{e}^{2\pi\mathrm{i}\deg(d)}q^{m\cdot d})}\tag{3}$$

is additive on graded R-modules A. For $i=0,\ldots,r$, let $C^i\subset\Sigma$ be the set of cones of codimension i. Consider the complex of R-modules

$$0\longrightarrow\bigoplus_{C\in C^0}\mathbb{C}[C]\longrightarrow\bigoplus_{C\in C^1}\mathbb{C}[C]\longrightarrow\cdots\longrightarrow\bigoplus_{C\in C^r}\mathbb{C}[C]\longrightarrow 0,$$

where the differentials are induced by the boundary maps from any cone C to its maximum proper subcones, with appropriate signs depending on the relative orientations of C and these subcones. It is straightforward to see that the cohomology of this complex occurs only at the C^0 spot, and in fact equals $\mathbb{C}[N]^\Sigma$. Since $r(A;q,m)$ is additive, this means that we only need to show

$$r(\mathbb{C}[C];q,m)=\text{a.c.}\Big(\sum_{n\in C}q^{m\cdot n}\mathrm{e}^{2\pi\mathrm{i}\deg(n)}\Big)$$

for each cone $C\in\Sigma$.

So fix $C\in\Sigma$. Let $D(C)$ be the set of generators of one-dimensional cones of C, and let R_C be the polynomial ring $\mathbb{C}[y^d\mid d\in D(C)]$. We claim that in the denominator of (3), we need to take the product only over $d\in D(C)$. Indeed, if $d\notin D(C)$, then y^d acts trivially on $\mathbb{C}[C]$. Hence $\mathrm{Tor}^*_R(\mathbb{C}[C],\mathbb{C})$ is equal to $\mathrm{Tor}^*_{R_C}(\mathbb{C}[C],\mathbb{C})$ tensored with the Koszul complex for $\{y^d\mid d\notin D(C)\}$. This Koszul complex gives rise to extra factors in the alternating sum of traces, which will cancel the factors

$$(1-\mathrm{e}^{2\pi\mathrm{i}\deg(d)}q^{m\cdot d}),\quad d\notin D(C)$$

in the denominator.

Hence we must now show that

$$\frac{\sum_i(-1)^i\mathrm{Trace}_{\mathrm{Tor}^i_{R_C}(\mathbb{C}[C],\mathbb{C})}\,q^{m\cdot\mathbf{n}}\mathrm{e}^{2\pi\mathrm{i}\mathbf{deg}}}{\prod_{d\in D(C)}(1-\mathrm{e}^{2\pi\mathrm{i}\deg(d)}q^{m\cdot d})}=\text{a.c.}\Big(\sum_{n\in C}q^{m\cdot n}\mathrm{e}^{2\pi\mathrm{i}\deg(n)}\Big).\tag{4}$$

Let $m \in M$ satisfy (2), so that the series on the right of (4) converges absolutely for all $|q| < \varepsilon$ for some $\varepsilon > 0$. We will prove the more general statement

$$\frac{\sum_i (-1)^i \mathrm{Trace}_{\mathrm{Tor}^i_{R_C}(A,\mathbb{C})} q^{m\cdot\mathbf{n}} \mathrm{e}^{2\pi \mathrm{i}\mathbf{deg}}}{\prod_{d\in D(C)}(1-\mathrm{e}^{2\pi\mathrm{i}\deg(d)}q^{m\cdot d})} = \mathrm{Trace}_A q^{m\cdot\mathbf{n}}\mathrm{e}^{2\pi\mathrm{i}\mathbf{deg}} \tag{5}$$

for all finitely generated graded R_C-modules A. Indeed, in (5) both sides are additive on finitely generated graded R_C-modules A, and coincide on R_C itself. Together with the existence of a free resolution, this implies (5), which finishes the proof of the lemma. $\square$

2.8. The following important lemma will allow us to show convergence of the series used to define $f_{N,\deg}$.

Lemma 2.9. *Let $n \in N$ be such that $q^{m\cdot n}$ appears with nonzero coefficient in*

$$\sum_i (-1)^i \mathrm{Trace}_{\mathrm{Tor}^i_R(\mathbb{C}[N]^\Sigma,\mathbb{C})} q^{m\cdot\mathbf{n}}\mathrm{e}^{2\pi\mathrm{i}\mathbf{deg}}. \tag{6}$$

Let $D = \{d_i\}$ be set of generators of one-dimensional cones of Σ. Then n lies in the interior of the convex hull of the finite set of points

$$\Big\{\sum_{i\in I} d_i \;\Big|\; I \text{ is any subset of the set of all } i\Big\}.$$

Proof. Without loss of generality, we may assume that Σ is simplicial. Indeed, we can freely subdivide Σ as long as we do not add any new one-dimensional cones. If we pick a generic collection of points on the one-dimensional faces of Σ, and construct the convex hull of these points for each cone, then we construct a simplicial refinement of Σ.

It is more convenient now to equip $\mathbb{C}[N]^\Sigma$ with a new $\mathbb{Q}_{\geq 0}^{\sharp D}$-grading that we will denote $\widehat{\deg}$. To define this grading, suppose that $n \in N$ lies in the cone $C \in \Sigma$, and write

$$n = \sum_{d\in D(C)} \alpha_d \cdot d, \quad \alpha_d \in \mathbb{Q}.$$

Then $\widehat{\deg}(n)$ is defined to have component α_d for $d \in D(C)$, and component 0 otherwise. Obviously, this definition is independent of C, and it is easy to see that the old $N \oplus \mathbb{C}$ grading is induced from this finer grading. Moreover, $\mathbb{C}[N]^\Sigma$ is a graded R-module in the new sense for $R = \mathbb{C}[y^d \mid d \in D]$.

Suppose that n is a lattice point that violates the statement of the lemma. This means that n appears with a nonzero coefficient in (6), and that there exists $h \in M_{\mathbb{C}}$ such that

$$h \cdot n \geq h \cdot (\sum_{i\in I} d_i), \quad \text{for all } I \subset \{1, \ldots, \sharp D\}. \tag{7}$$

Using the notation $n_I := n - \sum_{i\in I} d_i$, the condition (7) translates to

$$h \cdot n_I \geq 0, \quad \text{for all } I. \tag{8}$$

To calculate $\mathrm{Tor}^*(\mathbb{C}[N]^\Sigma, \mathbb{C})$, we can tensor $\mathbb{C}[N]^\Sigma$ with the Koszul complex and calculate the cohomology of the resulting complex. At the N-degree n we will have

$$0 \longrightarrow \cdots \longrightarrow \bigoplus_{i<j} (\mathbb{C}[N]^\Sigma[-d_i - d_j])_n \longrightarrow \bigoplus_i (\mathbb{C}[N]^\Sigma[-d_i])_n \longrightarrow (\mathbb{C}[N]^\Sigma)_n \longrightarrow 0. \tag{9}$$

For each I the space $(\mathbb{C}[N]^\Sigma[-\sum_{i\in I} d_i])_n$ is one-dimensional, with $\widehat{\deg}$ equal to

$$\widehat{\deg}(n_I) + \sum_{i\in I} \widehat{\deg}(d_i).$$

Now the complex (9) splits into a direct sum of subcomplexes according to $\widehat{\deg}$. If n appears with a nonzero coefficient in (6), then the Euler characteristic of one of these subcomplexes, say $\mathscr{C}_0$, must be nonzero. We will show that if this n satisfies (8), then in fact $\chi(\mathscr{C}_0) = 0$, which is a contradiction.

Let $\widehat{\deg}_0$ be the degree of $\mathscr{C}_0$. Consider the set $\mathscr{I}$ of all I such that

$$\widehat{\deg}(n_I) + \sum_{i\in I} \widehat{\deg}(d_i) = \widehat{\deg}_0.$$

We claim that $\mathscr{I}$ has a unique maximal element with respect to inclusion. Indeed, suppose $I, J \in \mathscr{I}$, so that

$$\widehat{\deg}(n_I) + \sum_{i\in I} \widehat{\deg}(d_i) = \widehat{\deg}(n_J) + \sum_{j\in J} \widehat{\deg}(d_j).$$

Then

$$\widehat{\deg}(n_I) - \widehat{\deg}(n_J) = \sum_{j\in(J\smallsetminus(J\cap I))} \widehat{\deg}(d_j) - \sum_{i\in(I\smallsetminus(I\cap J))} \widehat{\deg}(d_i),$$

which implies that d_j lies in the minimum cone containing n_I for all $j \in (J \smallsetminus (J \cap I))$, and that the corresponding coefficient of n_I is ≥ 1. Hence

$$\widehat{\deg}(n_{I\cup J}) + \sum_{j\in(J\smallsetminus(J\cap I))} \widehat{\deg}(d_i) = \widehat{\deg}(n_I),$$

and

$$\widehat{\deg}(n_I) + \sum_{i\in I} \widehat{\deg}(d_i) = \widehat{\deg}(n_{I\cup J}) + \sum_{j\in(I\cup J)} \widehat{\deg}(d_j).$$

Hence $I \cup J \in \mathscr{I}$. We will denote the maximum set for $\widehat{\deg}_0$ by $I^{\max}$.

