A SMOOTH SPACE OF TETRAHEDRA

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ABSTRACT. We construct a smooth symmetric compactification of the space of all labeled tetrahedra in $\mathbb{P}^3$.

1. INTRODUCTION

1.1. Let $\mathbb{P}^n$ be n-dimensional complex projective space, and let $P \subset \mathbb{P}^n$ be a set of $n+1$ labeled points in general position. By taking all possible linear spans of subsets of P, one obtains a configuration of flats in $\mathbb{P}^n$ arranged to form a simplex. The set X° of all such configurations is naturally a quasi-projective variety with a canonical singular compactification X (Definition 2.2). One is interested in the variety X for many reasons:

- For $n = 2$, the space X is the space of triangles in the plane. In [14], Schubert described a desingularization of X, and used it to study enumerative problems involving triangles [15, 13, 3].
- The space X is a *configuration variety* in the sense of Magyar [10, 18]. Such spaces arise naturally in the study of *generalized Schur modules.* These are (reducible) GL_n-modules that generalize the classical Schur modules, and have been studied in various guises by many authors [1, 9, 19, 8, 11, 12, 16]. One hopes that configuration varieties will play a role in a Borel-Weil theory for these modules.
- Let $\mathscr{B}$ be the Tits building for $\mathrm{SL}_{n+1}(\mathbb{C})$, and let $\mathscr{C}$ be the associated Coxeter complex [17]. Then X can be interpreted as the space of maps of $\mathscr{C}$ into $\mathscr{B}$. By considering other algebraic groups, one obtains a collection of natural configuration spaces related to the *Bott-Samelson varieties* of Demazure [4]. In particular, X can be regarded as a *canonical* Bott-Samelson variety associated to all reduced expressions of the longest word of the Weyl group of SL_{n+1}.
- The space X is a natural generalization of the Fulton-MacPherson space [5] $\mathbb{P}^n[n+1]$. This variety adds data to an open set of the product $\prod_{i=1}^{n+1} \mathbb{P}^n$ that records how points approach the (large) diagonals. In fact, $\mathbb{P}^n[n+1]$ is a desingularization of the space of all 1-*skeleta* of n-simplices in $\mathbb{P}^n$.

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1.2. In this paper we consider the case $n = 3$, where X is the space of tetrahedra in $\mathbb{P}^3$. We construct a symmetric compactification $\widetilde{X}$ of X° that we call the space of *complete tetrahedra* (Definition 3.7). It is obtained by embedding X° into a large ambient variety $E_\#$ and taking the closure of the image. The singular locus of the canonical compactification X is contained in the subvariety consisting of "collapsed tetrahedra"—that is, configurations of flats where certain faces coincide (Figure 1)—and $E_\#$ is constructed to capture the asymptotic behavior of a tetrahedron as it collapses. Our main theorem (Theorem 7.6) is that $\widetilde{X}$ is nonsingular.

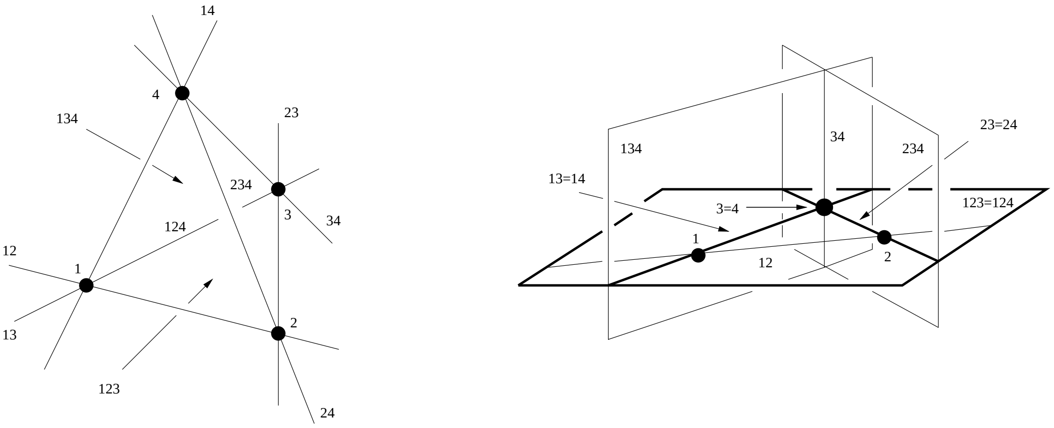

FIGURE 1. A point in X° and a point in $X \smallsetminus X^\circ$

In a later paper [2] we will make a more detailed study of the geometry of $\widetilde{X}$. We will show that the complement of X° in $\widetilde{X}$ is a divisor with normal crossings and compute the cohomology ring of $\widetilde{X}$.

1.3. Although this article considers the space of tetrahedra in $\mathbb{P}^3$, the definition of $\widetilde{X}$ makes sense for all n. Many of the results of the paper (in particular, sections 4–6) hold for arbitrary n, but we have avoided this generality since for $n \geq 4$ we cannot complete the proof that $\widetilde{X}$ is nonsingular (the combinatorial arguments in section 7 become unfeasible when $n \geq 4$). However, we conjecture that $\widetilde{X}$ provides a nonsingular compactification of X° for all n.

For $n = 2$, it is not hard to see that our variety $\widetilde{X}$ is nonsingular and coincides with certain triangle varieties found in the literature. More precisely, it is isomorphic to the Fulton-MacPherson space $\mathbb{P}^2[3]$, which in turn coincides with an auxiliary compactification constructed by Roberts-Speiser [13]. It is not, however, isomorphic to Schubert's compactification as a variety over X.[1]

[1] The difference between Schubert's space and Fulton-MacPherson-Roberts-Speiser's space appears when one considers the torus action on them, cf. [18].

1.4. We now give an overview of the definition of $\widetilde{X}$. The construction of $E_{\#}$ depends on the combinatorics of *hypersimplices* [6], polytopes intimately related to the geometry of Grassmannians. For our considerations, the relevant polytopes are the 3-dimensional hypersimplices Δ_1, Δ_2, and Δ_3 (Figure 2). The vertices of these hypersimplices are in bijection with the labeled faces of a tetrahedron, and the edges of the hypersimplices correspond to certain pairs of faces of the same dimension.

For each edge α in a hypersimplex, we form a $(\mathbb{P}^1\times\mathbb{P}^1)$-bundle $E_\alpha \to X$. The bundle E_α has a canonical section u_α and a diagonal subbundle D_α. The section u_α tracks the subspaces corresponding to the vertices of α in such a way that u_α intersects D_α precisely when these subspaces coincide. In order to record the asymptotic behavior in X near a collapsed tetrahedron, a natural idea is to form products of the E_α's and blow up the corresponding product of diagonals. The question is which products to take, and why.

1.5. Our main idea is that the relevant products are those indexed by the faces of dimension ≥ 2 of the hypersimplices. The motivation is that a configuration of flats in $\mathbb{P}^3$ arranged to form a tetrahedron contains certain "sub-" and "quotient" configurations corresponding to proper faces of the hypersimplices. For example, the three points and three lines in a given face of a tetrahedron form a subconfiguration that corresponds to a triangular face in the hypersimplex Δ_1, and the three lines and three planes containing a given point form a quotient configuration that corresponds to a triangular face in the hypersimplex Δ_3. Our motivation is that a nonsingular compactification of X° should add data recording the "infinitesimal shapes" of these sub- and quotient configurations. Hence, each locus we blow up corresponds to the collapsing together of the subspaces labeled by some face of a hypersimplex.

More precisely, our definition is as follows. Let $\mathcal{H}$ be the set of faces of dimension ≥ 2 of all the Δ_k. For each $\beta \in \mathcal{H}$, let $\mathcal{E}(\beta)$ be the set of edges in β. Let E_β be the product bundle

$$E_\beta := \prod_{\alpha\in\mathcal{E}(\beta)} E_\alpha,$$

and let D_β be the corresponding product of the diagonals D_α. The ambient variety $E_{\#}$ is then defined by blowing up each E_β along D_β and taking the product of the resulting blowups. The corresponding product of the sections u_α, then determines an embedding $X^\circ \to E_{\#}$, and we define $\widetilde{X}$ to be the closure.

1.6. The paper is organized as follows. Section 2 sets up notation, defines X, and contains background on hypersimplices. Section 3 contains the construction of $E_{\#}$ and $\widetilde{X}$. In section 4 we describe a collection of affine open subvarieties covering X and give equations defining a typical element $U \subset X$ in this collection. In section 5 we restrict the construction of $E_{\#}$ and $\widetilde{X}$ to the U's, obtaining a cover of $\widetilde{X}$ by open quasi-projective subvarieties. We also give defining equations for a typical element $\widetilde{U}$

of this cover. The point of sections 4–5 is that the nonsingularity of $\widetilde{X}$ follows from the nonsingularity of $\widetilde{U}$.

In the remaining sections we prove nonsingularity of $\widetilde{U}$. First, in section 6, we show that $\widetilde{U}$ has the structure of a vector bundle over a certain (multi-) projective variety Z that we call the *core*. We then study the GL_4-action on $\widetilde{U}$ to show that nonsingularity of Z follows from its nonsingularity at points in a certain subvariety $Z_{sp} \subset Z$. Finally in section 7, we give equations that cut out Z from projective space and use a graphical description of these defining relations to show that Z_{sp} consists of nonsingular points of Z; this proves Theorem 7.6.

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2. Notation and the basic variety X

2.1. Let $e_1, \dots, e_4$ be the standard basis of $\mathbb{C}^4$, and let $[\![4]\!]$ be the set $\{1, 2, 3, 4\}$. For any subset $I \subset [\![4]\!]$, let $E_I \subset \mathbb{C}^4$ be the subspace spanned by $\{e_i \mid i \in I\}$. Let $\mathbb{P}^3$ be the projective space of lines in $\mathbb{C}^4$, and let G be the algebraic group $\mathrm{GL}_4(\mathbb{C})$.

