HECKE OPERATORS AND $\mathbb{Q}$-GROUPS ASSOCIATED TO SELF-ADJOINT HOMOGENEOUS CONES

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Abstract. Let G be a reductive algebraic group associated to a self-adjoint homogeneous cone defined over $\mathbb{Q}$, and let $\Gamma \subset G$ be a appropriate neat arithmetic subgroup. We present two algorithms to compute the action of the Hecke operators on $H^i(\Gamma;\mathbb{Z})$ for all i. This simultaneously generalizes the modular symbol algorithm of Ash-Rudolph [7] to a larger class of groups, and provides techniques to compute the Hecke-module structure of previously inaccessible cohomology groups.

1. Introduction

1.1. Let G be a reductive algebraic group defined over $\mathbb{Q}$, and let $\Gamma \subset G(\mathbb{Q})$ be a neat arithmetic subgroup (§2.6). The group cohomology $H^*(\Gamma;\mathbb{Z})$ plays an important role in contemporary number theory, through its connection with automorphic forms and representations of the absolute Galois group. See [2] for an introduction to these ideas. This relationship is revealed in part through the action of the *Hecke operators* on the complex cohomology $H^*(\Gamma;\mathbb{C})$. These are endomorphisms induced from a family of correspondences associated to the pair $(\Gamma, G(\mathbb{Q}))$ (§2.7); the arithmetic nature of the cohomology is contained in the eigenvalues of these linear maps.

To compute the Hecke action in certain cases, one may use the *modular symbol algorithm.* One begins with a finite cell complex that computes $H^*(\Gamma;\mathbb{Z})$, and applies the Hecke operators to an easily understood set of dual homology classes, the *modular symbols* (§1.3). There is a finite set of modular symbols, distinguished by the choice of finite cell complex, that spans the homology classes. The modular symbol algorithm enables one to write the Hecke-image of a modular symbol as a sum of symbols taken from the finite spanning set.

Several research groups have used this technique to produce many corroborative examples of the "Langlands philosophy" [4, 6, 10, 23]. However, this technique has some shortcomings:

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- The group G must be either a linear group (SL_n or GL_n) [7] or a symplectic group (Sp_{2n}) [14].
- The group Γ must be associated to a euclidean domain. For example, $\Gamma \subset SL_n(\mathbb{Z})$ can be studied, but not $\Gamma \subset SL_n(\mathscr{O}_K)$, where $\mathscr{O}_K$ is the ring of integers in an algebraic number field $K/\mathbb{Q}$ with class number > 1.
- The modular symbol algorithm enables computation of the Hecke action only on $H^d(\Gamma;\mathbb{Z})$, where d is the *cohomological dimension* of Γ. This is the smallest integer d such that $H^*(\Gamma;M) = 0$ for $* > d$ and any $\mathbb{Z}\Gamma$-module M.

These limitations, particularly the last two, are real obstacles to continuing the experimental work cited above. For example:

- One is interested in H^2 of *Bianchi groups*, especially the cuspidal classes [12]. These arithmetic groups have the form $\Gamma \subset SL_2(\mathscr{O}_K)$, where $K/\mathbb{Q}$ is an imaginary quadratic extension . At present no systematic "motivic" explanation of the Hecke eigenvalues has been conjectured, although in some cases one can make the connection with the absolute Galois group [21].
- Work of Avner Ash and David Ginzburg suggests that one can find new automorphic L-functions by integrating rational cuspidal cohomology classes α for $G = GL_4$ over $(2,1,1)$-modular symbols, which are certain submanifolds of the associated locally symmetric space. Here $\alpha \in H^5$. If the $(2,1,1)$-modular symbols span a dual space to the cuspidal H^5 under this pairing, then the L-functions would exist and be non-zero. For $\Gamma_0(N) \subset SL_4(\mathbb{Z})$, Ash and the second author have computed $H^5(\Gamma_0(N))$ for a range of prime levels N. Computing the Hecke action on these groups will tell us, at least conjecturally, which classes are cuspidal, and which are lifts from smaller groups like Sp_4 or O_4. The modular symbol algorithm cannot be used because it works only in degree 6, not 5.

1.2. In this paper we overcome these obstacles for a special class of arithmetic groups, the groups $\Gamma \subset G(\mathbb{Q})$ such that G is the automorphism group of a *self-adjoint homogeneous cone* with a linear structure compatible with the Γ-action (§2.1). The real groups involved are $G(\mathbb{R}) = GL_n(\mathbb{R})$ and $GL_n(\mathbb{C})$, as well as more exotic examples (§2.2). The relevant arithmetic groups include $\Gamma \subset G(R)$, where R is

- $\mathbb{Z}$,
- the ring of integers in a totally real field, and
- the ring of integers in a CM field.

We present two algorithms (in Theorems 4.4, 4.11, and in Theorem 5.12) for the computation of the Hecke action.

1.3. In the remainder of this introduction, we compare our algorithms with the modular symbol algorithm of Ash-Rudolph [7]. For concreteness, we work with the simplest example with more than one interesting cohomology group, that of SL_3. First we establish notation.