It is easy to describe the subsets $I \subset I^{\max}$ satisfying

$$\widehat{\deg}(n_{I^{\max}\smallsetminus I}) + \sum_{i\in(I^{\max}\smallsetminus I)} \widehat{\deg}(d_i) = \widehat{\deg}_0. \tag{10}$$

Indeed, (10) happens if and only if there exists a cone $C \in \Sigma$ containing both $n_{I^{\max}}$ and $\{d_i \mid i \in I\}$. Let us say that I is an *allowed subset* of $I^{\max}$ if this happens.

Let $C_{\min}$ be the minimal cone containing $n_{I^{\max}}$. We will now show that if $i \in I^{\max}$, then d_i is not a generator of $C_{\min}$. Indeed, suppose d_i is a generator of $C_{\min}$, and let I be an allowed subset. If $i \notin I$, then $I \cup \{i\}$ is allowed, and if $i \in I$, then $I \smallsetminus \{i\}$ is allowed. Hence the set of all allowed subsets splits into pairs, and each pair contributes 0 to $\chi(\mathscr{C}_0)$. However, by assumption $\chi(\mathscr{C}_0) \neq 0$.

In addition, if any of the coordinates of $\widehat{\deg}(n_{I^{\max}})$ were at least 1, then we could replace $I^{\max}$ with $I^{\max} \cup \{i\}$, which contradicts its maximality. This means that $n_{I^{\max}}$ lies in the interior of the convex hull of all possible $\sum_{j\in J} d_j$, where J satisfies

$$J \subset \{i \mid d_i \in D(C_{\min})\}.$$

Therefore, 0 is in the interior of the convex hull of all $n_{I^{\max} \cup J}$, which implies

$$h \cdot n_{I^{\max}} = 0 \quad \text{and} \quad h \cdot d_j = 0$$

for all $d_j \in D(C_{\min})$. Because $h \cdot n_I \geq 0$ for all I, we conclude that $h \cdot d_i$ is nonnegative for all $i \in I^{\max}$, and is nonpositive for all other i.

Let us now calculate the Euler characteristic of $\mathscr{C}_0$. We have

$$(-1)^{\sharp(I^{\max})}\chi(\mathscr{C}_0) = \sum_{I \subseteq I^{\max}, I \text{ allowed}} (-1)^{\sharp(I)} \tag{11}$$

$$= \sum_{C\in\Sigma, C \ni n_{I^{\max}}} (-1)^{\operatorname{codim} C} \sum_{I \subseteq I^{\max}, d_i \in C \text{ for all } i \in I} (-1)^{\sharp(I)}. \tag{12}$$

Here we have used $\sum_{C\in\Sigma, C\ni n} (-1)^{\operatorname{codim} C} = 1$ for every lattice point n. This is easy to verify by looking at the simplicial complex Σ induces on a small sphere about n.

We can quotient $N_{\mathbb{R}}$ by the subspace generated by $C_{\min}$, and can induce a fan there. Hence without loss of generality we may assume $n_{I^{\max}} = 0$.

We now observe that the inner sum in (12) is zero, unless no d_i with $i \in I^{\max}$ lies in C. To see what this means, define

$$\widetilde{\Sigma} = \left\{ C \in \Sigma \;\middle|\; \text{there exists } d_i \in D(C) \text{ with } i \in I^{\max} \right\}.$$

Then

$$(-1)^{\sharp(I^{\max})}\chi(\mathscr{C}_0) = \sum_{C \notin \widetilde{\Sigma}} (-1)^{\operatorname{codim} C} \tag{13}$$

$$= 1 - \sum_{C \in \widetilde{\Sigma}} (-1)^{\operatorname{codim} C}. \tag{14}$$

Now define a subset U by

$$U := \bigcup_{C\in\widetilde{\Sigma}} \text{rel. int.}(C).$$

We claim that the sum in (14) is the Euler characteristic of U in Čech cohomology theory. Indeed, this cohomology could be calculated using the acyclic covering by the open stars

$$\operatorname{Star}(d_i) := \bigcup_{C_1\in\Sigma, C_1\ni d_i} \operatorname{rel.int.}(C_1),$$

for $i \in I^{\max}$, and the cones C above are in one-to-one correspondence with intersections of $\operatorname{Star}(d_i)$ via

$$\operatorname{Star}(C) := \bigcup_{C_1\in\Sigma, C_1\supseteq C} \operatorname{rel.int.}(C_1).$$

Therefore, to show that $\chi(\mathscr{C}_0) = 0$, it suffices to show that the space U is contractible.

First of all, notice that U contains the entire open half-space $h > 0$. Pick a point p inside this half-space. For every point $u \in U$ we will define a path $\nu\colon [0,1] \to U$ such that $\nu(0) = u$, $\nu(1) = p$ as follows. Consider any cone C with $u \in C$. There is a unique decomposition

$$u = u_1 + u_2$$

such that u_1 has nonzero coordinates only for $d_i \in I^{\max}$ and u_2 has nonzero coordinates only for $d_i \notin I^{\max}$. Then define

$$\nu(t) = \begin{cases} u_1 + (1-2t)u_2, & t \in [0, 1/2], \\ (2-2t)u_1 + (2t-1)p, & t \in [1/2, 1]. \end{cases}$$

It is easy to see that these paths assemble together into a continuous map $U \times [0,1] \to U$ providing a retraction of U onto p.

Hence we have shown that $\chi(\mathscr{C}_0) = 0$, which contradicts our assumption that n and $\widehat{\deg}_0$ disobey the conditions of the lemma. This completes the proof. □

Theorem 2.10. *The function $f_{N,\deg}(q)$ of Definition 2.2 is well-defined. More precisely,*

1. *There exists $\varepsilon > 0$ such that the outer sum over M converges absolutely and uniformly for all $|q| < \varepsilon$.*
2. *Each term $r(q,m)$ has a q-expansion with nonnegative powers of q, and only a finite number of m contribute nontrivially to q^n for any fixed n.*

Proof. The lattice M is a disjoint union of a finite number of regions, each characterized by the collection of signs of $m \cdot d_i$ (positive, zero, negative) for all i. Each such region is the interior of a rational polyhedral cone in some sublattice of M. It therefore suffices to show that each of the above convergence statements is true separately for m inside one such region $\mathscr{R}$.

Pick $\varepsilon < 1$ such that

$$\varepsilon < |\mathrm{e}^{2\pi\mathrm{i}\deg(d_i)}|^{-1}/2, \quad d_i \in D.$$

Then for every d_i with $\mathscr{R} \cdot d_i > 0$, we have

$$|1 - \mathrm{e}^{2\pi \mathrm{i} \deg(d_i)} q^{m \cdot d_i}| > 1/2.$$

For any d_i with $\mathscr{R} \cdot d_i = 0$ we just get a nonzero constant in the denominator, because we always assume that $\deg(d_i)$ are not integers. Finally, Lemma 2.9 implies that for every generator m_j of the closure of $\mathscr{R}$, and every n contributing to the numerator of $r(q, m)$ we get

$$m_i \cdot n > \sum_{k, m_i \cdot d_k < 0} m_i \cdot d_k.$$

Since for $m \cdot d_i < 0$ and $|q| < \varepsilon$, we have

$$|1 - \mathrm{e}^{2\pi \mathrm{i} \deg(d_i)} q^{m \cdot d_i}| > c_1 q^{m \cdot d_i}$$

for each m, it follows that the terms of the $\sum_m r(q, m)$ can be estimated by $c_2 q^{l(m)}$, where $l(m)$ is the minimum of the linear functions (one for each n in the numerator of $r(q, m)$) that are strictly positive on all generators of the closure of $\mathscr{R}$. This implies the first statement. The second statement is treated similarly. ☐

3. Theta functions and toric forms

3.1. The space $\mathscr{T}$ has a natural ring structure, with multiplication given by

$$f_{N_1, \deg_1} f_{N_2, \deg_2} = f_{N_1 \oplus N_2, \deg_1 \oplus \deg_2}.$$

In this section we study the ring $\mathscr{T}$ by expressing the toric form $f_{N,\deg}$ in terms of the theta function with characteristic [3]

$$\vartheta_{11}(z, \tau) = \vartheta(z, \tau) := \frac{1}{i} \sum_{n \in \mathbb{Z}} (-1)^n \mathrm{e}^{\pi \mathrm{i} \tau (n + \frac{1}{2})^2} \mathrm{e}^{\pi \mathrm{i} z (2n+1)}. \tag{15}$$

To do this, we give a topological interpretation of $f_{N,\deg}$. Associated to the pair $(N, \deg)$ is complete rational polyhedral fan $\Sigma \subset N \otimes \mathbb{R}$, and hence a complete toric variety X. Suppose that X is nonsingular. Then we show in Theorem 3.4 that $f_{N,\deg}$ can be computed as the graded Euler characteristic of certain graded infinite-dimensional vector bundle W over X. Moreover, we can compute this Euler characteristic $\chi(W)$ in two different ways:

1. We can use the Čech cohomology complex associated to the covering of X by toric affine charts. This yields the usual expression for $f_{N,\deg}$.
2. We can use the Hirzebruch-Riemann-Roch Theorem to compute $\chi(W)$ in terms of the Todd class $\mathrm{Td}(X)$ and the Chern character $\mathrm{ch}(W)$. Using the Jacobi triple product formula

$$\vartheta(z, \tau) = q^{\frac{1}{8}} (2 \sin \pi z) \prod_{l=1}^{l=\infty} (1 - q^l) \prod_{l=1}^{l=\infty} (1 - q^l \mathrm{e}^{2\pi \mathrm{i} z})(1 - q^l \mathrm{e}^{-2\pi \mathrm{i} z}), \tag{16}$$

we can interpret the resulting expressions in terms of ϑ.

