Modular symbols and Hecke operators

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Abstract. We survey techniques to compute the action of the Hecke operators on the cohomology of arithmetic groups. These techniques can be seen as generalizations in different directions of the classical modular symbol algorithm, due to Manin and Ash-Rudolph. Most of the work is contained in papers of the author and the author with Mark McConnell. Some results are unpublished work of Mark McConnell and Robert MacPherson.

1 Introduction

1.1

Let G be a semisimple algebraic group defined over $\mathbb{Q}$, and let $\Gamma \subset G(\mathbb{Q})$ be an arithmetic subgroup. The cohomology of Γ plays an important role in number theory, through its connection with automorphic forms and representations of the absolute Galois group $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$. This relationship is revealed in part through the action of the *Hecke operators* on the complex cohomology $H^*(\Gamma;\mathbb{C})$. These are endomorphisms induced from a family of correspondences associated to the pair $(\Gamma, G(\mathbb{Q}))$; the arithmetic nature of the cohomology is contained in the eigenvalues of these linear maps.

For $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$, the *modular symbols* and *modular symbol algorithm* of Manin [17] and Ash-Rudolph [8] provide a concrete method to compute the Hecke eigenvalues in $H^\nu(\Gamma;\mathbb{C})$, where $\nu = n(n+1)/2 - 1$ is the top degree (§2). These symbols have allowed many researchers to fruitfully explore the number-theoretic significance of this cohomology group, especially for $n = 2$ and 3 [3, 7, 5, 21, 22]. For all their power, though, modular symbols have limitations:

- The group G must be the linear group SL_n.
- The cohomology must be in the top degree ν.
- The group Γ must be a subgroup of $\mathrm{SL}_n(\mathbb{Z})$, or more generally $\mathrm{SL}_n(R)$, where R is a euclidean ring of integers of a number field.

1.2

In this article we discuss new techniques to compute the Hecke action on the cohomology of arithmetic groups that can be seen as generalizing the modular symbol algorithm by relaxing the three restrictions above. First in §3 we relax the first restriction by replacing the linear group SL_n with the symplectic group

Sp_{2n} [14]. Next in §4, we relax the second restriction and consider computations in $H^{\nu-1}(\Gamma)$, where $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ and $n \leq 4$ [13]. Finally, in the last two sections we relax all three restrictions, and consider arithmetic groups associated to *self-adjoint homogeneous cones* (§5) [12, 15], and arithmetic groups for which a *well-rounded retract* is defined (§6) [16]. The first class includes $\mathrm{SL}_n(\mathcal{O}_K)$, where $\mathcal{O}_K$ is the maximal order of a totally real or CM field, as well as arithmetic groups associated to the positive-definite 3×3 Hermitian octavic matrices. The second class includes arithmetic subgroups of $\mathrm{SL}_n(D)$, where D is a division algebra over $\mathbb{Q}$.

Most of this work is contained in papers of the author [14, 12, 13] or the author in joint work with Mark McConnell [15]. The last section is a summary of unpublished results of Robert MacPherson and Mark McConnell [16]. We have omitted other work, notably that of Bygott [10], Teitelbaum [20], and Merel [18], because of lack of space and/or author's expertise. It is a pleasure to thank Avner Ash, Robert MacPherson, and Mark McConnell for many conversations about these topics.

2 Classical modular symbols

2.1

We begin by recalling the classical modular symbol algorithm following Ash-Rudolph [8]. For simplicity we consider subgroups of $\mathrm{SL}_n(\mathbb{Z})$, although everything we say can be generalized to subgroups of $\mathrm{SL}_n(R)$, where R is a euclidean maximal order in a number field (cf. [11]).