For $k = 1, 2, 3$, let Gr_k be the Grassmannian of k-dimensional subspaces of $\mathbb{C}^4$, and for each proper nonempty subset $I \subset [\![4]\!]$, let $\mathrm{Gr}_I := \mathrm{Gr}_{|I|}$. (We use the notation $|I|$ for the cardinality of I.) Let Y be the product

$$Y := \prod_{\varnothing \subsetneq I \subsetneq [\![4]\!]} \mathrm{Gr}_I \cong (\mathrm{Gr}_1)^4 \times (\mathrm{Gr}_2)^6 \times (\mathrm{Gr}_3)^4,$$

and for each $I \subset [\![4]\!]$, let π_I be the projection to the Ith factor. The group G acts on Y by left multiplication, and each π_I is G-equivariant.

Definition 2.2. Let $p_0 \in Y$ be the point such that $\pi_I(p_0) = E_I$ for all $I \subset [\![4]\!]$, and let $X^\circ \subset Y$ be the G-orbit of p_0. Let $X \subset Y$ be $\overline{X^\circ}$ (the bar denotes Zariski closure). The space X (respectively X°) is called the *space of tetrahedra (resp. nondegenerate tetrahedra).*

Note that since the G-action preserves incidence relations among subspaces, for any $p \in X$ we have $\pi_I(p) \subset \pi_J(p)$ if $I \subset J$. Hence the configuration of subspaces $\{\pi_I(p) \mid I \subset [\![4]\!]\}$ satisfies the incidence relations corresponding to the faces of a tetrahedron.

The symmetric group S_4 acts on Y by permuting the factors, and this clearly induces an action on X° and X: given $\sigma \in S_4$ and $p \in X$, the point $\sigma \cdot p$ is determined by $\pi_I(\sigma \cdot p) = \pi_{\sigma^{-1}(I)}(p)$. This action can be viewed as "changing the labels" on the faces of a tetrahedron.

2.3. The construction of our resolution $\widetilde{X} \to X$ is based on the combinatorics of hypersimplices, so we recall basic facts about them. More details can be found in [6].

Let $\varepsilon_1, \dots, \varepsilon_4$ be the standard basis of $\mathbb{R}^4$, and for any subset $I \subset [\![4]\!]$, let $\varepsilon_I := \sum_{i \in I} \varepsilon_i$. Then the *hypersimplex (of rank k)* Δ_k is defined by

$$\Delta_k := \operatorname{Conv}\{\varepsilon_I \mid I \subset [\![4]\!] \quad \text{and} \quad |I| = k\},$$

where Conv denotes convex hull. The hypersimplices Δ_1 and Δ_3 are 3-simplices, and Δ_2 is an octahedron. Note that the vertices of a hypersimplex are indexed by proper nonempty subsets of $[\![4]\!]$ (Figure 2). In the remainder of the paper, we shall denote such a subset by the string of its elements listed in order, thus $\{2\} = 2$, $\{1, 3\} = 13$, etc. It will be convenient to fix a total ordering on the subsets of $[\![4]\!]$, and thus on the vertices of the hypersimplices:

$$\varnothing < 1 < 2 < 3 < 4 < 12 < 13 < 14 < 23 < 24 < 34 < 123 < 124 < 134 < 234 < 1234.$$

If I is a subset of $[\![4]\!]$ with $|I| = k$, we let I_0 be the set 1, 12, or 123, depending on whether $k = 1, 2$, or 3, respectively.

2.4. We identify faces of the hypersimplices with their corresponding sets of vertices. Let $\mathcal{E}$ be the set of pairs $\{I, J\}$ with $I, J \subset [\![4]\!]$ corresponding to edges of the hypersimplices, and for $k = 1, 2, 3$ let $\mathcal{E}_k \subset \mathcal{E}$ be the subset corresponding to edges of Δ_k. Let $\mathcal{H}$ be the set of vertex sets of all faces of dimension ≥ 2. Hence $\mathcal{H}$ contains the maximal 3-dimensional faces $\{1, 2, 3, 4\}$, $\{12, 13, 14, 23, 24, 34\}$, and $\{123, 124, 134, 234\}$, as well as 16 triangular faces. These triangular faces can be oriented as follows. Let $\beta = \{I, J, K\} \in \mathcal{H}$ be a triangular face, and let (JK, IK, IJ) be the corresponding triple of edges. We call this triple an *ordered triangle* if $I < J < K$.

We shall need notation for the edges of a given face in $\mathcal{H}$. For each $\beta \in \mathcal{H}$, we let $\mathcal{E}(\beta) \subset \mathcal{E}$ be the subset corresponding to the edges of β. For example, if β is the triangular face $\{12, 13, 23\}$ of the octahedron, then $\mathcal{E}(\beta) = \{\{12, 13\}, \{12, 23\}, \{13, 23\}\}$.

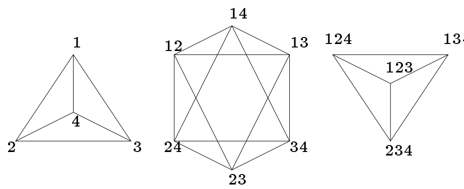

FIGURE 2.

3. The resolution $\widetilde{X}$

3.1. As the first step towards defining $\widetilde{X}$, we establish a correspondence between edges of the hypersimplices and certain $\mathbb{P}^1$-bundles over X. For each nonempty subset $I \subset [\![4]\!]$, let $F_I \to X$ be the pullback of the tautological $|I|$-plane bundle on $\mathrm{Gr}_{|I|}$ via the composition

$$X \longrightarrow Y \xrightarrow{\pi_I} \mathrm{Gr}_{|I|}\,.$$

Thus the fiber of F_I over a point $p \in X$ can be identified with the k-dimensional subspace $\pi_I(p) \subset \mathbb{C}^4$. The incidence conditions on X imply that if $I \subset J$, then $F_I \subset F_J$ is a subbundle.

For each $\alpha \in \mathcal{E}$, let F_α be the quotient $F_{I\cup J}/F_{I\cap J}$ where $\alpha = \{I, J\}$; this is a rank-2 vector bundle since $|I \cup J \smallsetminus I \cap J| = 2$. Let P_α be the projectivized bundle

$$P_\alpha = \mathbb{P}(F_\alpha).$$

3.2. The bundle P_α has canonical sections $u_\alpha^-, u_\alpha^+\colon X \to P_\beta$, defined geometrically as follows. The fiber of P_α over p can be identified with the set of lines in the 2-dimensional vector space $\pi_{I\cup J}(p)/\pi_{I\cap J}(p)$. We assume that $I < J$, and define $u_\alpha^-(p)$ to be the line $\pi_I(p)/\pi_{I\cap J}(p)$ and $u_\alpha^+(p)$ to be the line $\pi_J(p)/\pi_{I\cap J}(p)$. Since we will want to keep track of both sections simultaneously, we introduce the product bundle

$$E_\alpha = P_\alpha \times_X P_\alpha,$$

and let $u_\alpha\colon X \to E_\alpha$ be the product $u_\alpha^- \times u_\alpha^+$.

3.3. For each $\beta \in \mathcal{H}$, let E_β be the product bundle

$$E_\beta = \prod_{\alpha\in\mathcal{E}(\beta)} E_\alpha.$$

This is a $(\mathbb{P}^1 \times \mathbb{P}^1)^3$-bundle over X when β is a triangular face; for the maximal faces, P_{Δ_k} is a $(\mathbb{P}^1 \times \mathbb{P}^1)^6$-bundle for $k = 1, 3$ and a $(\mathbb{P}^1 \times \mathbb{P}^1)^{12}$-bundle for $k = 2$. Let $p_\beta\colon E_\beta \to X$ be the projection, and let $u_\beta\colon X \to E_\beta$ be the section obtained by taking the product of the sections u_α for all $\alpha \in \mathcal{E}(\beta)$.

3.4. We define the *ambient variety* E to be the product bundle

$$E = \prod_{\beta\in\mathcal{H}} E_\beta.$$

By Figure 2, there are 3 maximal elements of $\mathcal{H}$ with 6, 12, and 6 edges respectively, and there are 16 triangular faces with 3 edges each; thus, $E \to X$ is a locally trivial bundle with fiber isomorphic to

$$(\mathbb{P}^1 \times \mathbb{P}^1)^6 \times (\mathbb{P}^1 \times \mathbb{P}^1)^{12} \times (\mathbb{P}^1 \times \mathbb{P}^1)^6 \times \left((\mathbb{P}^1 \times \mathbb{P}^1)^3\right)^{16}.$$

Let $p\colon E \to X$ be the projection, and let $u\colon X \to E$ be the section obtained by taking the product of the sections u_β, $\beta \in \mathcal{H}$.

3.5. To build $\widetilde{X}$, we keep track of "limiting configurations" of the subspaces $\{\pi_I(p)\}$ as certain collections of them coincide. The relevant collections turn out to correspond to the faces $\mathcal{H}$ of the hypersimplices.

For each $\alpha \in \mathcal{E}$, let $D_\alpha \subset E_\alpha$ be the diagonal subbundle, and for each $\beta \in \mathcal{H}$, let $D_\beta \subset E_\beta$ be the subbundle

$$D_\beta = \prod_{\alpha \in \mathcal{E}(\beta)} D_\alpha.$$

The geometric significance of D_β is that the set of points $p \in X$ such that $u_\beta(p) \in D_\beta$ is precisely the set of p such that $\pi_I(p) = \pi_J(p)$ for all $I, J \in \beta$.

Let

$$b_\beta\colon (E_\beta)_\# \longrightarrow E_\beta$$

be the blowup of E_β along D_β. Since E_β is locally trivial over X, as is the subbundle D_β, the blowup $(E_\beta)_\#$ is also locally trivial over X; the fiber of this last bundle is isomorphic to the blowup of $(\mathbb{P}^1 \times \mathbb{P}^1)^n$ along the product of diagonals (where $n = 3$, 6, or 12 depending on β).

3.6. Since for any $p \in X^\circ$ the image $u_\beta(p)$ avoids the blowup center D_β, we have a regular map

$$b_\beta^{-1} \circ u_\beta\colon X^\circ \longrightarrow (E_\beta)_\#.$$

We define the *complete ambient variety* $E_\#$ to be the product

$$E_\# = \prod_{\beta \in \mathcal{H}} (E_\beta)_\#,$$

and let $b\colon E_\# \to E$ be the product of the blowup maps b_β, $\beta \in \mathcal{H}$.