This technique was used by Borisov-Libgober [2] to compute the elliptic genus of a smooth toric variety, and more details can be found there.

More generally, if X is singular, we can compute these quantities through a limiting process (Theorem 3.5). As a corollary of these results, we obtain that $f_{N,\deg}$ is meromorphic in $\mathfrak{H}$.

3.2. We begin with some notation. Let deg be a degree function with respect to the complete fan Σ, and let $\{d_i\}$ be the set of generators of one-dimensional cones of Σ. Let X be the toric variety associated to Σ, and for each d_i, let $D_i \subset X$ be the corresponding toric divisor. In what follows, we will usually abuse notation and use D_i to mean either the divisor or its cohomology class.

Let $\mathscr{O}(D_i)$ be the line bundle associated to D_i. Recall that the space of sections of $\mathscr{O}(D_i)$ over each open subset of X is by definition the space of all rational functions on X that have a pole of order at most 1 along D_i and no other poles. Moreover, we may extend the torus action from X to each $\mathscr{O}(D_i)$ by inducing from the action on the field of rational functions.

3.3. Let $\mathscr{O}$ be the trivial line bundle. For any $\alpha \in \mathbb{C}$, we define graded bundles

$$\begin{aligned} \Lambda^*_\alpha(\mathscr{O}(D_i)) &:= \mathscr{O} \oplus \mathscr{O}_{(\alpha)}(D_i) \\ \mathrm{Sym}^*_\alpha(\mathscr{O}(D_i)) &:= \mathscr{O} \oplus \mathscr{O}_{(\alpha)}(D_i) \oplus \mathscr{O}_{(\alpha^2)}(2D_i) \oplus \cdots \end{aligned}$$

For each generator d_i let $\alpha_i = \deg(d_i)$. Finally let W be the graded bundle

$$\bigotimes_i \bigotimes_{n=1}^{\infty} \Big(\Lambda^*_{-q^n \mathrm{e}^{-2\pi\mathrm{i}\alpha_i}} \mathscr{O}(D_i) \otimes \Lambda^*_{-q^{n-1}\mathrm{e}^{2\pi\mathrm{i}\alpha_i}} \mathscr{O}(-D_i) \otimes \mathrm{Sym}^*_{q^n} \mathscr{O}(D_i) \otimes \mathrm{Sym}^*_{q^n} \mathscr{O}(-D_i) \Big).$$

Although the bundle W is infinite-dimensional, each graded piece corresponding to a fixed power of q is finite-dimensional.

We are now ready to state the main result of this section.

Theorem 3.4. *Assume that the toric variety X is nonsingular, and that $\alpha_i \notin \mathbb{Z}$ for all generators of one-dimensional cones of Σ. Then*

$$f_{N,\deg}(q) = \int_X \prod_i \frac{(D_i/2\pi\mathrm{i})\vartheta(D_i/2\pi\mathrm{i} - \alpha_i, \tau)\vartheta'(0,\tau)}{\vartheta(D_i/2\pi\mathrm{i}, \tau)\vartheta(-\alpha_i, \tau)}. \tag{17}$$

Here the derivative is the partial derivative with respect to z, the product is taken over all d_i, and the right hand side of (17) *is interpreted as an expression in $H^*(X)$ using the Jacobi triple product* (16).

Proof. First consider the right-hand side of (17). Using

$$\mathrm{Td}(X) = \prod_i \frac{D_i}{1 - \mathrm{e}^{-D_i}},$$

and applying (16), and the product formula for $\vartheta'(0,\tau)$ [3], the right-hand side becomes

$$\int_X \prod_i \frac{D_i}{(1-\mathrm{e}^{-D_i})}\cdot\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-q^n\mathrm{e}^{D_i-2\pi\mathrm{i}\alpha_i})(1-q^{n-1}\mathrm{e}^{-D_i+2\pi\mathrm{i}\alpha_i})}{(1-q^n\mathrm{e}^{D_i})(1-q^n\mathrm{e}^{-D_i})(1-q^n\mathrm{e}^{-2\pi\mathrm{i}\alpha_i})(1-q^{n-1}\mathrm{e}^{2\pi\mathrm{i}\alpha_i})}.$$

This is seen to be

$$\prod_i\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^n\mathrm{e}^{-2\pi\mathrm{i}\alpha_i})(1-q^{n-1}\mathrm{e}^{2\pi\mathrm{i}\alpha_i})}\cdot\int_X \mathrm{Td}(X)\mathrm{ch}(W),$$

which by Hirzebruch-Riemann-Roch is

$$=\prod_i\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^n\mathrm{e}^{-2\pi\mathrm{i}\alpha_i})(1-q^{n-1}\mathrm{e}^{2\pi\mathrm{i}\alpha_i})}\cdot\chi(W).$$

Now we want to calculate $\chi(W)$ using the Čech complex associated to the covering of X by toric affine charts. Let x be a dummy multi-variable that keeps track of the torus action on $\mathscr{O}(D_i)$.

For every cone $C\in\Sigma$ with generators $d_1,\dots,d_k$, let U_C be the corresponding affine chart. Let

$$\dim_M(\Gamma(W,U_C))=\sum_{m\in M}\dim\Gamma(W,U_C)_m\cdot x^m$$

be the graded dimension of $\Gamma(W,U_C)$ with respect to M. Let $m_1,\dots,m_k$ be the dual basis to $d_1,\dots,d_k$. Then the space of sections of $\mathscr{O}(D_j)$ on U_C, as a module over $\Gamma(\mathscr{O},U_C)$, is generated by an element with grading $-m_j$ if d_j is one of the generators of C, and by an element with zero grading otherwise. Now it is easy to see that

$$\prod_i\prod_{n=1}^{\infty}\frac{(1-q^n)^2}{(1-q^n\mathrm{e}^{-2\pi\mathrm{i}\alpha_i})(1-q^{n-1}\mathrm{e}^{2\pi\mathrm{i}\alpha_i})}\cdot\dim_M(\Gamma(W,U_C))$$

$$=(\sum_{m\in M,m\cdot C=0}x^m)\cdot\prod_{i,d_i\in C}\prod_{n=1}^{\infty}\frac{(1-q^n)^2(1-q^n\mathrm{e}^{-2\pi\mathrm{i}\alpha_i}x^{-m_i})(1-q^{n-1}\mathrm{e}^{2\pi\mathrm{i}\alpha_i}x^{m_i})}{(1-q^n\mathrm{e}^{-2\pi\mathrm{i}\alpha_i})(1-q^{n-1}\mathrm{e}^{2\pi\mathrm{i}\alpha_i})(1-q^nx^{-m_i})(1-q^{n-1}x^{m_i})}$$

$$=(\sum_{m\in M,m\cdot C=0}x^m)\cdot\prod_{i,d_i\in C}(\sum_{l\in\mathbb{Z}}\frac{x^{lm_i}}{(1-\mathrm{e}^{2\pi\mathrm{i}\alpha_i}q^l)})=\sum_{m\in M}x^m\prod_{i,d_i\in C}\frac{1}{(1-\mathrm{e}^{2\pi\mathrm{i}\alpha_i}q^{m\cdot d_i})},$$

where we expand everything as a power series around $q=0$. Here we have used the identity of power series in q

$$\prod_{k\geq1}\frac{(1-tyq^{k-1})(1-t^{-1}y^{-1}q^k)}{(1-tq^{k-1})(1-t^{-1}q^k)}=\sum_{m\in\mathbb{Z}}\left(\frac{t^m}{1-yq^m}\right)\prod_{k\geq1}\frac{(1-yq^{k-1})(1-y^{-1}q^k)}{(1-q^k)^2},$$

whose proof can be found in [2].

Now all we need to prove the statement is to notice that, as power series in q,

$$\chi(W) = \sum_{C\in\Sigma}(-1)^{\operatorname{codim} C}\dim_M(\Gamma(W,U_C))\big|_{x=1}.$$

This follows easily from the description of Čech complex, see [2] for details. ☐

We will now combine the results of Theorems 2.10 and 3.4 to get similar expressions for an arbitrary toric form.

Theorem 3.5. *Let* deg *be a degree function with respect to a complete simplicial fan Σ in a lattice N. Let X be the toric variety associated to Σ, and $\widehat{\Sigma}$ be a refinement of Σ providing a desingularization $\widehat{X}$ of X. Let $\deg_1$ be a generic degree function with respect to Σ. Then*

$$f_{N,\deg}(q) = \lim_{\varepsilon\to 0}\int_{\widehat{X}}\prod_i \frac{(D_i/2\pi\mathrm{i})\vartheta(D_i/2\pi\mathrm{i}-\deg(d_i)-\varepsilon\deg_1(d_i),\tau)\vartheta'(0,\tau)}{\vartheta(D_i/2\pi\mathrm{i},\tau)\vartheta(-\deg(d_i)-\varepsilon\deg_1(d_i),\tau)},$$

where the product is taken over all generators of one-dimensional cones of $\widehat{\Sigma}$.