Let $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ be a torsion-free finite-index subgroup, and let $m \in M_n(\mathbb{Q})$, the $n \times n$ matrices over $\mathbb{Q}$. We want to show how to use m to construct a class in $H^\nu(\Gamma)$. To this end, let X be the symmetric space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$, let $\bar{X}$ be the bordification constructed by Borel-Serre [9], and let $\partial\bar{X} = \bar{X} \smallsetminus X$. Let $M = \Gamma\backslash X$, $\bar{M} = \Gamma\backslash\bar{X}$, and $\partial\bar{M} = \bar{M} \smallsetminus M$. Then $\bar{M}$ is a smooth manifold with corners, and $H^*(\Gamma) \cong H^*(\bar{M})$. We have an exact sequence

$$H_{n-1}(\partial\bar{X}) \to H_n(\bar{X}, \partial\bar{X}) \to H_n(\bar{M}, \partial\bar{M}) \to H^\nu(\bar{M}) \tag{1}$$

coming from the sequence of the pair $(\partial\bar{X}, \bar{X})$, the canonical projection $\bar{X} \to \bar{M}$, and Lefschetz duality. Moreover, the boundary $\partial\bar{X}$ has the homotopy type of the *Tits building* $\mathcal{B} = \mathcal{B}_{\mathrm{SL}}$ associated to $\mathrm{SL}_n(\mathbb{Q})$. This is an $(n-1)$-dimensional simplicial complex whose k-simplices Δ are in bijection with flags F of rational subspaces

$$F = \{0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_{k+1} \subsetneq \mathbb{Q}^n\};$$

we have $\Delta \subset \Delta'$ if and only if $F \subset F'$.

Any ordered tuple of nonzero rational vectors determines a maximal rational flag by defining F_k to be the span of the first k vectors. Hence if $m \in M_n(\mathbb{Q})$ has nonzero columns, the different orderings of the columns determine $n!$ different oriented $(n-1)$-simplices in $\mathcal{B}$. These simplices can be thought of as an oriented simplicial cycle giving a class $[m] \in H_{n-1}(\mathcal{B}) \cong H_{n-1}(\partial \bar{X})$. The class $[m]$ is called a *modular symbol*, and these classes span $H_{n-1}(\mathcal{B})$. According to Ash-Rudolph, the map $\Phi\colon H_{n-1}(\mathcal{B}) \to H^{\nu}(\Gamma)$ induced by (1) is surjective; hence the (duals of) the modular symbols span $H^{\nu}(\Gamma)$.

2.2

Write $[m] = [m_1, \ldots, m_n]$, where each column $m_i \in \mathbb{Q}^n \smallsetminus \{0\}$, and let $\mathcal{M}_n$ be the $\mathbb{Z}$-module generated by the classes of the symbols $[m]$. Using the description in §2.1, one can show that elements of $\mathcal{M}_n$ satisfy the following relations:

1. $[qm_1, m_2, \ldots, m_n] = [m]$, for $q \in \mathbb{Q}^{\times}$.
2. $[m_{\sigma(1)}, \ldots, m_{\sigma(n)}] = \operatorname{sgn}(\sigma)[m]$, for any permutation σ.
3. $[m] = 0$ if $\det m = 0$.
4. $\sum_{i=0}^{n} (-1)^i [m_0, \ldots, \hat{m}_i, \ldots, m_n] = 0$, for any $n+1$ vectors $m_0, \ldots, m_n$ (the "cocycle relation").

By the first relation, $\mathcal{M}_n$ is generated by those $[m]$ such that m_i is integral and primitive for all i. If $m \in \mathrm{SL}_n(\mathbb{Z})$, then $[m]$ is called a *unimodular symbol.* We have the following fundamental result of Manin ($n = 2$) and Ash-Rudolph ($n \geq 2$):

Theorem 1. *[17, 8] Any modular symbol is homologous to a finite sum of unimodular symbols.*

We sketch the proof. If $|\det m| > 1$, then one can show there exists $v \in \mathbb{Z}^n \smallsetminus \{0\}$ such that

$$0 \leq |\det m_i(v)| < |\det m|, \quad \text{for } i = 1, \ldots, n. \tag{2}$$

where $m_i(v)$ is the matrix obtained by replacing the column m_i with v. Such a v is called a *reducing point* for m. Then applying the cocycle relation to the tuple $v, m_1, \ldots, m_n$ yields an expression for $[m]$ in terms of the symbols $[m_i(v)]$. By induction this completes the proof.