Definition 3.7. Let $\widetilde{X}^\circ$ be the image of the embedding

$$X^\circ \longrightarrow E_\#$$

obtained by taking a product of the maps $b_\beta^{-1} \circ u_\beta$ for all $\beta \in \mathcal{H}$. The *complete space of tetrahedra*, denoted $\widetilde{X}$, is the closure of $\widetilde{X}^\circ$ in $E_\#$.

The composition $p \circ b\colon E_\# \to X$ restricts to a surjective birational morphism $\rho\colon \widetilde{X} \to X$.

3.8. Since the bundles E_β are constructed from tautological bundles, they admit natural G-actions lifting the action on X. Since the diagonals are preserved by these actions, the blown-up bundles $(E_\beta)_\#$ also admit natural G-actions lifting the action on X, and the blowdown maps b_β are equivariant. It follows that there are natural G-actions on E and $E_\#$, and that $b\colon E_\# \to E$ is equivariant. One can check that the section u is also equivariant, and thus $\widetilde{X}^\circ$ is G-stable. It follows that the action on $E_\#$ restricts to an action on $\widetilde{X}$ and that $\rho\colon \widetilde{X} \to X$ is G-equivariant.

Similar remarks apply to the S_4-action. This action also lifts to actions on E and $E_\#$ that permute the various factors of these product bundles. The map $b\colon E_\# \to E$ and the section $u\colon X \to E$ are both equivariant, so $\widetilde{X}^\circ$ is S_4-stable. Hence, the S_4-action on $E_\#$ restricts to an action on $\widetilde{X}$, and $\rho\colon \widetilde{X} \to X$ is S_4-equivariant.

4. The local variety U

4.1. Let Fl be the flag variety of full flags in $\mathbb{C}^4$, and let $V_* \in \mathrm{Fl}$ correspond to a flag

$$\{0\} = V_0 \subsetneq V_1 \subsetneq V_2 \subsetneq V_3 \subsetneq V_4 = \mathbb{C}^4,$$

where V_k is a subspace of dimension k. Let $U(V_*)$ be the set of all $p \in X$ in general position to V_*. In other words, $U(V_*)$ consists of all p such that for each proper nonempty subset $I \subset [\![4]\!]$, the $|I|$-plane $\pi_I(p)$ is transverse to V_k for all $1 \leq k \leq 3$.

The subset $U(V_*)$ can be described in terms of Schubert cells in the factors Gr_I of Y as follows. For each $k = 1, 2, 3$, let U_k be the open cell in Gr_k consisting of k-planes in general position to the fixed flag V_*. For each proper nonempty subset $I \subset [\![4]\!]$, let $U_I := U_{|I|}$. Then $\prod_I U_I$ is an open subvariety of Y isomorphic to an affine space. The variety $U(V_*)$ is the intersection of this open set and the subvariety X of Y. In particular, $U(V_*)$ is an affine open subset of X.

4.2. Let $E_\#|_{U(V_*)}$ be the restriction of the ambient bundle to $U(V_*)$. Let $U^\circ(V_*) = U(V_*) \cap X^\circ$, and let $\widetilde{U}^\circ(V_*)$ be the image of $U^\circ(V_*)$ under the embedding $U^\circ(V_*) \to E_\#|_{U(V_*)}$ of 3.6. Take $\widetilde{U}(V_*)$ to be the closure of $\widetilde{U}^\circ(V_*)$ in $E_\#|_{U(V_*)}$. We omit the proof of the following simple lemma:

Lemma 4.3. *The collection $\{U(V_*) \mid V_* \in \mathrm{Fl}\}$ (respectively, $\{\widetilde{U}(V_*) \mid V_* \in \mathrm{Fl}\}$) is an affine open cover of X (resp., $\widetilde{X}$). The group G acts transitively on both of these covers.*

To prove that $\widetilde{X}$ is nonsingular, it suffices by Lemma 4.3 to prove that $\widetilde{U}(V_*)$ is nonsingular for one particular choice of the flag V_*. We fix V_* to be the *flag at infinity*:

$$E_4 \subset E_{34} \subset E_{234},$$

and define U, U°, and $\widetilde{U}$ to be the varieties $U(V_*)$, $\widetilde{U}^\circ(V_*)$, and $\widetilde{U}(V_*)$ (respectively).

4.4. We put coordinates on U using Plücker coordinates on the Grassmannians Gr_I. For each $k = 1, 2, 3$, we have the Plücker embedding $\mathrm{Gr}_k \to \mathbb{P}(\bigwedge^k \mathbb{C}^4)$ with its usual coordinates $\{f_I \mid I \subset [\![4]\!], |I| = k\}$. The ratios $\{f_I/f_{I_0} \mid |I| = k\}$ provide coordinates on U_k. For any pair $I, J \subset [\![4]\!]$ with $|I| = |J|$, let $f_{I,J}$ be the regular function on U defined by

$$f_{I,J} := \pi_I^*(f_J/f_{J_0}).$$

It is clear that these functions generate the ring $\mathscr{O}_X(U)$, and that $f_{I,J_0} = 1$.

4.5. We can obtain a more symmetric set of generators for $\mathscr{O}_X(U)$ by observing that $p \in U$ can be constructed from functions on Fl and functions that measure the "difference" between the planes $\pi_I(p)$ and $\pi_J(p)$ for each edge $\{I, J\}$ of the appropriate hypersimplex.

The functions on the flag variety are defined as follows. There is a natural map $X \to \mathrm{Fl}$ given by $p \mapsto \{\pi_{I_0}(p)\}$. Let U_{op} be the open cell in Fl consisting of flags in general position to the flag at infinity. This cell has local coordinates

$$f_2/f_1,\ f_3/f_1,\ f_4/f_1,\ f_{13}/f_{12},\ f_{14}/f_{12},\ f_{124}/f_{123}.$$

The corresponding functions

$$f_{1,2},\ f_{1,3},\ f_{1,4},\ f_{12,13},\ f_{12,14},\ f_{123,124}$$

on U will be called *flag coordinates on U*.

4.6. The functions on U corresponding to edges in the hypersimplices are easiest to describe using certain local sections of the bundles of 3.1.

For each nonempty $I \subset [\![4]\!]$, let $\mathscr{F}_I$ be the sheaf of sections of the bundle $F_I \to X$. We define local sections $s_I \in \mathscr{F}_I(U)$ as follows. Let $k = |I|$. Then for each $p \in U$, the fiber of F_I over p can be identified with the k-dimensional subspace $\pi_I(p) \subset \mathbb{C}^4$. This subspace intersects the subspace V_{5-k} of our flag at infinity in a 1-dimensional subspace, and intersects V_{4-k} in the zero subspace. It follows that there is a unique vector $s_I(p) \in \pi_I(p) \cap V_{5-k}$ whose kth coordinate (with respect to the standard basis) is 1. This defines the section $s_I \colon U \to F_I|_U$.

In terms of Plücker coordinates, these sections can be expressed as

$$\begin{array}{lcccccccc}
s_i & = & e_1 & + & f_{i,2}\,e_2 & + & f_{i,3}\,e_3 & + & f_{i,4}\,e_4 \\
s_{ij} & = & & & e_2 & + & f_{ij,13}\,e_3 & + & f_{ij,14}\,e_4 \\
s_{ijk} & = & & & & & e_3 & + & f_{ijk,124}\,e_4 \\
s_{1234} & = & & & & & & & e_4 \ .
\end{array}$$

A priori, these are all sections of the trivial bundle $U \times \mathbb{C}^4$, but a simple verification shows that their images are contained in $F_I|_U$. The following lemma describes a crucial relation among these sections. We omit the straighforward proof.

Lemma 4.7. *Let $k = 1, 2, 3$. For each edge $\alpha = \{I, J\} \in \mathcal{E}_k$, we have*

$$s_J - s_I = (f_{J,K} - f_{I,K})s_{I\cup J},$$

where K is the subset 2, 13, *or* 124 *depending on whether k is* 1, 2, *or* 3, *respectively.*

For $k = 1, 2, 3$ and each edge $\alpha = \{I, J\} \in \mathcal{E}_k$ with $I < J$, we define the *edge coordinate x_α* by

$$x_\alpha = f_{J,K} - f_{I,K},$$

where K is the subset 2, 13, or 124 depending on whether k is 1, 2, or 3, respectively.

Lemma 4.8. *The ring $\mathscr{O}_X(U)$ is generated by the flag coordinates and the edge coordinates.*

Proof. We show that the functions $f_{I,J}$ can be expressed in terms of the flag coordinates and the x_α's. The proof is by induction on I, using the total ordering

$$1 \prec 12 \prec 123 \prec 2 \prec 13 \prec 3 \prec 23 \prec 124 \prec 14 \prec 4 \prec 24 \prec 134 \prec 34 \prec 234.$$

Note that this ordering differs from that defined in section 2.3. The key property of the ordering $\prec$ is that for each $J \succ 123$, there exists an edge $\{I,J\} \in \mathcal{E}$ such that I, $I \cap J$, $I \cup J \prec J$.

Using the formulas of 4.6, we can express the sections s_1, s_{12}, and s_{123} entirely in terms of the flag coordinates (and the basis e_1, e_2, e_3, e_4). Since s_1 determines the line π_1, $s_1 \wedge s_{12}$ determines the plane π_{12}, and $s_1 \wedge s_{12} \wedge s_{123}$ determines the 3-plane π_{123}, we can express all of the corresponding functions $f_{1,J}$, $f_{12,J}$, and $f_{123,J}$ in terms of the flag coordinates.