Proof. The point is that sometimes when we desingularize and add extra d_i, we may get $\deg(d_i)\in\mathbb{Z}$, so Theorem 3.4 cannot be directly applied. On the other hand, it is easy to see that in Theorem 2.10, the convergence is uniform in deg. Hence $f_{N,\deg}$ is continuous as a function of deg, and the result follows. ☐

Corollary 3.6. *The function $f_{N,\deg}$ is meromorphic on the upper halfplane.* ☐

4. Toric forms of level l

4.1. Let $l\geq 2$ be an integer, and let $\mathscr{M}_r(\Gamma_1(l))$ be the ring of modular forms of weight r with respect to the congruence subgroup $\Gamma_1(l)\subset SL_2(\mathbb{Z})$. Let $\mathscr{T}_r(l)\subset\mathscr{T}_r$ be the subspace spanned by

$$\Big\{f_{N,\deg}\in\mathscr{T}_r\;\Big|\;\deg(N)\subset\frac{1}{l}\mathbb{Z}\Big\}.$$

We say that $\mathscr{T}(l)=\bigoplus_r\mathscr{T}_r(l)$ is the ring of toric forms of *level* l.

In this section, we show that $\mathscr{T}_r(l)\subseteq\mathscr{M}_r(\Gamma_1(l))$, and describe an explicit set of generators for $\mathscr{T}(l)$. We relate toric forms of level l to the elliptic genera considered by Hirzebruch.

4.2. To begin, we show that toric forms of level l are automorphic with respect to $\Gamma_1(l)$. For later purposes, it is more convenient to phrase the statement in terms of $\Gamma_0(l)$, the subgroup of matrices satisfying

$$\begin{pmatrix} a & b\\ c & d\end{pmatrix} = \begin{pmatrix} * & *\\ 0 & *\end{pmatrix}\bmod l.$$

Proposition 4.3. *If* $f_{N,\deg} \in \mathscr{T}_r(l)$ *and* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(l)$*, then*

$$f_{N,\deg}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^r f_{N,d\cdot\deg}(\tau).$$

Proof. This will follow from Theorems 3.4 and 3.5 and the transformation properties of ϑ. For any modular transformation $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z})$, we have

$$\begin{aligned}\vartheta\Big(\frac{z}{c\tau+d},\frac{a\tau+b}{c\tau+d}\Big) &= \zeta(c\tau+d)^{\frac12}\mathrm{e}^{\frac{\pi\mathrm{i}cz^2}{c\tau+d}}\vartheta(z,\tau)\\ \vartheta'\Big(0,\frac{a\tau+b}{c\tau+d}\Big) &= \zeta(c\tau+d)^{\frac32}\vartheta'(0,\tau),\end{aligned}$$

where $\zeta^8=1$, and ζ depends on a,b,c,d but not on τ or z [13]. Now use the notation of Theorem 3.5. We denote $\deg(d_i)$ by α_i and $\deg_1(d_i)$ by β_i. We have

$$\begin{aligned} f_{N,\deg}\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-r} &= \lim_{\varepsilon\to 0}\int_{\hat X}\prod_i \frac{(\frac{D_i/2\pi\mathrm{i}}{c\tau+d})\vartheta(\frac{D_i/2\pi\mathrm{i}}{c\tau+d}-\alpha_i-\varepsilon\beta_i,\frac{a\tau+b}{c\tau+d})\vartheta'(0,\frac{a\tau+b}{c\tau+d})}{\vartheta(\frac{D_i/2\pi\mathrm{i}}{c\tau+d},\frac{a\tau+b}{c\tau+d})\vartheta(-\alpha_i-\varepsilon\beta_i,\frac{a\tau+b}{c\tau+d})}\\
&= \lim_{\varepsilon\to 0}\int_{\hat X}\prod_i \frac{(\frac{D_i}{2\pi\mathrm{i}})\mathrm{e}^{\frac{\pi\mathrm{i}c}{c\tau+d}(\frac{D_i}{2\pi\mathrm{i}}-(\alpha_i+\varepsilon\beta_i)(c\tau+d))^2}\vartheta(\frac{D_i}{2\pi\mathrm{i}}-(\alpha_i+\varepsilon\beta_i)(c\tau+d),\tau)\vartheta'(0,\tau)}{\vartheta(\frac{D_i}{2\pi\mathrm{i}},\tau)\vartheta(-(\alpha_i+\varepsilon\beta_i)(c\tau+d),\tau)\mathrm{e}^{\frac{\pi\mathrm{i}c}{c\tau+d}((\frac{D_i}{2\pi\mathrm{i}})^2+(\alpha_i+\varepsilon\beta_i)^2(c\tau+d)^2)}}\\
&= \lim_{\varepsilon\to 0}\int_{\hat X}\prod_i \frac{(\frac{D_i}{2\pi\mathrm{i}})\vartheta(\frac{D_i}{2\pi\mathrm{i}}-(\alpha_i+\varepsilon\beta_i)(c\tau+d),\tau)\vartheta'(0,\tau)}{\vartheta(\frac{D_i}{2\pi\mathrm{i}},\tau)\vartheta(-(\alpha_i+\varepsilon\beta_i)(c\tau+d),\tau)}\mathrm{e}^{-cD_i(\alpha_i+\varepsilon\beta_i)}\\
&= \lim_{\varepsilon\to 0}\int_{\hat X}\prod_i \frac{(\frac{D_i}{2\pi\mathrm{i}})\vartheta(\frac{D_i}{2\pi\mathrm{i}}-d\alpha_i-\varepsilon\beta_i(c\tau+d),\tau)\vartheta'(0,\tau)}{\vartheta(\frac{D_i}{2\pi\mathrm{i}},\tau)\vartheta(-d\alpha_i-\varepsilon\beta_i(c\tau+d),\tau)}\mathrm{e}^{-cD_i\varepsilon\beta_i},\end{aligned}$$

where in the last step we used the elliptic property

$$\vartheta(z+\tau) = -\mathrm{e}^{-\pi\mathrm{i}(2z+\tau)}\vartheta(z,\tau).$$

We can assume that the values β_i are chosen so that $\beta_i = m\cdot d_i$ for some generic $m\in M_{\mathbb{C}}$. This guarantees $\sum\beta_i D_i$ is trivial in $H^*(X)$. It is easy to see that the arguments of Theorem 3.5 still work, provided $\beta_i\neq 0$.

Now we have

$$\begin{aligned} f_{N,\deg}\left(\frac{a\tau+b}{c\tau+d}\right)(c\tau+d)^{-r} &= \lim_{\varepsilon\to 0}\int_{\hat X}\prod_i \frac{(\frac{D_i}{2\pi\mathrm{i}})\vartheta(\frac{D_i}{2\pi\mathrm{i}}-d\alpha_i-\varepsilon\beta_i(c\tau+d),\tau)\vartheta'(0,\tau)}{\vartheta(\frac{D_i}{2\pi\mathrm{i}},\tau)\vartheta(-d\alpha_i-\varepsilon\beta_i(c\tau+d),\tau)}\\ &= f_{N,d\cdot\deg}(\tau).\end{aligned}$$

$\square$

4.4. To ease notation, we often suppress τ from ϑ and write simply $\vartheta(z)$. We introduce functions

$$s^{(k)}_{a/l}(q) = (2\pi \mathrm{i})^{-k}\left(\frac{\partial^k}{\partial z^k}\right)_{z=0}\log\left(\frac{z\vartheta(z+a/l)\vartheta'(0)}{\vartheta(z)\vartheta(a/l)}\right)$$

for all $a = 1,\dots,l-1$ and all integers $k \geq 1$. Formally we will also consider them for all integers a not divisible by l because of the obvious periodicity.

Also we introduce

$$r^{(k)}(q) = \left(\frac{\partial^k}{\partial z^k}\right)_{z=0}\log\left(\frac{\vartheta(z)}{z\vartheta'(0)}\right)$$

for all $k \geq 1$. We have $r^{(k)} = 0$ for odd k, and $s^{(k)}_{-a/l} = (-1)^k s^{(k)}_{a/l}$.

Lemma 4.5. 1. *For all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_1(l)$,

$$s^{(k)}_{a/l}\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^k s^{(k)}_{a/l}(\tau).$$

2. *For all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z})$,

$$r^{(k)}\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^k r^{(k)}(\tau),\ \ k \geq 4$$

$$r^{(2)}\Big(\frac{a\tau+b}{c\tau+d}\Big) = (c\tau+d)^2 r^{(2)}(\tau) + \frac{c(c\tau+d)}{2\pi \mathrm{i}}.$$

Proof. This can be shown by arguments similar to those used to prove Proposition 4.3, and so we omit the details. □

Proposition 4.6. *Every element of $\mathscr{T}(l)$ is a polynomial with constant coefficients of $s^{(k)}_{a/l}$ and $r^{(k)}$. Moreover, the grading of $\mathscr{T}(l)$ by the rank of the lattice coincides with the one defined by the superscripts in brackets.*

Proof. In the smooth case this result follows immediately from Theorem 3.4 and Taylor's formula. Let $\alpha_i = \deg(d_i)$. Then for each i we have

$$\frac{(D_i/2\pi\mathrm{i})\vartheta(D_i/2\pi\mathrm{i}-\alpha_i)\vartheta'(0)}{\vartheta(D_i/2\pi\mathrm{i})\vartheta(-\alpha_i)} = \exp\Big(\sum_{k\geq 1}\frac{(-1)^k}{k!}D_i^k s^{(k)}_{\alpha_i}\Big).$$

Hence $f_{N,\deg}$ is polynomial in $s^{(k)}_{a/l}$.