This process of rewriting a modular symbol as a sum of unimodular symbols is called the *modular symbol algorithm.* Using this algorithm one can compute the action of the Hecke operators on $H^{\nu}(\Gamma)$ as follows. There are only finitely many unimodular symbols mod Γ, and from them one can select a subset dual to a basis of $H^{\nu}(\Gamma)$. A Hecke operator acts on the modular symbols by taking a unimodular symbol into a sum of nonunimodular symbols. Hence the modular symbol algorithm allows one to compute the Hecke action on a basis, from which one can easily compute the eigenvalues.

3 Symplectic modular symbols

3.1

For the first generalization we replace the linear group with the symplectic group [14]. Let V be a $2n$-dimensional $\mathbb{Q}$-vector space with basis $\{e_1, \ldots, e_n, e_{\bar{n}}, \ldots, e_{\bar{1}}\}$, where $\bar{\imath} := 2n + 1 - i$. Let $\langle\ ,\ \rangle: V \times V \to \mathbb{Q}$ be the nondegenerate, alternating bilinear form defined by

$$\langle e_i, e_j \rangle = \begin{cases} 1 & \text{if } j = \bar{\imath} \text{ with } i < j \\ -1 & \text{if } j = \bar{\imath} \text{ with } i > j \\ 0 & \text{otherwise.} \end{cases}$$

The form $\langle\ ,\ \rangle$ is called a *symplectic form*, and the *symplectic group* $\mathrm{Sp}_{2n}(\mathbb{Q})$ is defined to be the subgroup of $\mathrm{SL}_{2n}(\mathbb{Q})$ preserving $\langle\ ,\ \rangle$.

3.2

Much of §2 carries over without change, but there are some new wrinkles coming from the geometry of the symplectic form. Recall that an *isotropic subspace* is one on which the symplectic form vanishes, and that maximal (necessarily n-dimensional) isotropic subspaces are called *Lagrangian*. Then the symplectic building $\mathcal{B}_{\mathrm{Sp}}$ has a k-simplex for every length $(k+1)$ flag of isotropic subspaces. Since the columns of a symplectic matrix m satisfy

$$\langle m_i, m_j \rangle = 0 \quad \text{if and only if} \quad i \neq \bar{\jmath}, \tag{3}$$

it is easy to see that m determines $2^n \cdot n!$ oriented simplices of maximal dimension in $\mathcal{B}_{\mathrm{Sp}}$.

Furthermore, the arrangement of these simplices in $\mathcal{B}_{\mathrm{Sp}}$ differs from the linear case. Suppose we use the columns of m to induce points in the projective space $\mathbb{P}^{2n-1}(\mathbb{Q})$. Then the Lagrangian subspaces spanned by the columns of m become $(n-1)$-dimensional flats arranged in the configuration of a *hyperoctahedron*.[1] This time m determines a class $[m] \in H_{n-1}(\mathcal{B}_{\mathrm{Sp}})$, and as m ranges over all rational matrices with columns satisfying (3), the duals of the classes $[m]$ span $H^{\nu}(\Gamma)$.

3.3

As a first step towards a symplectic modular symbol algorithm, one must understand the analogues of the relations from §2.2. The analogues of 1–3 are only slightly different to reflect the hyperoctahedral symmetry. The cocycle relation, however, is more interesting. A symbol $[m]$ and a generic nonzero rational point $v \in V$ determine $2n$ modular symbols $[m_i(v)]$ as follows. For any pair (i, j) with $i \neq \bar{\jmath}$, we define points m_{ij} by

$$m_{ij} := \langle v, m_j \rangle m_i - \langle v, m_i \rangle m_j.$$

Let $[m_i(v)]$ be the modular symbol obtained by replacing $m_{\bar{\imath}}$ with v, and replacing the m_j with $j \notin \{i, \bar{\imath}\}$ by m_{ij}. Then one can show $[m] = \sum \varepsilon_i [m_i(v)]$ for appropriate signs ε_i.

For an example of this relation, consider Figure 1. The figure on the left shows the cocycle relation for Sp_4 in terms of a configuration in $\mathbb{P}^3$. The black dots are the points corresponding to the m_i, the grey dot correspond to v, and the triangles to the points m_{ij}.