For the case $I = 2$, since $s_2 = s_1 + x_{1,2}s_{12}$ (by Lemma 4.7) and s_2 determines π_2, we can express the functions $f_{2,J}$ in terms of $x_{1,2}$ and the flag coordinates. For $I = 13$, $s_{13} = s_{12} + x_{12,13}s_{123}$ and $s_1 \wedge s_{13}$ determines π_{13}, so we can express the functions $f_{13,J}$ in terms of $x_{12,13}$ and the flag coordinates. Expressions for the remaining functions are obtained similarly. □

4.9. Define polynomial rings

$$R_{op} := \mathbb{C}[f_{1,2}, f_{1,3}, f_{1,4}, f_{12,13}, f_{12,14}, f_{123,124}],$$
$$R_{\mathcal{E}} := \mathbb{C}[x_\alpha \mid \alpha \in \mathcal{E}],$$

and let $\mathbb{A}_{op} = \operatorname{Spec} R_{op}$, $\mathbb{A}_{\mathcal{E}} = \operatorname{Spec} R_{\mathcal{E}}$. Lemma 4.8 says that the natural homomorphism $R_{op} \otimes R_{\mathcal{E}} \to \mathscr{O}_X(U)$ is surjective, so $U \subset \mathbb{A}_{op} \times \mathbb{A}_{\mathcal{E}}$. We now describe the ideal that set-theoretically cuts out U. This ideal is generated by linear, quadric, cubic, and quartic polynomials in the flag and edge coordinates.

First consider the flag coordinates. The map $X \to \mathrm{Fl}$ is actually a locally trivial fibration, and our coordinates define a trivialization over U_{op}. Since U_{op} is nonsingular and the flag coordinates on U are pulled back from a system of local parameters on U_{op}, there are no relations among the flag coordinates holding on U.

Now consider the edge coordinates. Recall that a triple of edges (JK, IK, IJ) is an ordered triangle if $\{I, J, K\}$ is a triangular face and $I < J < K$.

Lemma 4.10. *The subvariety $U \subset \mathbb{A}_{op} \times \mathbb{A}_{\mathcal{E}}$ is defined set-theoretically by the following polynomials:*

1. *The linear functions*
$$x_{\alpha_1} - x_{\alpha_2} + x_{\alpha_3},$$
for all ordered triangles $(\alpha_1, \alpha_2, \alpha_3)$ (Figure 3, left).
2. *The quadric functions*
$$x_{\alpha_1}x_{\alpha_2^*} - x_{\alpha_2}x_{\alpha_1^*},$$
where $\alpha_1 = \{i,j\}, \alpha_2 = \{j,k\}, \alpha_1^* = \{ik, jk\}, \alpha_2^* = \{ij, ik\}$ or $\alpha_1 = \{il, jl\}, \alpha_2 = \{jl, kl\}, \alpha_1^* = \{ikl, jkl\}, \alpha_2^* = \{ijl, ikl\}$ (Figure 3, right).

3. *The cubic functions*

$$x_{\alpha_1}x_{\alpha_2}x_{\alpha_3} - x_{\alpha_1^*}x_{\alpha_2^*}x_{\alpha_3^*},$$

where $\alpha_1 = \{ij, il\}, \alpha_2 = \{ik, kl\}, \alpha_3 = \{jk, jl\}, \alpha_1^* = \{jk, kl\}, \alpha_2^* = \{ij, jl\}, \alpha_3^* = \{ik, il\}$ *(Figure 4, left).*

4. *The quartic functions*

$$x_{\alpha_1}x_{\alpha_3}x_{\alpha_2^*}x_{\alpha_4^*} - x_{\alpha_2}x_{\alpha_4}x_{\alpha_1^*}x_{\alpha_3^*},$$

where $\alpha_1 = \{i,j\}, \alpha_2 = \{j,k\}, \alpha_3 = \{k,l\}, \alpha_4 = \{l,i\}, \alpha_1^* = \{ikl, jkl\}, \alpha_2^* = \{ijl, ikl\}, \alpha_3^* = \{ijk, ijl\}, \alpha_4^* = \{jkl, ijk\}$ *(Figure 4, right).*

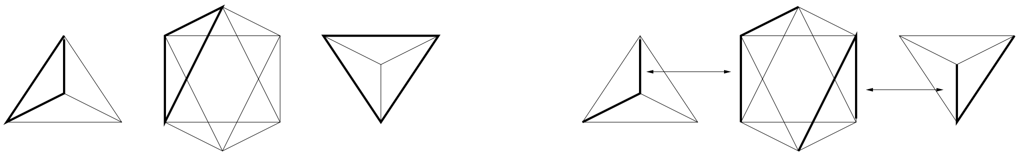

FIGURE 3. Edges in the linear and quadric relations

FIGURE 4. Edges in the cubic and quartic relations

Proof. The vanishing of the linear polynomials follows from the definition of the edge coordinates. The quadric relations follow from this definition and Lemma 4.7. The cubic (resp., quartic) relations can be obtained from the quadric relations by eliminating coordinates corresponding to edges in $\mathcal{E}_1$ and $\mathcal{E}_3$ (resp., $\mathcal{E}_2$). It follows that U is a subvariety of the variety defined by the given polynomials.

To see that U coincides with this variety, it suffices to show that they are both irreducible and have the same dimension. Since U is the closure of the connected 12-dimensional nonsingular variety U°, it is a 12-dimensional irreducible variety. Let U_{inc} be the variety defined by the vanishing of the linear and quadric polynomials. A simple computation using Macaulay2 [7] shows that U_{inc} has three irreducible components, two of which correspond to the ideals

$$\langle x_\alpha \mid \alpha \in \mathcal{E}_2\rangle \quad \text{and} \quad \langle x_\alpha \mid \alpha \in \mathcal{E}_1 \cup \mathcal{E}_3\rangle.$$

The remaining component U_1 is 12-dimensional. Since U contains points where all x_α are nonzero, we have $U = U_1$. □

Remark 4.11. Let X_{inc} be the *incidence variety* consisting of all $p \in Y$ such that $\pi_I(p) \subset \pi_J(p)$ whenever $I \subset J$. Then one can show $U_{inc} = X_{inc} \cap \prod_I U_I$, where $\prod_I U_I$ is the affine cell of 4.1, although we will not need this here. Hence the linear and quadric relations provide a very simple description of the incidence variety.

In the case under study, Lemma 4.10 shows that X_{inc} has two other components besides X, corresponding to the following types of configurations in $\mathbb{C}^4$:

1. Four 3-planes containing a 2-plane that contains four lines.
2. Six 2-planes containing a line and contained in a 3-plane.

Figure 5 shows general points in these components as configurations in $\mathbb{P}^3$. For the general case of simplex configurations in $\mathbb{C}^n$, the components of X_{inc} are unknown.

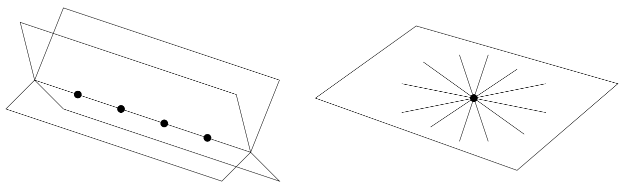

FIGURE 5. Other components of the incidence variety.

5. THE LOCAL RESOLUTION $\widetilde{U}$

5.1. In this section we describe an open subbundle W (with affine space fibers) of the restricted ambient bundle $E|_U$. The section u restricts to a section of W, thus we can apply the blowup construction to W, obtaining an open subbundle $W_\#$ of $E_\#|_U$ that contains the local resolution $\widetilde{U}$. To get functions on $\widetilde{U}$, we then show that $\widetilde{U}$ is contained in a certain closed subvariety of $W_\#$ defined in terms of the edge coordinates.

To describe W, we first need to describe some trivializations of the various bundles restricted to U.

5.2. For each $\alpha = \{I, J\} \in \mathcal{E}$, let $\mathscr{F}_\alpha$ be the sheaf of sections of the quotient bundle $F_\alpha = F_{I\cup J}/F_{I\cap J}$. Letting $\mathscr{F}_\alpha^*$ be the dual module, and restricting to U, we then have (by definition)

$$F_\alpha|_U = \operatorname{Spec}_{\mathscr{O}_X(U)} S_\alpha$$

where S_α is the symmetric $\mathscr{O}_X(U)$-algebra $\operatorname{Sym} \mathscr{F}_\alpha^*(U)$.

There are natural maps $\mathscr{F}_I \to \mathscr{F}_\alpha$, $\mathscr{F}_J \to \mathscr{F}_\alpha$, and $\mathscr{F}_{I\cup J} \to \mathscr{F}_\alpha$ induced by the corresponding inclusions of F_I, F_J, and $F_{I\cup J}$ (respectively) into $F_{I\cup J}$. With respect to these maps, the local sections $s_I \in \mathscr{F}_I(U)$, $s_J \in \mathscr{F}_J(U)$, and $s_{I\cup J} \in \mathscr{F}_{I\cup J}(U)$ all have images in $\mathscr{F}_\alpha(U)$, which we denote by $\overline{s}_I$, $\overline{s}_J$, and $\overline{s}_{I\cup J}$, respectively. It follows from the explicit descriptions in 4.6 that $\overline{s}_I + \overline{s}_J$ and $\overline{s}_{I\cup J}$ are nonzero and linearly independent.