Things are more complicated in the general case. Let $\beta_i = \deg_1(d_i)$, where $\deg_1$ is defined in Theorem 3.5. If $\alpha_i \notin \mathbb{Z}$, then

$$\log\frac{(D_i/2\pi\mathrm{i})\vartheta(D_i/2\pi\mathrm{i}-\alpha_i-\varepsilon\beta_i)\vartheta'(0)}{\vartheta(D_i/2\pi\mathrm{i})\vartheta(-\alpha_i-\varepsilon\beta_i)} = \log\frac{(D_i/2\pi\mathrm{i}-\varepsilon\beta_i)\vartheta(D_i/2\pi\mathrm{i}-\alpha_i-\varepsilon\beta_i)\vartheta'(0)}{\vartheta(D_i/2\pi\mathrm{i}-\varepsilon\beta_i)\vartheta(-\alpha_i)}$$

$$-\log\frac{(-\varepsilon\beta_i)\vartheta(-\alpha_i-\varepsilon\beta_i)\vartheta'(0)}{\vartheta(-\varepsilon\beta_i)\vartheta(-\alpha_i)}+\log\frac{\vartheta(D_i/2\pi\mathrm{i}-\varepsilon\beta_i)}{(D_i/2\pi\mathrm{i}-\varepsilon\beta_i)\vartheta'(0)}-\log\frac{\vartheta(D_i/2\pi\mathrm{i})}{(D_i/2\pi\mathrm{i})\vartheta'(0)}$$
$$-\log\frac{\vartheta(-\varepsilon\beta_i)}{(-\varepsilon\beta_i)\vartheta'(0)}.$$

This allows us to write

$$\frac{(D_i/2\pi\mathrm{i})\vartheta(D_i/2\pi\mathrm{i}-\alpha_i-\varepsilon\beta_i)\vartheta'(0)}{\vartheta(D_i/2\pi\mathrm{i})\vartheta(-\alpha_i-\varepsilon\beta_i)}$$

as a power series in D_i and ε, whose coefficients are polynomials in $s_{\alpha_i}^{(k)}$ and $r^{(k)}$.

Now suppose $\alpha_i\in\mathbb{Z}$. Then

$$\log\frac{(D_i/2\pi\mathrm{i})\vartheta(D_i/2\pi\mathrm{i}-\alpha_i-\varepsilon\beta_i)\vartheta'(0)}{\vartheta(D_i/2\pi\mathrm{i})\vartheta(-\alpha_i-\varepsilon\beta_i)}=\log\frac{(D_i/2\pi\mathrm{i})\vartheta(D_i/2\pi\mathrm{i}-\varepsilon\beta_i)\vartheta'(0)}{\vartheta(D_i/2\pi\mathrm{i})\vartheta(-\varepsilon\beta_i)}$$
$$=\log(D_i/2\pi\mathrm{i}-\varepsilon\beta_i)-\log(-\varepsilon\beta_i)+\log\frac{\vartheta(D_i/2\pi\mathrm{i}-\varepsilon\beta_i)}{(D_i/2\pi\mathrm{i}-\varepsilon\beta_i)\vartheta'(0)}-\log\frac{\vartheta(D_i/2\pi\mathrm{i})}{(D_i/2\pi\mathrm{i})\vartheta'(0)}$$
$$-\log\frac{\vartheta(-\varepsilon\beta_i)}{(-\varepsilon\beta_i)\vartheta'(0)}.$$

Thus we can write

$$\prod_i\frac{(D_i/2\pi\mathrm{i})\vartheta(D_i/2\pi\mathrm{i}-\alpha_i-\varepsilon\beta_i)\vartheta'(0)}{\vartheta(D_i/2\pi\mathrm{i})\vartheta(-\alpha_i-\varepsilon\beta_i)}$$

as ε^{-l} times a power series of D and ε whose coefficients are polynomial of $s_{a/l}^{(k)}$ and $r^{(k)}$. Since the limit as $\varepsilon\to 0$ exists, after we integrate we find no negative degrees of ε.

Finally, the statement about grading is proved by looking at the total degree in D and ε and observing that at the end we put $\varepsilon=0$. ☐

4.7. We will now show that the $s_{a/l}^{(k)}$ can be expressed as polynomials in $s_{a/l}^{(1)}$ with minor exceptions.

Lemma 4.8. *Let N be any natural number and let $a_1,\ldots,a_{N+1}$ be a set of nonzero residues* $\bmod l$ *so that $a_1+\cdots+a_{N+1}=0 \bmod l$. Then the coefficient of t^N in*

$$\exp\Big(\sum_{k\geq 0}(t^k/k!)\big(\sum_{j=1}^{N+1}s_{a_j/l}^{(k)}\big)\Big)$$

is zero.

Proof. ¿From the definition of $s_{a/l}^{(n)}$ and Taylor's formula,

$$\exp\Big(\sum_{k\geq 0}(t^k/k!)\big(\sum_{j=1}^{N+1}s_{a_j/l}^{(k)}\big)\Big)=\frac{t^{N+1}}{(2\pi\mathrm{i})^{N+1}}\prod_{j=1}^{N+1}\frac{\vartheta(t/2\pi\mathrm{i}+a_j/l)\vartheta'(0)}{\vartheta(t/2\pi\mathrm{i})\vartheta(a_j/l)}.$$

Let

$$F(t,\tau)=\prod_{j=1}^{N+1}\frac{\vartheta(t+a_j/l)\vartheta'(0)}{\vartheta(t)\vartheta(a_j/l)}.$$

We want to show the residue of F at $t=0$ is zero. But notice that F is a doubly-periodic function due to the condition on $\sum a_j$. Therefore, the sum of its residues in a fundamental domain is zero. But F has a unique pole at $t=0$, which finishes the proof. $\square$

Proposition 4.9. 1. *If $l\geq 5$ then all $s^{(k)}_{a/l}$ are polynomials of $s^{(1)}_{a/l}$.*

2. *If $l=4$ then all $s^{(k)}_{a/4}$ are polynomials of $s^{(1)}_{1/4}$ and $s^{(2)}_{1/4}$.*
3. *If $l=3$ then all $s^{(k)}_{a/3}$ are polynomials of $s^{(1)}_{1/3}$ and $s^{(3)}_{1/3}$.*
4. *If $l=2$ then all $s^{(k)}_{1/2}$ are polynomials of $s^{(2)}_{1/2}$ and $s^{(4)}_{1/2}$.*

Proof. Case $l\geq 5$. We need to show that for a fixed $N\geq 2$ all $s^{(N)}_{a/l}$ can be expressed in terms of $s^{(\leq N-1)}_{a/l}$. Denote the ring generated by $s^{(\leq N-1)}_{a/l}$ by R_{N-1}. Because of Lemma 4.8, for every set of nonzero residues $a_1,\ldots,a_{N+1}$ with $\sum_j a_j=0 \bmod l$, we have

$$\sum_j s^{(N)}_{a/l}=0 \bmod R_{N-1}.$$

If $N\geq 3$, then this implies that for any four residues a,b,c,d with $a+b=c+d$, we have

$$s^{(N)}_{a/l}+s^{(N)}_{b/l}=s^{(N)}_{c/l}+s^{(N)}_{d/l} \bmod R_{N-1}.$$

This implies

$$s^{(N)}_{a/l}=(2-a)s^{(N)}_{1/l}+(a-1)s^{(N)}_{2/l} \bmod R_{N-1},\quad a=1,\ldots,l-1.$$

We now go back to the original relation and see that if $\sum_{j=1}^{N+1}a_j=kl$, then

$$(2(N+1)-kl)s^{(N)}_{1/l}+(kl-(N+1))s^{(N)}_{2/l}=0 \bmod R_{N-1}.$$

Since $l\geq 5$ we can find two sets of a_j with different k, which shows that $s^{(N)}_{1/l},s^{(N)}_{2/l}\in R_{N-1}$.

If $N=2$, we have

$$s^{(2)}_{a/l}+s^{(2)}_{b/l}+s^{(2)}_{c/l}=0 \bmod R_1,$$

if $a+b+c=0 \bmod l$. Together with the symmetry $s^{(2)}_{a/l}=s^{(2)}_{(l-a)/l}$, this implies

$$s^{(2)}_{(a+1)/l}=-s^{(2)}_{a/l}-s^{(2)}_{1/l} \bmod R_1$$

if $a,a+1\neq 0 \bmod l$. Therefore,

$$s^{(2)}_{a/l}=(\frac{-1-3(-1)^a}{2})s^{(2)}_{1/l} \bmod R_1$$

for $a = 1, \ldots, l-2$. It remains to consider $a = b = 2, c = l-4$ to show that all $s^{(2)}_{a/l}$ lie in R_1.