Let G be the split form of SL_3, so that $G(\mathbb{Q}) = SL_3(\mathbb{Q})$. Let V be the $\mathbb{R}$-vector space of all 3×3 real symmetric matrices, and let $C \subset V$ be the open cone of positive-definite matrices. Then $G(\mathbb{R}) = SL_3(\mathbb{R})$ acts on C by $c \mapsto g \cdot c \cdot g^t$, and the stabilizer of c is isomorphic to $SO_3(\mathbb{R})$, the maximal compact subgroup of $SL_3(\mathbb{R})$. The group $\mathbb{R}^{>0}$ acts on C by homotheties, and we denote the quotient by X. We have an isomorphism $SL_3(\mathbb{R})/SO_3(\mathbb{R}) \xrightarrow{\sim} X$ given by $gK \mapsto \mathbb{R}^{>0}gg^t$, and X is a smooth noncompact manifold of real dimension 5.

Let $L \subset V$ be the lattice of integral symmetric matrices, and let $\Gamma_L = SL_3(\mathbb{Z})$ be the stabilizer of L. We fix a neat $\Gamma \subset \Gamma_L$. The space X is contractible, so we may identify the group cohomology $H^*(\Gamma; \mathbb{Z})$ with $H^*(\Gamma\backslash X; \mathbb{Z})$. The space $\Gamma\backslash X$ is 5-dimensional, and the cohomological dimension of Γ is 3.

1.4. We may use modular symbols to study $H^3(\Gamma)$. Let $\bar{C}$ be the closure of C. Any nonzero primitive vector $v \in \mathbb{Z}^3$ determines a rank-one semidefinite quadratic form $q(v) \in \bar{C} \smallsetminus C$ as follows: if we write $v = (a, b, c)^t \in \mathbb{Z}^3$, then

$$q(v) := \begin{pmatrix} a \\ b \\ c \end{pmatrix} \begin{pmatrix} a & b & c \end{pmatrix}.$$

The rays $\{\mathbb{R}^{>0} q(v) \mid v \in \mathbb{Z}^3 \smallsetminus \{0\}\}$ are called the *cusps* of C.

Let $\mathbf{v} = (v_1, v_2, v_3)$ be an ordered triple of distinct nonzero primitive integral vectors. Then if $\mathbf{v}$ is linearly independent, it determines an open oriented 3-cone $\sigma(\mathbf{v}) \subset C$, by

$$\sigma(\mathbf{v}) = \Big\{ \sum \rho_i q(v_i) \;\Big|\; \rho_i > 0 \Big\}.$$

Under the composition

$$C \longrightarrow X \longrightarrow \Gamma\backslash X,$$

this cone is taken to an open submanifold of $\Gamma\backslash X$. The closure of this submanifold in $\Gamma\backslash\bar{X}$, the Borel-Serre compactification of $\Gamma\backslash X$ (§3.9), determines a class $[\mathbf{v}] \in H_2(\Gamma\backslash\bar{X}, \partial(\Gamma\backslash\bar{X}))$. Such a class is by definition a *modular symbol.* Via Lefschetz duality we may identify $[\mathbf{v}]$ with a class in $H^3(\Gamma\backslash\bar{X}) = H^3(\Gamma; \mathbb{Z})$. It can be shown that the duals of the modular symbols span $H^3(\Gamma; \mathbb{Z})$. However, there are infinitely many of them.

If $\det \mathbf{v} = \pm 1$, then $[\mathbf{v}]$ is called a *unimodular symbol.* There are only finitely many unimodular symbols modulo Γ, and one can show that their duals span H^3 as follows [7]. Suppose $|\det \mathbf{v}| > 1$. Using the euclidean algorithm, one can construct a nonzero $w \in \mathbb{Z}^3$ such that

$$0 \leq |\det(w, v_i, v_j)| < |\det \mathbf{v}|$$

for any $1 \leq i < j \leq 3$. Since modular symbols satisfy the relation

$$[v_1, v_2, v_3] = [w, v_2, v_3] - [w, v_1, v_3] + [w, v_1, v_2],$$

and since $[\mathbf{v}] = 0$ if $\det \mathbf{v} = 0$, by iterating one can write a modular symbol as a sum of unimodular symbols. This is the *modular symbol algorithm*.

The image of a unimodular symbol α under a Hecke operator T is a finite sum of modular symbols, which in general are not unimodular. Using the modular symbol algorithm, we may write $T(\alpha)$ as a sum of unimodular symbols. Thus we may compute the Hecke action on $H^3(\Gamma;\mathbb{Z})$ using the finite spanning set given by the unimodular symbols.