Since F_α is a rank-2 vector bundle, these sections determine dual sections $g_\alpha, h_\alpha \in \mathscr{F}^*_\alpha(U)$, giving an isomorphism

$$F_\alpha|_U \cong U \times \operatorname{Spec} \mathbb{C}[g_\alpha, h_\alpha].$$

Passing to the projectivized bundle $P_\alpha = \mathbb{P}(F_\alpha)$ restricted to U, we obtain

$$P_\alpha|_U = \operatorname{Proj}_{\mathscr{O}_X(U)} S_\alpha \cong U \times \operatorname{Proj} \mathbb{C}[g_\alpha, h_\alpha].$$

5.3. Let $D(g_\alpha) \subset P_\alpha|_U$ be the divisor determined by g_α, and let V_α be the corresponding open set $P_\alpha|_U - D(g_\alpha)$. Then V_α is an affine line bundle over U, and we have an isomorphism

$$V_\alpha \cong U \times \operatorname{Spec} \mathbb{C}[h_\alpha/g_\alpha].$$

5.4. For each $\alpha \in \mathcal{E}$, let W_α be the product $V_\alpha \times_U V_\alpha$. To distinguish the two factors of W_α, we denote the first by V_α^- and the second by V_α^+. For any $\beta \in \mathcal{H}$, let W_β be the product of the W_α as α runs over all edges in $\mathcal{E}(\beta)$. Finally, let W be the product of the W_β as β ranges over all faces in $\mathcal{H}$. Combining the previous sections, we then have the following:

Lemma 5.5. *Let R be the polynomial ring*

$$R := \mathbb{C}[x^+_{\alpha,\beta}, x^-_{\alpha,\beta} \mid \beta \in \mathcal{H}, \alpha \in \mathcal{E}(\beta)],$$

and let $\mathbb{A}_{amb} = \operatorname{Spec} R$. Then W is an open subbundle of $E|_U$, and is isomorphic to the product $U \times \mathbb{A}_{amb}$. The indeterminate $x^-_{\alpha,\beta}$ (resp., $x^+_{\alpha,\beta}$) corresponds to the function h_α/g_α on the factor V_α^- (resp., V_α^+) of W_β.

5.6. We now restrict the blowup construction of 3.5 to the affine space bundle $W \to U$. For each $\beta \in \mathcal{H}$, we let $(W_\beta)_\#$ be the blowup of W_β along the product of diagonals $D_\beta \cap W_\beta$. In terms of the coordinates in Lemma 5.5, the ideal defining this product of diagonals is $\langle x^+_{\alpha,\beta} - x^-_{\alpha,\beta} \mid \alpha \in \mathcal{E}(\beta)\rangle$. Thus, if R_β is the polynomial ring $\mathbb{C}[y_{\alpha,\beta} \mid \alpha \in \mathcal{E}(\beta)]$ and $\mathbb{P}_\beta := \operatorname{Proj} R_\beta$, we have a closed embedding

$$(W_\beta)_\# \longrightarrow W_\beta \times \mathbb{P}_\beta,$$

where the ideal defining the image is

$$\big\langle y_{\alpha,\beta}(x^+_{\alpha^*,\beta} - x^-_{\alpha^*,\beta}) - y_{\alpha^*,\beta}(x^+_{\alpha,\beta} - x^-_{\alpha,\beta}) \bigm| \alpha, \alpha^* \in \mathcal{E}(\beta)\big\rangle.$$

5.7. Now we take products over $\mathcal{H}$. Let $W_\#$ be the product over U of the blowups $\{(W_\beta)_\# \mid \beta \in \mathcal{H}\}$. We then have an embedding

$$W_\# \longrightarrow W \times \prod_{\beta \in \mathcal{H}} \mathbb{P}_\beta,$$

and the blowdown map $b\colon W_\# \to W$ is simply the restriction of the projection to the first factor. Moreover, in terms of coordinates in Lemma 5.5, the image of this embedding is cut out by the multihomogeneous polynomials

$$y_{\alpha,\beta}(x^+_{\alpha^*,\beta} - x^-_{\alpha^*,\beta}) - y_{\alpha^*,\beta}(x^+_{\alpha,\beta} - x^-_{\alpha,\beta})$$

for all $\beta \in \mathcal{H}$ and $\alpha, \alpha^* \in \mathcal{E}(\beta)$.

5.8. We now consider the subvariety $\widetilde{U} \subset W_{\#}$. This is, by definition, the closure of $\widetilde{U}^\circ = b^{-1}(u(U^\circ))$.

Lemma 5.9. *The image of the section $u\colon U \to E|_U$ is contained in the open subvariety W. In terms of the coordinates of Lemma 5.5 the section $u\colon U \to W$ is defined by the $\mathscr{O}_X(U)$-module homomorphism defined by $x^{\pm}_{\alpha,\beta} \mapsto \pm x_\alpha$.*

Proof. Let $\alpha = \{I, J\} \in \mathcal{E}$ with $I < J$. Then the section u^-_α, (respectively, u^+_α) is defined, at each point $p \in U$, to be the linear span of the nonzero vector $\overline{s}_I(p)$ (resp., $\overline{s}_J(p)$). With g_α as in 5.2, it follows from the formulas in 4.6 that $g_\alpha(\overline{s}_I)$ and $g_\alpha(\overline{s}_J)$ are nonzero on U. Hence the image $u_\alpha(U)$ is contained in W_α. Taking suitable products of these sections, we then have $u(U) \subset W$.

Using the various bundle trivializations above, one can show that the section u is given by $x^-_{\alpha,\beta} \mapsto h_\alpha(\overline{s}_I)/g_\alpha(\overline{s}_I)$ and $x^+_{\alpha,\beta} \mapsto h_\alpha(\overline{s}_J)/g_\alpha(\overline{s}_J))$. By Lemma 4.7, we have

$$\overline{s}_I - f_{I,K}\overline{s}_{I\cup J} = \overline{s}_J - f_{J,K}\overline{s}_{I\cup J}.$$

Applying g_α to this equation, and using $g_\alpha(\overline{s}_I + \overline{s}_J) = 1$, we have $g_\alpha(\overline{s}_I) = g_\alpha(\overline{s}_J) = 1/2$. Applying h_α to this equation, and using $h_\alpha(\overline{s}_I + \overline{s}_J) = 0$, we have $h_\alpha(\overline{s}_J) = -h_\alpha(\overline{s}_I) = (f_{J,K} - f_{I,K})/2 = x_\alpha/2$. Thus $u\colon U \to W$ is given by $x^{\pm}_{\alpha,\beta} \mapsto \pm x_\alpha$. $\square$

5.10. Combining the trivialization of Lemma 5.5 with the embedding of Lemma 4.10, we can view W as a subvariety of the affine space $\mathbb{A}_{op} \times \mathbb{A}_{\mathcal{E}} \times \mathbb{A}_{amb}$. It follows from Lemma 5.9 that the section $u\colon U \to W$ is the restriction of the inclusion $\mathbb{A}_{op} \times \mathbb{A}_{\mathcal{E}} \to \mathbb{A}_{op} \times \mathbb{A}_{\mathcal{E}} \times \mathbb{A}_{amb}$ defined by $x^{\pm}_{\alpha,\beta} \mapsto \pm x_\alpha$; thus, $u(U)$ is defined set theoretically by the polynomials of Lemma 4.10 together with the linear polynomials

$$x^+_{\alpha,\beta} - x_\alpha \qquad \text{and} \qquad x^-_{\alpha,\beta} + x_\alpha$$

for all $\beta \in \mathcal{H}$ and $\alpha \in \mathcal{E}(\beta)$.

5.11. By combining the embedding of 5.7 with 5.10, the blowup $W_{\#}$ (and hence $\widetilde{U}$) can be regarded as a subvariety of

$$\mathbb{A}_{op} \times \mathbb{A}_{\mathcal{E}} \times \mathbb{A}_{amb} \times \prod_{\beta\in\mathcal{H}} \mathbb{P}_\beta.$$

It follows from the relations in 5.10 that $\widetilde{U}^\circ$ (and hence its closure $\widetilde{U}$) will be contained in the subvariety defined by $x^{\pm}_{\alpha,\beta} = \pm x_\alpha$ for all $\beta \in \mathcal{H}$ and $\alpha \in \mathcal{E}(\beta)$. Since the projection

$$\mathbb{A}_{op} \times \mathbb{A}_{\mathcal{E}} \times \mathbb{A}_{amb} \times \prod_{\beta\in\mathcal{H}} \mathbb{P}_\beta \longrightarrow \mathbb{A}_{op} \times \mathbb{A}_{\mathcal{E}} \times \prod_{\beta\in\mathcal{H}} \mathbb{P}_\beta$$

is an isomorphism when restricted to this subvariety, the further restriction to $\widetilde{U}$ defines a closed embedding

$$\widetilde{U} \longrightarrow \mathbb{A}_{op} \times \mathbb{A}_{\mathcal{E}} \times \prod_{\beta \in \mathcal{H}} \mathbb{P}_\beta.$$

The image of this embedding is cut out by the polynomials of Lemma 4.10 together with the multihomogeneous polynomials

$$y_{\alpha,\beta} x_{\alpha^*} - y_{\alpha^*,\beta} x_\alpha, \quad \beta \in \mathcal{H}, \quad \alpha, \alpha^* \in \mathcal{E}(\beta).$$

6. The core Z

6.1. In this section we use the embedding of 5.11 to show that $\widetilde{U}$ is isomorphic to a 9-dimensional vector bundle over a certain 3-dimensional multi-projective variety.

Definition 6.2. Let $\widetilde{U} \to \prod_\beta \mathbb{P}_\beta$ be the composition of the embedding of 5.11 with the projection to the projective spaces. The image of this map will be called the *core*, and denoted Z. We let $\eta : \widetilde{U} \to Z$ denote the induced map.

6.3. For each $k = 1, 2, 3$, we consider the projection $Z \to \mathbb{P}_{\Delta_k}$, and let $L_k \to Z$ be the pullback of the tautological line bundle. Since $\mathbb{P}_{\Delta_k} = \operatorname{Proj} R_{\Delta_k}$ (see 5.6), L_k is naturally a subvariety of $(\operatorname{Spec} R_{\Delta_k}) \times Z$. By identifying the ring $R_{\mathcal{E}}$ with $R_{\Delta_1} \otimes R_{\Delta_2} \otimes R_{\Delta_3}$ (via $y_{\alpha,\Delta_k} \mapsto x_\alpha$), we can identify the 3-dimensional bundle $L_1 \times_Z L_2 \times_Z L_3$ with a subvariety of $\mathbb{A}_{\mathcal{E}} \times Z$. Let $N \to Z$ be the 9-dimensional vector bundle obtained by taking the product (over Z) of the trivial bundle $\mathbb{A}_{op} \times Z$ and the bundle $L_1 \times_Z L_2 \times_Z L_3$. There is a natural embedding

$$N \longrightarrow \mathbb{A}_{op} \times \mathbb{A}_{\mathcal{E}} \times \prod_{\beta \in \mathcal{H}} \mathbb{P}_\beta,$$

and it follows from the equations in 5.11 that $\widetilde{U}$ is contained in the image. Thus, we have an embedding

$$\widetilde{U} \longrightarrow N$$

whose composition with the bundle projection to Z coincides with the map η.