Case $l = 4$. The only difference is that for $N = 2$ one can no longer use $a = b = 2, c = l-4$.

Case $l = 3$. At $N = 3$ we can no longer find two sets of four a_j with different k. On the other hand at $N = 2$ we easily conclude that $s^{(2)}_{1/3} \in R_1$, because of $1+1+1 = 0 \bmod l$.

Case $l = 2$. By symmetry, all we have is $s^{(2k)}_{1/2}$ for $k \geq 1$. We notice that $s^{(2k)}_{1/2}$ is proportional to G^*_{2k} of [15], and is therefore a modular form. It is also known (see [9]) that the ring of modular forms for $\Gamma_0(2)$ is freely generated by an element of degree 2 and an element of degree 4. It remains to observe that $(s^{(2)}_{1/2})^2$ is not proportional to $s^{(4)}_{1/2}$. $\square$

Lemma 4.10. *For all l and $n \geq 4$, any $r^{(n)}$ is a polynomial in $s^{(k)}_{a/l}$.*

Proof. Consider the function

$$f(z,\tau) = \frac{\partial^2}{\partial z^2} \log(\vartheta(z,\tau)).$$

This is easily seen to be a doubly-periodic function. Its only pole in the fundamental domain is at the origin, and the Laurent expansion around it is

$$f(z,\tau) = \frac{(-1)}{z^2} + \sum_{k \geq 0} r^{(k+2)}(\tau) \frac{z^k}{k!}.$$

On the other hand, consider the function

$$g(z,\tau) = \left(\frac{\vartheta(z+1/l,\tau)\vartheta'(0,\tau)}{\vartheta(1/l,\tau)\vartheta(z,\tau)} \right) \cdot \left(\frac{\vartheta(z-1/l,\tau)\vartheta'(0,\tau)}{\vartheta(-1/l,\tau)\vartheta(z,\tau)} \right)$$

The function g is even, so it has no residue at $z = 0$. One easily sees that its Laurent expansion is

$$g(z) = \frac{1}{z^2} + \varphi(z),$$

where $\varphi(z)$ is holomorphic. Thus, $f + g$ is an even doubly-periodic holomorphic function, which implies that it is constant in z. As a result, the coefficients of its Laurent expansion at z^n, $n \geq 2$ are zero. It remains to notice that all coefficients of g at $z = 0$ are polynomials in $s^{(k)}_{1/l}$. $\square$

Theorem 4.11. *The ring $\mathscr{T}(l)$ is a subring of the ring of modular forms for $\Gamma_1(l)$.*

1. *If $l \geq 5$, then it is generated by $s^{(1)}_{a/l}$.*
2. *If $l = 4$, then it is generated by $s^{(1)}_{1/4}$ and $s^{(2)}_{1/4}$.*
3. *If $l = 3$, then it is generated by $s^{(1)}_{1/3}$ and $s^{(3)}_{1/3}$.*

4. *If* $l = 2$, *then it is generated by* $s^{(2)}_{1/2}$ *and* $s^{(4)}_{1/2}$.

Proof. Because of Lemma 4.10, we can express every element of $\mathscr{T}(l)$ as a polynomial in $r^{(2)}$ and $s^{(k)}_{a/l}$. The automorphic properties allow us to conclude that $r^{(2)}$ is absent from these polynomial expressions. Together with Proposition 4.9, this implies that $\mathscr{T}(l)$ is contained inside the ring generated by $s^{(1)}_{a/l}$ if $l \geq 5$, and by appropriate $s^{(k)}_{a/l}$ for $l = 2, 3, 4$.

To prove the opposite inclusion, we first notice that $\mathscr{T}(l)$ contains all $s^{(1)}_{a/l}$. Indeed, consider the degree function on the one-dimensional lattice $N = \mathbb{Z}$ that equals a/l on $n = 1$ and $n = -1$. There is only one possible complete fan, and the corresponding toric variety is $\mathbb{P}^1$. By Theorem 3.4, we get

$$f_{N,\deg} = -2s^{(1)}_{a/l}.$$

Similarly, by using $\mathbb{P}^k$ we conclude that $s^{(k)}_{a/l} \in \mathscr{T}(l)$, which helps us at levels less than 5.

To finish the proof, we must show that toric forms have the expected behavior at all cusps. Let $\alpha \in \mathbb{Q}$. Here is the argument for $s^{(1)}_{\alpha}$. It is easy to see that

$$s^{(1)}_{\alpha}(\tau) = \frac{\vartheta'(\alpha, \tau)}{2\pi \mathrm{i} \vartheta(\alpha, \tau)}. \tag{18}$$

We need to show that for any $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$, the function

$$s^{(1)}_{\alpha}\Big(\frac{a\tau + b}{c\tau + d}\Big)(c\tau + d)^{-1}$$

is bounded in any neighborhood of $\mathrm{i}\infty$. Using the transformation properties of ϑ, we can deduce that

$$s^{(1)}_{\alpha}\Big(\frac{a\tau + b}{c\tau + d}\Big)(c\tau + d)^{-1} = \frac{\vartheta'(c\alpha\tau + d\alpha, \tau)}{2\pi \mathrm{i} \vartheta(c\alpha\tau + d\alpha, \tau)} + c\alpha.$$

Using elliptic properties of ϑ, we can reduce $c\alpha\tau + d\alpha$ to $\lambda\tau + \nu$ where $0 \leq \lambda, \nu < 1$ and $(\lambda, \nu) \neq (0, 0)$. This introduces an extra summand that is irrelevant, since all we want to show is boundedness. We will closely examine the q-expansion of $\vartheta(\lambda\tau + \nu, \tau)$.

¿From the definition of ϑ (15), we have

$$\vartheta(\lambda\tau + \nu, \tau) = \frac{1}{i} \sum_{n \in \mathbb{Z}} (-1)^n \mathrm{e}^{\pi \mathrm{i} \tau ((n + \frac{1}{2})^2 + \lambda(2n+1))} \mathrm{e}^{\pi \mathrm{i} \nu (2n+1)}.$$

Since $0 \leq \lambda < 1$, the smallest (rational) power of q can occur at $n = -1$ if $\lambda \neq 0$ or at $n = -1, 0$ if $\lambda = 0$.

In the first case, it is clear that the coefficient is nonzero. Since differentiation by z contributes a nonzero factor of $\pi\mathrm{i}(2n + 1)$, the ratio in the formula above starts with q^0, and has nonnegative powers of q.

In the second case, two terms that contribute to the smallest power of q have nonzero coefficients whose ratio is $(-1)e^{2\pi i\nu}$. Hence there is no cancellation. As a result, the ratio again has only nonnegative powers of q.

A similar argument works for all $s^{(k)}_{a/l}$, and so this completes the proof. $\square$

Remark 4.12. It is easy to see that all elliptic genera of Hirzebruch that are modular with respect to $\Gamma_1(l)$ (see [8]) take values in $\mathscr{T}(l)$ for appropriate l. In general, however, they form a subspace, because in their definition one only uses $s^{(k)}_\alpha$ for a fixed α. Nevertheless, $\mathscr{T}(l)$ could be alternatively defined as the space spanned by all products of all Hirzebruch elliptic genera that are modular with respect to $\Gamma_1(l)$.

Remark 4.13. Even though we will not need it for our arguments, it is worth mentioning that $r^{(k)} = -2E_k$ for all (even) k where E_k is the standard Eisenstein series. To prove it, use the product formula for $\vartheta(z,\tau)$, take its logarithm, differentiate once with respect to z, expand the result as a geometric series and differentiate some more. So the ring generated by $r^{(k)}$ is the ring of almost-modular forms, see [15].

Remark 4.14. It is natural to ask whether $\mathscr{T}(l) = \mathscr{M}(\Gamma_1(l))$. For example, it is not hard to show that $\mathscr{T}(l) = \mathscr{M}(\Gamma_1(l))$ for $l = 2$, 3, and 4. In general, however, the answer is no. In particular, we can never construct any cusp forms of weight one this way.

We have investigated this question in more detail for the case of weight 2 and prime level. Specifically, we have considered the $\Gamma_0(l)/\Gamma_1(l)$-invariant part of $\mathscr{T}_2(l)$, which consists of modular forms for $\Gamma_0(l)$. Extensive computer calculations suggest that $(\mathscr{T}_2(l))^{\Gamma_0(l)/\Gamma_1(l)}$ coincides with the span of the Eisenstein series and the eigenforms of analytic rank zero[1].

5. Hecke stability

5.1. To conclude the paper, we will show that toric forms of level l are well-behaved with respect to the Hecke operators, the involution operator w_l, and Atkin-Lehner lifting. For background, we refer to Lang [11].

We begin with the Hecke operators. These are endomorphisms of $\mathscr{M}_r(\Gamma_1(l)) \otimes \mathbb{C}$ generated as follows. For a prime p with $(p,l) = 1$, the operator T_p acts by

$$(f \mid T_p)(\tau) = p^{-1}\sum_{i=0}^{p-1} f((\tau+i)/p) + p^{r-1}(f \mid \varepsilon_p)(p\tau),$$

Here $f \mid \varepsilon_p$ denotes the action of an element of $\Gamma_0(l)/\Gamma_1(l) \cong (\mathbb{Z}/l\mathbb{Z})^\times$ given by a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} p^{-1} & b \\ 0 & p \end{pmatrix} \bmod l.$$

[1]Note added in proof: the above statement is true; we refer to [1] for details.