[1] Recall that a hyperoctahedron is the convex hull of the $2n$ points $\{\pm e \mid e \in E\}$, where E is the standard basis of $\mathbb{R}^{2n}$.

3.4

Now we can describe the symplectic modular symbol algorithm. Let $m \in M_{2n}(\mathbb{Z})$ have columns satisfying (3). Then $\det m = \prod_{i=1}^{n} \langle m_i, m_{\bar{\imath}} \rangle$, and one can show that if $|\det m| > 1$, there exists a vector $v \in \mathbb{Z}^n \smallsetminus \{0\}$ such that

$$0 \leq |\langle m_i, v \rangle| < \langle m_i, m_{\bar{\imath}} \rangle, \quad \text{for } i = 1, \ldots, 2n.$$

We can apply v to $[m]$ in the cocycle relation alluded to in §3.3, but we will unfortunately find that $|\det m_i(v)| > |\det m|$ in general. However, all is not lost. It turns out that for fixed i and fixed v, the $2n-2$ vectors $\{m_{ij} \mid j \neq i, \bar{\imath}\}$ form a tuple that can be regarded as a symplectic modular symbol associated to Sp_{2n-2}. By induction one knows how to make these symbols unimodular, and this allows one to further reduce the $[m_i(v)]$ (cf. the right of Figure 1).

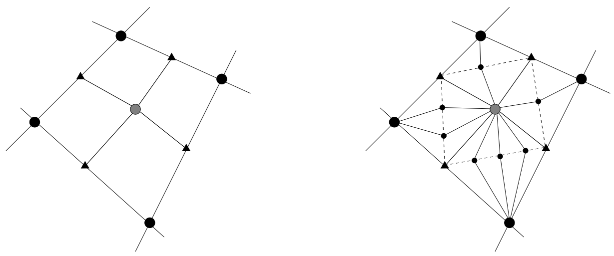

Fig. 1. $G = \mathrm{Sp}_4$. On the left, the outer square is the original symbol $[m]$, and the four smaller squares are the symbols $[m_i(v)]$. On the right, each modular symbol has been further reduced by applying the modular symbol algorithm to $\mathrm{Sp}_2 = \mathrm{SL}_2$ modular symbols.

4 Below the cohomological dimension

4.1

We return to the case of SL_n. As said before, a limitation of the modular symbol algorithm is that one can compute the Hecke action only on the top degree cohomology. For $n \leq 3$ this cohomology group is very interesting: it contains *cuspidal* classes, i.e. classes associated to cuspidal automorphic forms. If $n \geq 4$, however, the top degree cohomology group no longer contains cuspidal classes. In particular, if $n = 4$, one is really interested in computing the Hecke action on $H^5(\Gamma)$. For instance, recent work Jasper Scholten has constructed 4-dimensional representations of $\mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$ that should be related to a cuspidal Hecke eigenform in $H^5(\Gamma)$ for some $\Gamma \subset \mathrm{SL}_4(\mathbb{Z})$ [19]. The modular symbol algorithm, however, applies only to $H^6(\Gamma)$.

In this section we describe an algorithm that for $n \leq 4$ allows computation of the Hecke action on $H^{\nu-1}(\Gamma)$ [13]. However, there is one caveat: we cannot prove the algorithm will terminate. In practice, happily, the algorithm has always converged, and has permitted investigation of this cohomology [4].

4.2

To compute with lower degree cohomology groups, we use the *sharbly complex* S_* [2]. For $k \geq 0$, let S_k be the $\mathbb{Z}\Gamma$-module generated by the symbols $\mathbf{u} = [v_1, \ldots, v_{n+k}]$, where $v_i \in \mathbb{Q} \smallsetminus \{0\}$, modulo the analogues of relations 1–3 in §2.2. Elements of S_k are called *k-sharblies*. Let $\partial\colon S_k \to S_{k-1}$ be the map $\mathbf{u} \mapsto \sum_i (-1)^i [v_1, \ldots, \widehat{v_i}, \ldots, v_{n+k}]$, linearly extended to all of S_k. There is a map $S_0 \to \mathcal{M}_n$ giving a $\mathbb{Z}\Gamma$-free resolution of $\mathcal{M}_n$, and one can show that this implies $H^{\nu-k}(\Gamma; \mathbb{C}) \cong H_k(S_* \otimes \mathbb{C})$.