6.4. To prove that $\widetilde{U}$ coincides with N, we use the G-action on X. The stabilizer of the flag at infinity is the subgroup $B \subset G$ consisting of lower triangular matrices. The group B acts on the varieties U° and U by the usual action on the Plücker coordinates. In this section, we describe a B-action on the bundle $N \subset \mathbb{A}_{op} \times \mathbb{A}_{\mathcal{E}} \times Z$, with the property that the embedding $\widetilde{U}^\circ \to N$ is B-equivariant.

The action on Z is trivial. The action on $\mathbb{A}_{op}$ is the usual action of the Borel on the corresponding big cell U_{op} in the flag variety. The action on $\mathbb{A}_{\mathcal{E}}$ is given in terms of the characters $t_k : B \to \mathbb{C}^\times$ defined by $t_1(b) = b_{22}/b_{11}$, $t_2(b) = b_{33}/b_{22}$, and $t_3(b) = b_{44}/b_{33}$ where b is the matrix (b_{ij}). For each $\alpha \in \mathcal{E}_k$, the action of b on x_α is then the diagonal action $x_\alpha \mapsto t_k(b) x_\alpha$. It is clear that N is a B-stable subvariety of $\mathbb{A}_{op} \times \mathbb{A}_{\mathcal{E}} \times Z$.

Lemma 6.5. *The embedding $\widetilde{U} \to N$ is a B-equivariant isomorphism.*

Proof. Since the flag coordinates on U are pulled back from the coordinates on the flag variety, the composition $\widetilde{U}^\circ \to N \to \mathbb{A}_{op}$ is equivariant. An explicit calculation using the sections of 4.6 and the definition of the edge coordinates shows that the composition $\widetilde{U}^\circ \to N \to \mathbb{A}_{\mathcal{E}}$ is equivariant. And finally, since for each $\beta \in \mathcal{H}$, the group B acts via the same character on x_α, for all $\alpha \in \mathcal{E}(\beta)$, the induced action on each $\mathbb{P}_\beta$ will be trivial. It follows that $\widetilde{U}^\circ$ embeds equivariantly into N; hence, so does its closure.

To see that the embedding is an isomorphism, we let $Z^\circ = \eta(\widetilde{U}^\circ)$. Since $\widetilde{U}$ is the closure of $\widetilde{U}^\circ$ in $W_\#$, Z° is dense in Z. It follows from the description of the B-action that the unipotent subgroup of B acts freely and transitively on $\mathbb{A}_{op}$, and that the diagonal subgroup of B acts fiberwise on the product of the complements of the zero sections in $L_1 \times_Z L_2 \times_Z L_3$. Thus B acts with dense orbit on each fiber of N. Since each x_α is nonzero on the image of $\widetilde{U}^\circ$ in N, the image of $\widetilde{U}^\circ$ intersects this B-orbit for every fiber of $N|_{Z^\circ} \to Z^\circ$. It follows that the image of $\widetilde{U}^\circ$ is dense in N, so the image of its closure $\widetilde{U}$ coincides with N. □

6.6. By Lemma 6.5, we know that $\widetilde{U}$ is isomorphic to a vector bundle over Z, hence $\widetilde{U}$ will be smooth if and only if Z is smooth. We next show that nonsingularity of the core Z follows from nonsingularity along a certain subvariety, called the locus of *special points*. We begin with some notation.

For any point $\widetilde{p} \in \widetilde{X}$, let p be its image in X. We define the *number of k-planes in* $\widetilde{p}$ by

$$n_k(\widetilde{p}) := \mathrm{Card}\bigl\{\pi_I(p) \bigm| I \subset \{1,2,3,4\},\ |I| = k\bigr\}.$$

The point $\widetilde{p}$ is called *split* (resp. *minimally split*) if $n_k(\widetilde{p}) \geq 2$ (resp. $n_k(\widetilde{p}) = 2$) for all $k \leq 3$. A point $z \in Z$ is called *special* if there exists a minimally split point $\widetilde{p} \in \widetilde{U}$ with $\eta(\widetilde{p}) = z$. We let $Z_{sp} \subset Z$ be the subvariety of special points.

Proposition 6.7. *The set of split points is open in each fiber of $\eta \colon \widetilde{U} \to Z$.*

Proof. For any fiber, we can choose $\widetilde{p} \in \widetilde{U}$ whose B-orbit is open in that fiber. The description of the B-action in 6.4 therefore implies that for any $k \leq 3$, there will be some $\alpha \in \mathcal{E}_k$ such that $x_\alpha \neq 0$ on the image of $\widetilde{p}$ in N. But $x_\alpha \neq 0$ implies $\pi_I(p) \neq \pi_J(p)$, where $\alpha = \{I, J\}$. Therefore $\widetilde{p}$ is split, and the result follows since n_k is constant on B-orbits. □

Proposition 6.8. *If $\widetilde{p} \in \widetilde{U}$ is split, then $\overline{G \cdot \widetilde{p}} \cap \widetilde{U}$ contains a minimally split point.*

To prove the proposition we require some lemmas.

Lemma 6.9. *Let $k \in \{1,2,3\}$ and let $F_1, F_2, F_3 \in \mathrm{Gr}_k$ be three distinct points. Then there exists a one-parameter subgroup $\mu : \mathbb{C}^\times \to G$ such that*

$$\lim_{t \to 0} \mu(t) \cdot F_2 = F_1 \qquad \textit{and} \qquad \lim_{t \to 0} \mu(t) \cdot F_3 \neq F_1.$$

Proof. We can find a subspace $F_4 \subset \mathbb{C}^4$ such that $F_4 \oplus F_1 = F_4 \oplus F_2 = \mathbb{C}^4$, and such that $\dim F_4 \cap F_3 > 0$. Then for μ we can take any one-parameter subgroup that scales in F_1 with a negative weight and scales in F_4 with a positive weight. $\square$

Lemma 6.10. *Let $\widetilde{p}, \widetilde{s} \in \widetilde{U}$ with $\widetilde{p}$ split and $\widetilde{s} \in \overline{G \cdot \widetilde{p}}$. Then there exists a split point $\widetilde{r} \in \overline{G \cdot \widetilde{p}} \cap \widetilde{U}$ such that $\widetilde{s} \in \overline{B \cdot \widetilde{r}}$. Moreover, if $n_k(\widetilde{s}) > 1$, then $n_k(\widetilde{s}) = n_k(\widetilde{r})$.*

Proof. Let W be the set of split points in $G \cdot \widetilde{p}$. This set is open, and thus $\widetilde{s}$ is in its closure. But since the B-orbit of any point in W lies in W, the entire fiber $\eta^{-1}(\eta(\widetilde{s}))$ must also be in the closure of W. Letting $\widetilde{r}$ be a split point in this fiber completes the proof of the first statement. For the second statement, a look at the B-action shows that passing to a point in $\widetilde{U}$ that is in an orbit closure either preserves n_k or drops it down to 1. $\square$

Proof of Proposition 6.8. We use the lemmas above to collapse the configuration associated to $\widetilde{p}$ so that only two subspaces of each dimension remain. We use implicitly that in passing to a point in the closure of a G-orbit, the number of planes in any given dimension cannot increase.

We begin with the subspaces of dimension 1. By assumption, $n_1(\widetilde{p}) > 1$. If $n_1(\widetilde{p}) > 2$, then we can use Lemma 6.9 to find $\widetilde{o} \in \overline{G \cdot \widetilde{p}}$ such that $n_1(\widetilde{p}) > n_1(\widetilde{o}) \geq 2$. The orbit $G \cdot \widetilde{o}$ must lie in $\overline{G \cdot \widetilde{p}}$, and since G acts transitively on our charts that cover $\widetilde{X}$ (see 4.3), we can find a G-translate $\widetilde{s}$ of $\widetilde{o}$ such that $\widetilde{s} \in \overline{G \cdot \widetilde{p}} \cap \widetilde{U}$. Since the functions n_k are constant on G-orbits, we have $n_1(\widetilde{s}) = n_1(\widetilde{o})$.

Lemma 6.10 implies that we can find a split point $\widetilde{r} \in \overline{G \cdot \widetilde{p}}$ with $\widetilde{s} \in \overline{B \cdot \widetilde{r}} \cap \widetilde{U}$, and such that $n_1(\widetilde{r}) = n_1(\widetilde{s})$ is ≥ 2 and is $< n_1(\widetilde{p})$. Since any point in $\overline{G \cdot \widetilde{r}}$ is also in $\overline{G \cdot \widetilde{p}}$, we can repeat this procedure until we find a split point $\widetilde{p}_1 \in \overline{G \cdot \widetilde{p}} \cap \widetilde{U}$ with $n_1(\widetilde{p}_1) = 2$.

Now we use induction on k to produce points $\widetilde{p}_2$ and $\widetilde{p}_3$. The key point is that we can apply the lemmas to reduce n_k while preserving $n_l > 1$ for $l \neq k$. Since the final point $\widetilde{p}_3 \in \overline{G \cdot \widetilde{p}} \cap \widetilde{U}$ is minimally split, this completes the proof. $\square$

6.11. Propositions 6.7 and 6.8 imply that nonsingularity of $\widetilde{U}$ follows from nonsingularity at minimally split points. Since $\widetilde{U}$ is a vector bundle over Z, nonsingularity at the minimally split points follows from the nonsingularity of Z_{sp}.

7. Nonsingularity

7.1. Recall that the core Z is a subvariety of

$$\prod_{\beta \in \mathcal{H}} \mathbb{P}_\beta \cong \mathbb{P}^5 \times \mathbb{P}^{11} \times \mathbb{P}^5 \times (\mathbb{P}^2)^4 \times (\mathbb{P}^2)^8 \times (\mathbb{P}^2)^4,$$

where the index set $\mathcal{H}$ corresponds to faces of the hypersimplices of dimension ≥ 2. Each factor $\mathbb{P}_\beta$ has homogeneous coordinates $\{y_{\alpha,\beta} \mid \alpha \in \mathcal{E}(\beta)\}$, corresponding to the edges of the face β. By combining the polynomials of 5.11 with the polynomials defining U in 4.10, we obtain polynomials defining Z.