For a prime p with $(p,l)>1$, the operator U_p acts by

$$(f \mid U_p)(\tau) = p^{-1}\sum_{i=0}^{p-1} f((\tau+i)/p)$$

In terms of q-expansions, if $f=\sum_n a_n q^n$, we have

$$f \mid U_p = \sum_{p\mid n} a_n q^{n/p}.$$

If we define an operator V_p by

$$f \mid V_p = \sum_n a_n q^{np}, \tag{19}$$

then we have for $(p,l)=1$

$$T_p = U_p + p^{r-1}\varepsilon_p V_p. \tag{20}$$

It is easy to see how ε_p acts on the toric forms.

Lemma 5.2. *Let $f_{N,\deg}\in \mathscr{T}(l)$. If $(p,l)=1$, then*

$$f_{N,\deg} \mid \varepsilon_p = f_{N,p\deg}.$$

Proof. This follows from Proposition 4.3. □

Theorem 5.3. *The space $\mathscr{T}_r(l)$ is stable under the action of the Hecke operators.*

Proof. First we assume $(p,l)=1$ and consider the operator T_p. Let $f_{N,\deg}\in\mathscr{T}_r(l)$, and assume deg is linear with respect to Σ.

Using Lemma 5.2 and (20), we find

$$\begin{aligned} f_{N,\deg} \mid T_p &= \sum_{m\in M}\sum_{C\in\Sigma}(-1)^{\operatorname{codim} C}\text{a.c.}\Big(\sum_{n\in C\cap N}\delta_{m\cdot n}^{0 \bmod p\mathbb{Z}} q^{m\cdot n/p} e^{2\pi i\deg(n)}\Big)\\ &\quad + p^{r-1}\sum_{m\in M}\sum_{C\in\Sigma}(-1)^{\operatorname{codim} C}\text{a.c.}\Big(\sum_{n\in C\cap N} q^{m\cdot np} e^{2\pi i p\deg(n)}\Big)\\ &= \sum_{m\in M}\sum_{C\in\Sigma}(-1)^{\operatorname{codim} C}\text{a.c.}\Big(\sum_{n\in C\cap N}(\delta_{m\cdot n}^{0 \bmod p\mathbb{Z}} q^{m\cdot n/p} e^{2\pi i\deg(n)}\\ &\quad + p^{r-1}\delta_{m}^{0 \bmod pM} q^{m\cdot n} e^{2\pi i p\deg(n)})\Big). \end{aligned}$$

Here δ_β^α takes the value 1 if β is the residue α, and is 0 otherwise.

We will compare this expression with

$$\sum_S f_{S,p\deg},$$

where the sum is taken over lattices S satisfying $N \subset S \subset \frac{1}{p}N$ and $[S : N] = p^{r-1}$. The dual of each S lies in M, and we have

$$\sum_S f_{S,p\deg} = \sum_{m\in M}\sum_{C\in\Sigma}(-1)^{\operatorname{codim} C}\text{a.c.}\Big(\sum_S \sum_{n\in C\cap S} \delta_{m\cdot S}^{0 \bmod \mathbb{Z}} q^{m\cdot n}\mathrm{e}^{2\pi \mathrm{i} p\deg(n)}\Big) \tag{21}$$

$$= \sum_{m\in M}\sum_{C\in\Sigma}(-1)^{\operatorname{codim} C}\text{a.c.}\Big(\sum_R \sum_{n\in C\cap R} \delta_{m\cdot R}^{0 \bmod p\mathbb{Z}} q^{m\cdot n/p}\mathrm{e}^{2\pi \mathrm{i}\deg(n)}\Big), \tag{22}$$

Here the sum over R in (22) is taken over lattices with $N \supset R \supset pN$ and $[N : R] = p$.

Now consider the contributions to the R-sum by different $m \in M$. If $m \notin pM$, then there is only one R for which $m \cdot R$ is always divisible by p: the set of all n with $m \cdot n = 0 \bmod p$. So for every $m \notin pM$, each cone $C \in \Sigma$ contributes

$$\text{a.c.}\Big(\sum_{n\in C\cap N} \delta_{m\cdot n}^{0 \bmod p\mathbb{Z}} q^{m\cdot n/p}\mathrm{e}^{2\pi \mathrm{i}\deg(n)}\Big).$$

On the other hand, if $m \in pM$, then the sum (22) is taken over all R. Letting $\mu(n) = \sharp\{R \mid n \in R\}$, we have for fixed m and C

$$\begin{aligned}
&\text{a.c.}\Big(\sum_R \sum_{n\in C\cap R} q^{m\cdot n/p}\mathrm{e}^{2\pi \mathrm{i}\deg(n)}\Big) = \text{a.c.}\Big(\sum_{n\in C\cap N} \mu(n) q^{m\cdot n/p}\mathrm{e}^{2\pi \mathrm{i}\deg(n)}\Big)\\
&= \frac{p^{r-1}-1}{p-1}\text{a.c.}\Big(\sum_{n\in C\cap N} q^{m\cdot n/p}\mathrm{e}^{2\pi \mathrm{i}\deg(n)}\Big) + p^{r-1}\text{a.c.}\Big(\sum_{n\in C\cap pN} q^{m\cdot n/p}\mathrm{e}^{2\pi \mathrm{i}\deg(n)}\Big)\\
&= \frac{p^{r-1}-1}{p-1}\text{a.c.}\Big(\sum_{n\in C\cap N} q^{m\cdot n/p}\mathrm{e}^{2\pi \mathrm{i}\deg(n)}\Big) + p^{r-1}\text{a.c.}\Big(\sum_{n\in C\cap N} q^{m\cdot n}\mathrm{e}^{2\pi \mathrm{i} p\deg(n)}\Big).
\end{aligned}$$

Collecting these facts, we see

$$\begin{aligned}
f_{N,\deg} \,\Big|\, T_p - \sum_S f_{S,p\deg}(q) &= -\frac{p^{r-1}-p}{p-1}\sum_{m\in pM}\sum_{C\in\Sigma}(-1)^{\operatorname{codim} C}\text{a.c.}\Big(\sum_{n\in C} q^{m\cdot n/p}\mathrm{e}^{2\pi \mathrm{i}\deg(n)}\Big)\\
&= -\frac{p^{r-1}-p}{p-1}\sum_{m\in M}\sum_{C\in\Sigma}(-1)^{\operatorname{codim} C}\text{a.c.}\Big(\sum_{n\in C} q^{m\cdot n}\mathrm{e}^{2\pi \mathrm{i}\deg(n)}\Big)\\
&= \frac{p-p^{r-1}}{p-1} f_{N,\deg}.
\end{aligned}$$

This shows that $f_{N,\deg} \,\big|\, T_p$ is a linear combination of toric forms.

Now let $(p, l) > 1$, and consider the operator U_p. To show that $\mathscr{T}_r(l)$ is stable under U_p, we use a technique similar to the proof of Theorem 3.5. Let ε be a parameter, and let $\deg'$ be a generic degree function with respect to Σ. By the argument above, we have

$$f_{N,\deg+\varepsilon\deg'} \,\Big|\, T_p = \sum_{S_i} \alpha_i f_{S_i,\deg_i+\varepsilon\deg_i'},$$

where the sum is taken over some collection of lattices $\{S_i\}$ with $\deg_i(S_i) \subset \frac{1}{l}\mathbb{Z}$ and $\deg_i'$ generic for all i.

By (19), we have

$$f_{N,\deg+\varepsilon\deg'} \bigm| U_p = f_{N,\deg+\varepsilon\deg'} \bigm| T_p - p^{r-1} f_{N,p\deg+\varepsilon p\deg'} \bigm| V_p. \tag{23}$$

Let $m_1, \ldots, m_k \subset M$ be a complete set of residues for M/pM. For any $m \in M$, we write $[m/p]$ for the function $n \mapsto (m \cdot n)/p$.

Lemma 5.4. *We have*

$$\sum_k f_{N,\deg+\varepsilon\deg'+[m_k/p]} = p^r f_{N,p\deg+\varepsilon p\deg'} \bigm| V_p.$$

Proof of Lemma 5.4. We have

$$\sum_k f_{N,\deg+\varepsilon\deg'+[m_k/p]}$$
$$= \sum_{m\in M}\sum_{C\in\Sigma}(-1)^{\operatorname{codim} C}\text{a.c.}\Big(\sum_{n\in C} q^{m\cdot n}\sum_k \mathrm{e}^{2\pi\mathrm{i}(\deg+\varepsilon\deg'+[m_k/p])(n)}\Big).$$

Using

$$\sum_k \mathrm{e}^{2\pi\mathrm{i}(m_k\cdot n)/p} = \begin{cases} 0 & n \notin pN \\ p^r & n \in pN \end{cases}$$

this becomes

$$p^r \sum_{m\in M}\sum_{C\in\Sigma}(-1)^{\operatorname{codim} C}\text{a.c.}\Big(\sum_{n\in C\cap pN} q^{m\cdot n}\mathrm{e}^{2\pi\mathrm{i}(\deg+\varepsilon\deg')(n)}\Big) = p^r f_{N,p\deg+\varepsilon p\deg'} \bigm| V_p.$$

□

Lemma 5.4 and (23) imply

$$f_{N,\deg+\varepsilon\deg'} \bigm| U_p = \sum_i \alpha_i f_{S_i,\deg_i+\varepsilon\deg_i'}, \tag{24}$$

for some collection of lattices S_i and degree functions $\deg_i, \deg_i'$. We can desingularize Σ separately with respect to each S_i. Then by Theorem 3.4, (24) equals

$$\sum_i \int_{X_i} \Phi_i(\varepsilon),$$

where each X_i is a nonsingular toric variety, and $\Phi_i(\varepsilon)$ is an ε-family of rational functions in ϑ and ϑ'.