As in §2.2, it suffices to consider k-sharblies $\mathbf{u} = [v_1, \ldots, v_{n+k}]$ with all v_i integral and primitive. Any modular symbol of the form $[v_{i_1}, \ldots, v_{i_n}]$, where $\{i_1, \ldots, i_n\} \subset \{1, \ldots, n+k\}$, is called a *submodular symbol* of $\mathbf{u}$.

Let $\xi = \sum n(\mathbf{u})\mathbf{u}$ be a sharbly chain. We denote by $\|\xi\|$ the maximum absolute value of the determinant of any submodular symbol of ξ. The chain ξ is called *reduced* if $\|\xi\| = 1$. It is known that reduced 1-sharbly cycles provide a finite spanning set of $H^{\nu-1}(\Gamma; \mathbb{C})$ for $n \leq 4$.

Since the Hecke operators take reduced sharbly cycles to nonreduced cycles, our goal is to apply the modular symbol algorithm *simultaneously* over a nonreduced 1-sharbly cycle ξ to lower the determinants of the submodular symbols. Hence we are faced with two problems: first, how do we combine reducing points with the original 1-sharbly ξ to produce a new 1-sharbly ξ' homologous to ξ; second, how do we choose the reducing points so that $\|\xi'\| < \|\xi\|$?

4.3

To address the first issue we do the following. Suppose $\mathbf{u} = [v_1, \ldots, v_{n+1}]$ satisfies $n(\mathbf{u}) \neq 0$, and for $i = 1, \ldots, n+1$, let $\mathbf{v}_i$ be the submodular symbol $[v_1, \ldots, \widehat{v_i}, \ldots, v_{n+1}]$. Assume that all these submodular symbols are nonunimodular, and for each i let w_i be a reducing point for $\mathbf{v}_i$.

For any subset $I \subset \{1, \ldots, n+1\}$, let $\mathbf{u}_I$ be the 1-sharbly $[u_1, \ldots, u_{n+1}]$, where $u_i = w_i$ if $i \in I$, and $u_i = v_i$ otherwise. Then we have a relation in S_1 given by

$$\mathbf{u} = -\sum_{I \neq \varnothing} (-1)^{\# I} \mathbf{u}_I. \tag{4}$$

Geometrically this relation can be expressed using the combinatorics of the hyperoctahedron [13, §4.4]. More generally, if some $\mathbf{v}_i$ happen to be unimodular, then one can construct a similar relation using an iterated cone on a hyperoctahedron.

4.4

Now we apply the construction in §4.3 to all the 1-sharblies $\mathbf{u}$ with $n(\mathbf{u}) \neq 0$, and we choose reducing points Γ-equivariantly. Specifically, if $\mathbf{v}$ and $\mathbf{v}'$ are two submodular symbols of ξ with $\gamma\mathbf{v} = \mathbf{v}'$, then we choose the corresponding reducing points such that $\gamma w = w'$. After applying (4) to all the $\mathbf{u}$ we determine a new 1-sharbly cycle ξ'. Clearly ξ' is homologous to ξ. We claim that $\|\xi'\|$ should be less than $\|\xi\|$.

To see why this should be true, consider the 1-sharblies $\mathbf{u}_I$ on the right of (4). Of these 1-sharblies, those with $\#I = 1$ contain the $\mathbf{v}_i$ among their submodular symbols. We claim that since ξ is a *cycle* mod Γ, and since the reducing points were chosen Γ-equivariantly over ξ, these 1-sharblies will not appear in ξ'. Hence by construction we have eliminated some of the "bad" submodular symbols from ξ.