Lemma 7.2. *The subvariety Z of $\prod_\beta \mathbb{P}_\beta$ is defined set-theoretically by the following multihomogeneous polynomials:*

1. *The linear polynomials*
$$y_{\alpha_1,\beta} - y_{\alpha_2,\beta} + y_{\alpha_3,\beta},$$
where $\alpha_1, \alpha_2, \alpha_3$ are as in Lemma 4.10(item 1) and $\beta \in \mathcal{H}$ is such that $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{E}(\beta)$.
2. *The quadric polynomials*
$$y_{\alpha_1,\beta}y_{\alpha_2,\beta^*} - y_{\alpha_2,\beta}y_{\alpha_1,\beta^*},$$
where α_1 and α_2 are any two edges that share a vertex and $\beta, \beta^* \in \mathcal{H}$ are such that $\alpha_1, \alpha_2 \in \mathcal{E}(\beta)$ and $\alpha_1, \alpha_2 \in \mathcal{E}(\beta^*)$.
3. *The quadric polynomials*
$$y_{\alpha_1,\beta}y_{\alpha_2^*,\beta^*} - y_{\alpha_2,\beta}y_{\alpha_1^*,\beta^*},$$
where $\alpha_1, \alpha_2, \alpha_1^*, \alpha_2^*$ are as in Lemma 4.10(item 2) and $\beta, \beta^* \in \mathcal{H}$ are such that $\alpha_1, \alpha_2 \in \mathcal{E}(\beta)$ and $\alpha_1^*, \alpha_2^* \in \mathcal{E}(\beta^*)$.
4. *The cubic polynomials*
$$y_{\alpha_1,\beta}y_{\alpha_2,\beta}y_{\alpha_3,\beta} - y_{\alpha_1^*,\beta}y_{\alpha_2^*,\beta}y_{\alpha_3^*,\beta}$$
where $\alpha_1, \alpha_2, \alpha_3, \alpha_1^*, \alpha_2^*, \alpha_3^*$ are as in Lemma 4.10(item 3) and β is the hypersimplex Δ_2.
5. *The quartic polynomials*
$$y_{\alpha_1,\beta}y_{\alpha_3,\beta}y_{\alpha_2^*,\beta^*}y_{\alpha_4^*,\beta^*} - y_{\alpha_2,\beta}y_{\alpha_4,\beta}y_{\alpha_1^*,\beta^*}y_{\alpha_3^*,\beta^*}$$
where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_1^*, \alpha_2^*, \alpha_3^*, \alpha_4^*$ are as in Lemma 4.10(item 4) and β, β^* are the hypersimplices Δ_1, Δ_3, respectively.

7.3. We represent points in $\prod_\beta \mathbb{P}_\beta$ combinatorially using the graph Γ in Figure 6. The edges of Γ are in bijection with the variables $y_{\alpha,\beta}$, and we encode a point in $\prod_\beta \mathbb{P}_\beta$ by assigning values to the edges modulo $(\mathbb{C}^\times)^{19}$ (since Γ has 19 connected components). It will be convenient to abuse language slightly by identifying points in $\prod_\beta \mathbb{P}_\beta$ with Γ. In doing so we shall always assume that values have been assigned to the variables $y_{\alpha,\beta}$.

Let T and T^* be two triangular subgraphs of Γ. Then T (respectively, T^*) corresponds to a choice of an ordered triangle and a choice of a face in $\mathcal{H}$ to designate the component of Γ that contains T (resp., T^*). We shall say that T and T^* are *related* if one of the following two conditions holds:

1. The two ordered triangles for T and T^* coincide, or
2. The faces in Figure 2 that correspond to the two ordered triangles for T and T^* are in adjacent hypersimplices and one is a 180° rotated copy of the other.

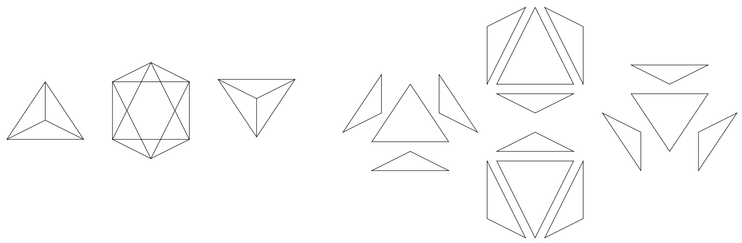

FIGURE 6. The graph Γ

In either case, there is a natural correspondence between the three edges of T and the three edges of T^*, and we say that T and T^* have the *same shape* if the corresponding triples of values are proportional. Our calculations will involve the use of this notion together with setting various $y_{\alpha,\beta}$ to 0; we indicate the latter by marking in bold the corresponding edge of Γ.

As a first step towards showing that special points are nonsingular, we consider the equations in Lemma 7.2 and their meaning in the context of Figure 6.

Lemma 7.4. *Suppose that Γ represents a point in Z. Then any two related triangular subgraphs T and T^* have the same shape and they must appear as one of the five possibilities shown in Figure 7.*

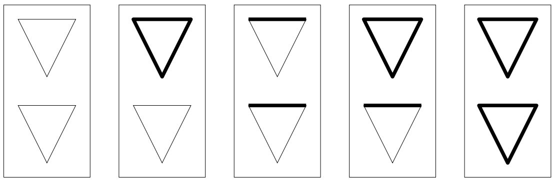

FIGURE 7. Possible related triangles.

Proof. The quadric relations in Lemma 7.2 imply that related triangles will have the same shape. Using this fact and the linear relations of Lemma 7.2 it is then easy to verify that the only combinations of zero values for such a pair are those shown in Figure 7. □

Proposition 7.5. *The subvariety Z_{sp} consists of* 66 *isolated points and* 4 *subvarieties of positive dimension. Modulo the action of the symmetric group and the duality exchanging lines and* 3*-spaces, there are five types of points of Z_{sp}. In terms of Γ, these types appear in Figures 8–12. Figure 12 represents a point in the positive-dimensional locus.* [2]

[2]The labels of the figures refer to certain divisors in Z, cf. [2]. The numbers in parentheses indicate how many of each type of component appear, without modding out by the action of S_4. The notation $(a+a)$ indicates that there are $2a$ components of this type; we have only depicted one of each dual pair.

Proof. Let $\widetilde{p} \in \widetilde{U}$ be minimally split, and let p be is its image in U. Then $n_k(\widetilde{p}) = 2$ for $k \leq 3$, which implies $\mathrm{Card}\{\pi_I(\widetilde{p})\} = 6$. Up to symmetry a minimally split point must have its subspaces partitioned as follows:

1. The lines must collapse together as $(3,1)$ or $(2,2)$. (The notation (p,q) means that the two distinct lines are the image of p and q lines, where $p+q=4$.)
2. The 2-planes must collapse together as $(5,1)$, $(4,2)$, or $(3,3)$.
3. The 3-planes, like the lines, must collapse together as $(3,1)$ or $(2,2)$.

With these facts and Figures 6 and 7 in hand, computing Z_{sp} becomes a combinatorial exercise. We leave the pleasure of this computation to the reader. □

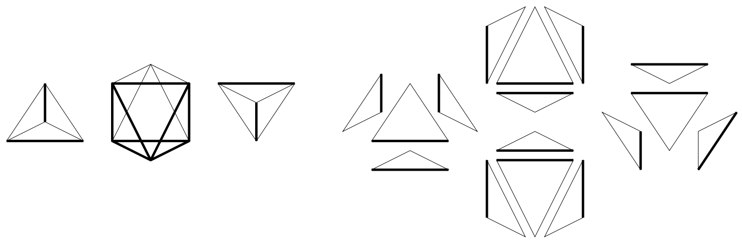

FIGURE 8. DDE (6).

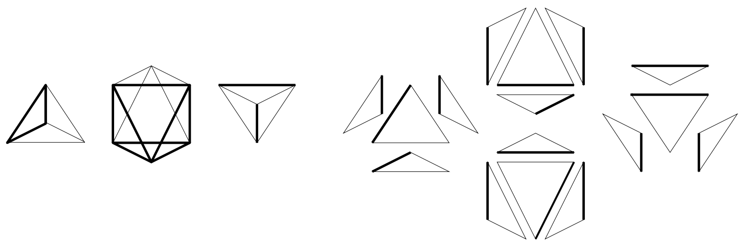

FIGURE 9. CDE $(12+12)$.

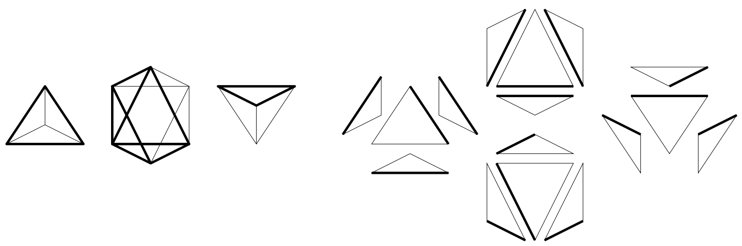

FIGURE 10. CC^*E (24).

Theorem 7.6. *The core Z (and thus $\widetilde{X}$) is nonsingular.*

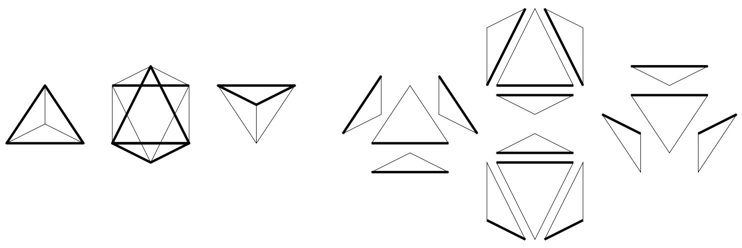

FIGURE 11. $CC^*_{nop}D$ (12).