Since the action of U_p commutes with the ε-limit, we see that

$$\begin{aligned} f_{N,\deg} \bigm| U_p &= \lim_{\varepsilon\to 0}\sum_i \int_{X_i} \Phi_i(\varepsilon) \\ &= \lim_{\varepsilon\to 0} P(\varepsilon). \end{aligned}$$

Here $P(\varepsilon)$ is a Laurent series in ε with coefficients that are polynomials in $s^{(1)}_{a/l}$ and $r^{(2)}$ ($l \geq 5$), and by appropriate $s^{(k)}$ for other l, as in Proposition 4.6. This limit exists and converges to a modular form. Hence $r^{(2)}$ doesn't appear in the final expression, and by Theorem 4.11

$$f_{N,\deg} \mid U_p \in \mathscr{T}_r(l).$$

□

5.5. As a by-product of the above argument, we get that toric forms are stable under lifting of oldforms.

Corollary 5.6. *Let $f(q) = f(\tau)$ be a toric form of level l. Then $f(p\tau)$ is a sum of toric forms of level pl.*

Proof. As in Lemma 5.4,

$$f_{N,\deg}(p\tau) = p^{-r} \sum_k f_{N,\deg/p+[m_k/p]}(\tau).$$

□

Remark 5.7. At this point we do not know any direct proof of Hecke stability in terms of $s^{(k)}_{a/l}$. Because Hecke operators are generally not compatible with multiplication, we find it fascinating that there exists a Hecke stable finitely generated subring of the ring of modular forms.

5.8. Finally, we consider the Fricke involution. Let W_l be the matrix

$$\begin{pmatrix} 0 & -1 \\ l & 0 \end{pmatrix}$$

Theorem 5.9. *The space $\mathscr{T}(l)$ is stable under the action of W_l.*

Proof. By Remark 4.14, we need only consider $l \geq 5$, and for these levels, it suffices to verify the statement for $s^{(1)}_{a/l}$.

We have

$$\log\vartheta(z/l\tau + a, -1/l\tau) = f(\tau) + \frac{\pi\mathrm{i}}{l\tau}(z + a\tau)^2 + \log\vartheta(z + a\tau, l\tau).$$

After differentiating with respect to z and evaluating at $z = 0$, we obtain

$$\frac{1}{l\tau} s^1_{a/l}\left(-\frac{1}{l\tau}\right) = \frac{a}{l} + \frac{\vartheta'(a\tau, l\tau)}{2\pi\mathrm{i}\vartheta(a\tau, l\tau)}. \tag{25}$$

Now differentiate (16) with respect to z to obtain

$$\frac{1}{2\pi\mathrm{i}}\frac{\partial}{\partial z}\log\vartheta = \frac{1}{2} - \sum_{n\geq 1}\frac{\mathrm{e}^{2\pi\mathrm{i}z}q^n}{1 - \mathrm{e}^{2\pi\mathrm{i}z}q^n} + \sum_{n\geq 0}\frac{\mathrm{e}^{-2\pi\mathrm{i}z}q^{n-1}}{1 - \mathrm{e}^{-2\pi\mathrm{i}z}q^{n-1}}.$$

The right-hand side of (25) becomes

$$C - \sum_{n\geq 1} \frac{q^{a+nl}}{1-q^{a+nl}} + \sum_{n\geq 0} \frac{q^{-a+nl}}{1-q^{-a+nl}}, \tag{26}$$

where C is a constant.

In (26) we can take the first sum over $n \geq 0$ and the second over $n \geq 1$, and absorb the correction in the constant. After simplifying, we find that (26) becomes

$$C - \sum_{d\geq 1} q^d \big(\sum_{a+nl|d} -1 + \sum_{-a+nl|d} +1 \big) = C - \sum_{d\geq 1} q^d \sum_{k|d} (\delta_k^{a \bmod l} - \delta_k^{-a \bmod l}),$$

where δ is defined as in the proof of Theorem 5.3.

Now it is not hard to show, using (18), that

$$s^{(1)}_{b/l} = C_1(b) - \sum_d q^d \sum_{k|d} (\mathrm{e}^{2\pi \mathrm{i} kb/l} - \mathrm{e}^{-2\pi \mathrm{i} kb/l}).$$

Convolving with roots of unity, we obtain

$$\begin{aligned} \sum_{j=1}^{l-1} s^{(1)}_{j/l} \mathrm{e}^{-2\pi \mathrm{i} ja/l} &= C_2 - \sum_d q^d \sum_{k|d} \big(\sum_{j=1}^{N-1} \mathrm{e}^{-2\pi \mathrm{i} ja/l} (\mathrm{e}^{2\pi \mathrm{i} kj/l} - \mathrm{e}^{-2\pi \mathrm{i} kj/l}) \big) \\ &= C_2 - l \sum_d q^d \sum_{k|d} \left(\delta_k^{a \bmod l} - \delta_k^{-a \bmod l} \right). \end{aligned}$$

Hence after subtracting a linear combination of $s^{(1)}_{b/l}$ from $s^{(1)}_{a/l} \mid W_l$, we obtain a constant. Since this difference is modular, the constant must vanish, and the proof is complete. □

Remark 5.10. Notice that (26) implies that $s^{(1)}_{a/l} \mid W_l$ is the same as the Hecke-Eisenstein form $G_{1,a}$ appearing in [11, Ch. 15, §1].

6. Appendix: background on toric geometry

6.1. In this section we collect basic facts from toric geometry. For more details, the reader may consult [6].

6.2. Let N be a lattice, and let $N_{\mathbb{R}} = N \otimes \mathbb{R}$. A subset $C \subset N_{\mathbb{R}}$ is a *cone* if C is closed under homotheties and contains no line.

Let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice, and let $m \cdot n$ be the natural pairing. A cone C is *rational polyhedral* if there exist $m_1, \ldots, m_s \in M$ such that

$$C = \bigcap_i \big\{ x \in N_{\mathbb{R}} \bigm| m_i \cdot x \geq 0 \big\}. \tag{27}$$

A *face* of C is the subset of C obtained by making some of the inequalities in (27) equalities. A rational polyhedral cone C is *simplicial* if

$$C = \sum_{i=1}^{k} \mathbb{R}_{\geq 0} n_i, \quad n_i \in N,$$

where k is the dimension of the subspace generated by C.

6.3. Let Σ be a set of rational polyhedral cones. Then Σ is called a *fan* if

1. each face of a cone in Σ is also in Σ, and
2. the intersection of any two cones in Σ is a face of each.

A fan is *complete* if

$$\bigcup_{C \in \Sigma} C = N_{\mathbb{R}},$$

and is simplicial if all $C \in \Sigma$ are simplicial.

6.4. Let Σ be a fan. We can associate a *toric variety* X_Σ to Σ as follows.

If $C \subset N_{\mathbb{R}}$, let $C^* \subset M_{\mathbb{R}}$ be the dual cone. Then we have a $\mathbb{C}$-algebra S_C defined by

$$S_C = \mathbb{C}[C^* \cap M],$$

and we let U_C be the affine variety $\operatorname{Spec} S_C$. The variety U_C is the *toric chart* associated to C.

We can glue these affine varieties together using the combinatorics of Σ. Let $C_1, C_2 \in \Sigma$ satisfy $C_1 \cap C_2 \neq \varnothing$. Then we can identify $U_{C_1 \cap C_2}$ with a principal open subvariety of U_{C_1} and of U_{C_2}. These identifications are compatible as we range over all the cones of Σ, and so after gluing we obtain a variety X_Σ. One can show that X_Σ is separated, and has an open set isomorphic to an algebraic torus T. Moreover, X_Σ is complete if and only if Σ is complete, and X_Σ is nonsingular if and only if each cone of Σ is generated by part of a basis of N.

6.5. Let $\{d_i\} \subset \Sigma$ be the set of one-dimensional cones. Associated to each d_i is a T-invariant divisor $D_i \subset X_\Sigma$. One can show that if X_Σ is nonsingular, then the cohomology classes of the D_i generate $H^*(X_\Sigma, \mathbb{C})$. The linear relations among these classes are of the form

$$\sum_i (m \cdot d_i) D_i = 0,$$

where $m \in M_{\mathbb{C}}$. The total Chern class of X is given by

$$c(TX) = \prod_i (1 + D_i).$$

¿From this it follows that the Todd class is given by

$$\mathrm{Td}(TX) = \prod_i \frac{D_i}{1 - \mathrm{e}^{-D_i}}.$$

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