4.5

Unfortunately, this doesn't guarantee that $\|\xi'\| < \|\xi\|$. The problem is that we have no way of knowing that the submodular symbols of the $\mathbf{u}_I$ with $\#I > 1$ don't have large determinants. Indeed, this brings us back to the second question raised in §4.2, since if the reducing points are chosen naïvely, these submodular symbols *will* have large determinants. However, we claim that one can (conjecturally) choose the reducing points "uniformly" over ξ in a sense by using LLL-reduction, and that this problem doesn't occur in practice. In fact, in thousands of computer tests and in applications, we have always found $\|\xi'\| < \|\xi\|$ if $n \leq 4$ and $\|\xi\| > 1$. We refer the interested reader to [13] for details.

5 Self-adjoint homogeneous cones

5.1

Now we describe a different approach to computing the Hecke action that can be found in [12, 15]. The main idea is to replace modular symbols and sharbly chains with chains built from rational polyhedral cones, and to replace "unimodularization" with moving the support of a chain into a certain canonically defined set of rational polyhedral cones. The results of this section apply to any arithmetic group that is associated to a *self-adjoint homogeneous cone*; the reduction theory in this generality is due to Ash [6, Ch. 2]. However, for simplicity we describe the results in the context of Voronoĭ's work reduction theory of real positive-definite quadratic forms [23].

Let V be the real vector space of all real symmetric $n \times n$ matrices, and let C be the subset of positive-definite matrices. Then C is a cone, i.e. C is a convex set closed under homotheties and containing no straight line. The group $\mathrm{SL}_n(\mathbb{Z})$ acts on V preserving C, and the action commutes with homotheties. In fact, modulo homotheties C is isomorphic to $X = \mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n)$; this exhibits a hidden linear structure of the symmetric space X.

Let $\bar{C}$ be the closure of C in V. Voronoĭ showed how to construct a set $\mathcal{V}$ of rational polyhedral cones in $\bar{C}$ such that

1. Γ acts on $\mathcal{V}$.

2. If $\sigma \in \mathcal{V}$ then so is any face of σ.
3. If $\sigma, \tau \in \mathcal{V}$, then $\sigma \cap \tau$ is a face of each.
4. Modulo Γ, the set $\mathcal{V}$ is finite.
5. The intersections $\sigma \cap C$ cover C.

The cones $\mathcal{V}$ provide a reduction theory for C in the following sense: any $x \in C$ lies in a unique cone $\sigma(x) \in \mathcal{V}$, and the number of $\gamma \in \Gamma$ such that $\gamma \cdot \sigma(x) = \sigma(x)$ is bounded. Given $x \in C$, there is an explicit algorithm, the *Voronoĭ reduction algorithm*, to find $\sigma(x)$.

The Voronoĭ cones descend modulo homotheties to induce a decomposition of X into cells. Furthermore, we can enlarge C to a cone $\widetilde{C}$ such that, if $\widetilde{X}$ denotes $\widetilde{C}$ modulo homotheties, then the quotient $\Gamma\backslash\widetilde{X}$ is compact. This *Satake compactification* of $\Gamma\backslash X$ is singular in general, but nevertheless can still be used to compute $H^*(\Gamma;\mathbb{C})$. For us, the salient points are that the images of the Voronoĭ cones induce a decomposition of $\widetilde{C}$, with all the properties listed above, and that the Voronoĭ reduction algorithm extends to the boundary $\partial\widetilde{C} := \widetilde{C} \smallsetminus C$.

5.2

Now let $\mathbf{C}_*^R$ be the complex over $\mathbb{C}$ generated by *all* simplicial rational polyhedral cones in $\widetilde{C}$, and let $\mathbf{C}_*^V$ be the subcomplex generated by Voronoĭ cones.[2] For any chain $\xi \in \mathbf{C}_*^R$, let supp ξ be the set of cones supporting ξ. The complex $\mathbf{C}_*^R$ is analogous to the sharbly complex, and the subcomplex $\mathbf{C}_*^V$ to the subcomplex generated by the reduced sharblies. In general, however, $\mathbf{C}_*^V$ is not isomorphic to the complex of reduced sharblies. Cycles $\xi \in \mathbf{C}_*^V$ can be used to compute $H^*(\Gamma)$, but the image $T(\xi)$ of ξ under a Hecke operator won't be supported on Voronoĭ cones. Hence we must show how to push $T(\xi)$ back into $\mathbf{C}_*^V$.