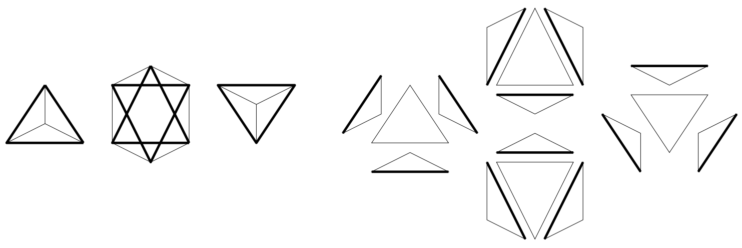

FIGURE 12. $CC^*_{op}D$ (4).

Proof. We apply the Jacobian condition for nonsingularity in an affine neighborhood of each point of Z_{sp}. Let $z = \eta(\widetilde{p}) \in Z_{sp}$. Fix an affine neighborhood of z in $\prod_{\beta \in \mathcal{H}} \mathbb{P}_\beta$ as follows. For each $\beta \in \mathcal{H}$, choose one $\alpha(\beta) \in \mathcal{E}(\beta)$ with $y_{\alpha(\beta),\beta} \neq 0$, and set this coordinate equal to 1. The remaining variables $\{y_{\alpha,\beta} \mid \alpha \neq \alpha(\beta)\}$ form a system of local parameters at z in $\prod_\beta \mathbb{P}_\beta$.

Let Ω^1_z be the $\mathbb{C}$-vector space of differentials of Z at the point z. Since Z is a threefold, we have $\dim \Omega^1_z \geq 3$, with equality only if Z is nonsingular at z. Furthermore,

$$\Omega^1_z = \Big(\bigoplus_{\substack{(\alpha,\beta) \\ \alpha \neq \alpha(\beta)}} \mathbb{C} \cdot dy_{\alpha,\beta} \Big) / J,$$

where J is the subspace generated by the differentials (evaluated at z) of all functions vanishing on Z. To study this quotient, we will use the following combinatorial rules for computing with differentials in Ω^1_z. These follow immediately from the equations in Lemma 7.2; we omit the proof.

Lemma 7.7. *Suppose that* $y_{\alpha_1,\beta_1} y_{\alpha_2,\beta_2} - y_{\alpha_2,\beta_1} y_{\alpha_1,\beta_2}$ *vanishes on* Z.

1. *If* $y_{\alpha_1,\beta_1} = y_{\alpha_2,\beta_1} = y_{\alpha_1,\beta_2} = 0$ *and* $y_{\alpha_2,\beta_2} \neq 0$, *then* $dy_{\alpha_1,\beta_1} = 0$.
2. *If* $y_{\alpha_1,\beta_1} = y_{\alpha_1,\beta_2} = 0$ *and* $y_{\alpha_2,\beta_2} = y_{\alpha_2,\beta_1} \neq 0$, *then* $dy_{\alpha_1,\beta_1} = dy_{\alpha_1,\beta_2}$.

Using these rules, we add data for Ω^1_z to Γ as in Figure 13. The 0 means that the differential of the variable corresponding to the edge is 0, and the two da's indicate that the two differentials coincide in Ω^1_z.

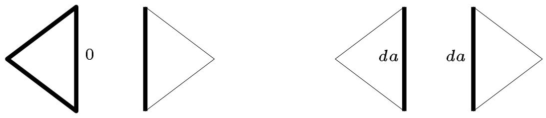

FIGURE 13. Rules for differentials.

First we verify nonsingularity at the isolated points of Z_{sp}. Since the computations for the various points are all very similar, we explain the case DDE in detail and leave the others to the reader. We fix an affine neighborhood of a point of type DDE by assigning the value 1 to exactly one thin edge in each of the 19 components in Figure 8. Since the differentials of the linear polynomials are in J, the differential corresponding to any thin edge is a linear combination of differentials corresponding to bold edges. Thus, Ω^1_z is generated by $dy_{\alpha,\beta}$, where α is a bold edge of the component β.

Consider the hypersimplex connected components of Figure 8. Applying Lemma 7.7, we find three independent differentials da, db, and dc in these components; the other differentials in these components are 0. Now consider the other connected components of Figure 8. Using Lemma 7.7 we see that the remaining differentials are either 0 or are equal to da, db, or dc. The result is summarized in Figure 14. Hence Ω^1_z is 3-dimensional, and all the points of type DDE are nonsingular points of Z.

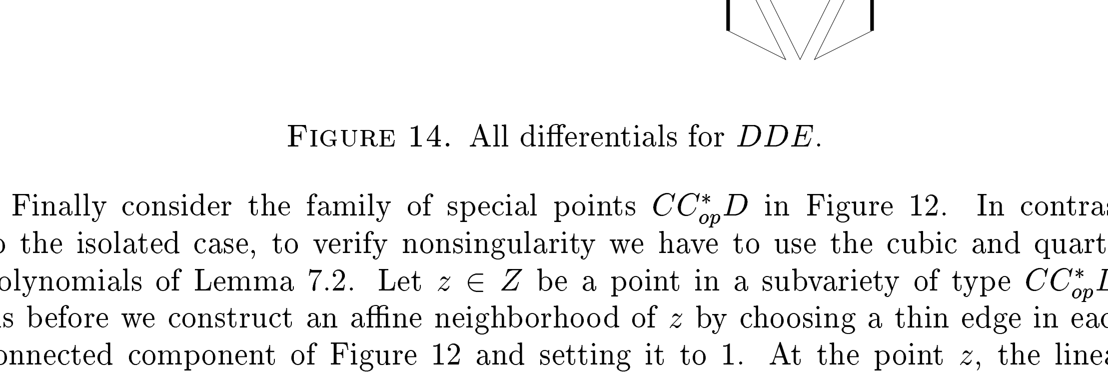

FIGURE 14. All differentials for DDE.

Finally consider the family of special points $CC^*_{op}D$ in Figure 12. In contrast to the isolated case, to verify nonsingularity we have to use the cubic and quartic polynomials of Lemma 7.2. Let $z \in Z$ be a point in a subvariety of type $CC^*_{op}D$. As before we construct an affine neighborhood of z by choosing a thin edge in each connected component of Figure 12 and setting it to 1. At the point z, the linear relations imply that all of the thin edges *except those in the four thin triangles* will also have value one. To complete the graph Γ to represent the point z, we apply the quadratic relations to find $u, v \in \mathbb{C}$ such that the values are as in Figure 15 (up to the choice of the edges with values 1). Hence, a priori, this is a 2-dimensional component of Z_{sp}.

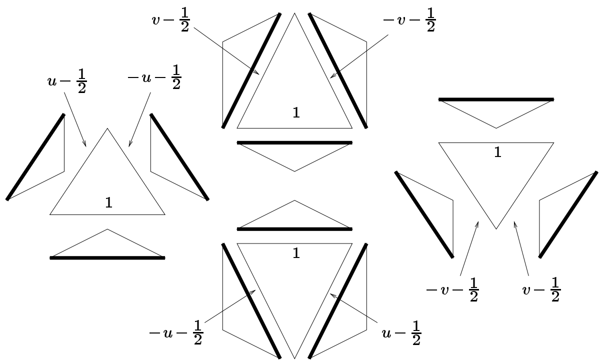

FIGURE 15. A typical point in $CC^*_{op}D$.

The quartic relations from Lemma 7.2, however, imply that the two parameters u and v satisfy a linear relation. For the choice of parameters in Figure 15, for example, this relation is

$$1 \cdot (u - \frac{1}{2}) \cdot 1 \cdot 1 = 1 \cdot (v - \frac{1}{2}) \cdot 1 \cdot 1, \quad \text{or} \quad u = v.$$

Thus this component of Z_{sp} is in fact a curve.

We now complete the proof of the theorem. As in the isolated case, the differentials on all thin edges, except for those in the four thin triangles, can be expressed as linear combinations of the differentials on bold edges. Moreover the differentials on edges of the thin triangles can be expressed as linear combinations of du and dv. Thus, using Lemma 7.7, we can find 8 differentials that span Ω^1_z (Figure 16):

$$da, db, dc, da^*, db^*, dc^*, du, dv.$$

Note that the span of these is at most 5-dimensional, because of the relations $da + db + dc = da^* + db^* + dc^* = 0$ induced by the differentials of the linear relations, and the relation $du = dv$ induced by the linear relation between u and v.

To finish, we claim that the spans of da, db, dc and da^*, db^*, dc^* are each 1-dimensional. Indeed, a quadric relation implies that the front face of the octahedron in Figure 16 has the same shape as the corresponding face in Figure 15, which implies

$$db - (u - \frac{1}{2})\, dc = 0.$$

This relation, and a similar one involving db^* and dc^*, shows that $\dim \Omega^1_z = 3$. This completes the proof of the main theorem. $\square$

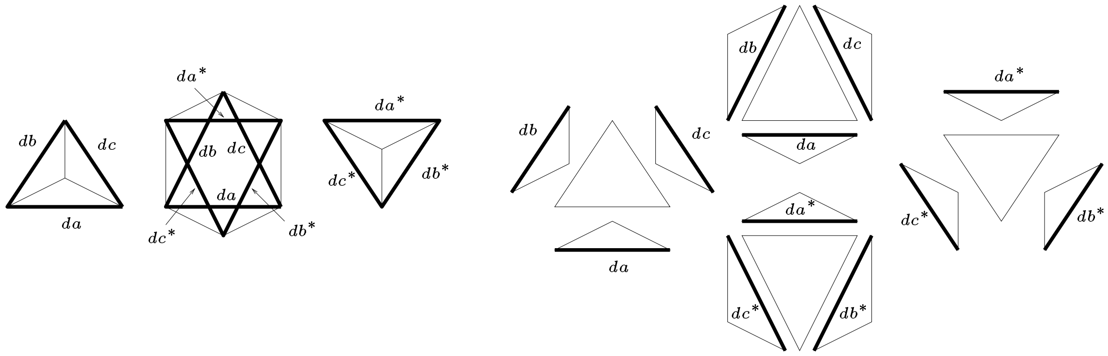

FIGURE 16. Differentials for $CC^*_{op}D$.

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