To accomplish this we have essentially two tools—we can subdivide the cones in supp $T(\xi)$, and we can use the Voronoĭ reduction algorithm to determine the cone any point lies in. We apply these as follows. Using the linear structure on $\widetilde{C}$, we first subdivide $T(\xi)$ very finely into a chain ξ'. Then to each 1-cone $\tau \in$ supp ξ', we assign a 1-cone $\rho_\tau \in \partial\widetilde{C}$, and we use the combinatorics of ξ' to assemble the ρ_τ into a cycle ξ'' homologous to ξ. We claim that if ξ' is constructed so that 1-cones $\tau \in$ supp ξ' lie in the *same or adjacent Voronoĭ cones,* then the ρ_τ can be chosen to ensure $\xi'' \in \mathbf{C}_*^V$.

5.3

We illustrate this process for SL_2; more details can be found in [12]. Modulo homotheties the three-dimensional cone $\widetilde{C}$ becomes the extended upper halfplane $\mathfrak{H}^* := \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$, with $\partial\widetilde{C}$ passing to the cusps $\mathfrak{H}^* \smallsetminus \mathfrak{H}$. The 3-cones in $\mathcal{V}$ tiling C pass to the $\mathrm{SL}_2(\mathbb{Z})$-translates of the ideal triangle with vertices at $0, 1, \infty$. Let us call these ideal triangles *Voronoĭ triangles.*

[2] Although the Voronoĭ cones aren't necessarily simplicial, we can assume that they have been Γ-equivariantly subdivided.

If $\xi \in \mathbf{C}_*^R$ is dual to a class in $H^1(\Gamma)$ and is supported on one 2-cone, then $\operatorname{supp} \xi$ passes to a geodesic μ between two cusps u_1, u_2 (Figure 2). We can subdivide μ into geodesic segments $\{\mu_i\}$ so that the endpoints e_i, e_{i+1} of μ_i lie in the same or adjacent Voronoĭ triangles. Then we assign cusps to the e_i as follows. If e_i is not an endpoint of ξ, then we assign any cusp c_i of the Voronoĭ triangle containing e_i. Otherwise, if $e_i = u_1$ or u_2 and hence is an endpoint of μ, then we assign e_i to itself. This determines a homology between ξ and a chain ξ'' supported on cones passing to the segments $[c_i, c_{i+1}]$. These cones are Voronoĭ cones, and thus $\xi'' \in \mathbf{C}_*^V$.

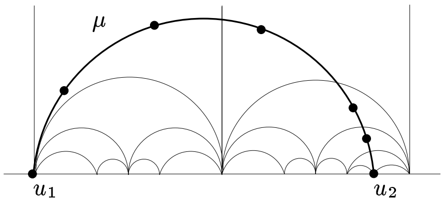

Fig. 2. A subdivision of μ; the solid dots are the e_i. Since the e_i lie in the same or adjacent Voronoĭ triangles, we can assign cusps to them to construct a homology to a cycle in $\mathbf{C}_*^V$.

6 Well-rounded retracts

6.1

To conclude this article, we describe unpublished work of MacPherson and McConnell [16] that allows one to compute the Hecke action on those Γ for which a *well-rounded retract* W is available. Again for simplicity we focus on $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$; our first task is to explain what W is.

Let $V = \mathbb{R}^n$ with the standard inner product preserved by $\mathrm{SO}(n)$, and let $L \subset V$ be a lattice. For any $v \in V$, write $\|v\|$ for the length of v. Let $m(L)$ be the minimal nonzero length attained by any vector in L, and let $M(L) = \{v \in L \mid \|v\| = m(L)\}$. Then L is said to be *well-rounded* if $M(L)$ spans V.

6.2

Consider the space of cosets $Y = \mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})$. This space can be interpreted as the space of oriented lattices in $\mathbb{R}^n$ modulo homotheties. Let $W \subset Y$ be the subset of well-rounded lattices, and for any $j = 0, \ldots, n$, let $Y_j = \{L \in Y \mid \dim \operatorname{span} M(L) \geq j\}$. Clearly $Y_0 = Y$ and $Y_n = W$.

According to Ash [1], there is an $\mathrm{SO}(n)$-equivariant retraction $r\colon Y \to W$ constructed as follows. Let $L \in Y_j$, and write $V = V_1 \oplus V_2$, where $V_1 = (\operatorname{span} M(L)) \otimes \mathbb{R}$, and V_2 is the orthogonal complement of V_1. For $0 < \lambda \leq 1$, let

$T(\lambda)$ be the linear transformation $(v_1, v_2) \mapsto (v_1, \lambda v_2)$, and let $L[\lambda]$ be the image of L under $T(\lambda)$. There is a critical value λ_0 for which $\dim \operatorname{span} M(L[\lambda]) > j$. Then we can define $r_j\colon Y_j \to Y_{j+1}$ by $r_j(L) = L[\lambda_0]$. These retractions can be composed to define the retraction $r\colon Y \to W$, and the space W is the well-rounded retract.

Since r is $\mathrm{SO}(n)$-equivariant, it induces a retraction $\mathrm{SL}_n(\mathbb{Z})\backslash\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}(n) \to W/\mathrm{SO}(n)$. Moreover, W can be given the structure of a locally-finite regular cell-complex. In a certain sense, these cells are dual to the Voronoĭ cones from §5: Voronoĭ cones of codimension k are in bijection with W-cells of dimension k. The construction works if Γ is replaced with any finite-index subgroup of $\mathrm{SL}_n(\mathbb{Z})$, and hence one has a convenient topological model to study the cohomology of any such Γ.

6.3

Now we consider how the ideas used in the construction of W can be applied to compute the action of the Hecke operators on cohomology. Let $d = (d_1, \dots, d_n)$ be a tuple of strictly positive integers, and let $g(d) \in \mathrm{GL}_n(\mathbb{Q})$ be the diagonal matrix with entries d. Let $\Gamma' := \Gamma \cap g^{-1}\Gamma g$. The *Hecke correspondence* associated to this data is the diagram $(c_1, c_2)\colon \Gamma'\backslash X \to \Gamma\backslash X$, where the two maps are defined by $c_1(\Gamma' x) = \Gamma x$ and $c_2(\Gamma' x) = \Gamma g x$. In terms of the above description, $c_1^{-1} \circ c_2$ is the (multivalued) map that takes any lattice L to the set of sublattices $\{M \subset L \mid L/M \cong \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_n\mathbb{Z}\}$. A Hecke correspondence induces a map $c_1^* \circ (c_2)_*$ on cohomology that is exactly a classical Hecke operator. For example, if $n = 2$, p is a prime, and $d = (1, p)$, then the induced Hecke operator is the usual T_p.

6.4

Fix a tuple d and a pair of lattices $M \subset L$ as above. Choose $u \in [1, \infty)$. For $v \in L$, let $\| \ \|_u$ be $\|v\|$ if $v \in M$, and $u \cdot \|v\|$ otherwise. Now we can consider the retraction r described in §6.2, but using $\| \ \|_u$ instead of $\| \ \|$ as the notion of length. When $u = 1$, the result is the usual retract W. But for $u = u_0$ sufficiently large, only vectors in M will be detected in the retraction. Since M is itself a lattice, we have $W_{u_0} \cong W$.

These two complexes W_1 and W_{u_0} appear in a larger complex $\mathcal{W}$ that depends on n and d and is fibered over the interval $[1, u_0]$ with fiber W_u. The fibers W_1 and W_{u_0} map to W by the maps c_1 and c_2, respectively. One computes the action of the Hecke operator by lifting a class on $\Gamma\backslash W$ to $\Gamma'\backslash\mathcal{W}$, pushing the lift across $\Gamma'\backslash\mathcal{W}$ to the face $\Gamma\backslash W_{u_0}$, and then pushing down via c_2 to $\Gamma\backslash W$.

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