1.5. Now we describe our approach to computing the Hecke action. We begin by shifting attention from the tuple $\mathbf{v}$ to the cone $\sigma(\mathbf{v})$. Instead of using $\det \mathbf{v}$ as a measure of non-unimodularity, we use the relative position of $\sigma(\mathbf{v})$ with respect to a distinguished collection of cones in $\bar{C}$, the set $\mathscr{V}$ of *Voronoĭ cones* (§3.3). In this example, $\mathscr{V}$ is the $SL_3(\mathbb{Z})$-orbit of the closed cone generated by the six cusps $\mathbb{R}^{>0}q(e_i)$ and $\mathbb{R}^{>0}q(e_i - e_j)$, where $\{e_i\} \subset \mathbb{Z}^3$ is the standard basis, and $1 \leq i < j \leq 3$. The intersection of each of these cones with C is a weak fundamental domain for $SL_3(\mathbb{Z})$, in the following sense. Let $\sigma \in \mathscr{V}$ and $x \in C$, and let $O(x)$ be the $SL_3(\mathbb{Z})$ orbit of x. Then the cardinality of the intersection of $\sigma \cap C$ with $O(x)$ is bounded by a constant not depending on σ or x.

One may use $\mathscr{V}$ for cohomology calculations as follows. First we let $\mathbf{C}_*^R$ be the complex over $\mathbb{Z}$ generated by *all* simplicial rational cones in $\bar{C}$ of all dimensions, with the obvious boundary maps (§3.12). This complex maps by linear projection to the singular chain complex of a certain Satake compactification $\Gamma\backslash\tilde{X}$ of $\Gamma\backslash X$, and is surjective on homology. The subcomplex $\mathbf{C}_*^V$ generated by the Voronoĭ cones is a finite complex mod Γ, and also maps surjectively to homology. The relative homology $H_*(\Gamma\backslash\tilde{X}, \Gamma\backslash\partial\tilde{X})$ can then be identified with the cohomology $H^{5-*}(\Gamma\backslash X) = H^{5-*}(\Gamma)$ (Proposition 3.10).

The complex $\mathbf{C}_*^R$ is a variant of a complex that first appeared in the literature in [17] and [3][1], and it plays the role of the modular symbols—it is infinite mod Γ and is preserved by the Hecke operators. The Voronoĭ subcomplex $\mathbf{C}_*^V$, on the other hand, plays the role of the unimodular symbols—it is finite mod Γ, yet is not preserved by the Hecke operators. Thus to compute the Hecke action, we must show how to write any cycle in $\mathbf{C}_*^R$ that is a Hecke image of a Voronoĭ cycle as a sum of Voronoĭ cycles. Hence we have replaced the algebraic problem of reducing a determinant with the geometric problem of moving a cycle built of "generic" cones into a cycle supported on Voronoĭ cones.

1.6. We give two techniques to do this. Our first technique (Theorem 4.4) replaces $\sigma(\mathbf{v})$ with a set of cones $F(\mathbf{v})$ constructed by refining the intersection of $\sigma(\mathbf{v})$ with the Voronoĭ cones. Then for each 1-dimensional cone of $F(\mathbf{v})$ we choose a cusp w. These cusps can be combined with the original $v \in \mathbf{v}$ into tuples, and this yields a relation

[1] This latter complex is called the *sharbly complex*, in honor of the authors of [17]. The name is due to Lee Rudolph.

in homology. For example, in Figure 1, we show a cone spanned by $\mathbf{v} = (v_1, v_2, v_3)$, which represents a class in $H^3(\Gamma\backslash X)$. This class is equal to the sum (with appropriate orientations) of the classes corresponding to (v_1, w_1, w_2), (w_1, w_2, w_4), … , (v_3, w_3, w_4). The new cones are Voronoĭ cones, so we have reached our goal: up to homology, we may replace $\sigma(\mathbf{v})$ with a sum of Voronoĭ cycles.

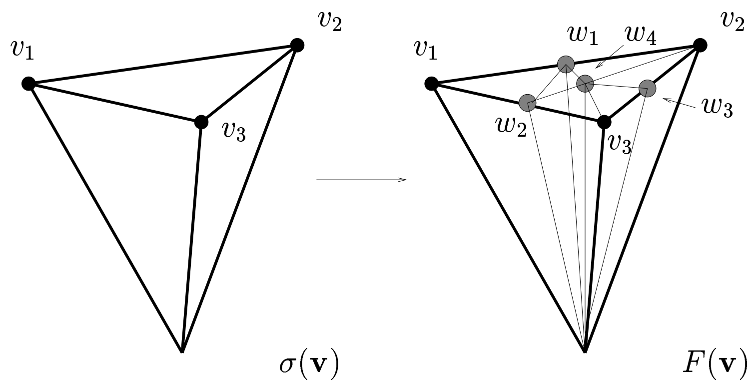

FIGURE 1.

In general, a cycle $\xi \in \mathbf{C}_*^V$ will be a sum of cones with vanishing boundary mod Γ, and its image under a Hecke operator $T(\xi)$ will be a similar object in $\mathbf{C}_*^R$. We show that the new cusps can be chosen Γ-equivariantly over $T(\xi)$, which ensures that the result is a cycle (Theorem 4.11). We further show that the cusps can be chosen so that the resulting cones are Voronoĭ cones.

Our first technique has the disadvantage that the fans $F(\mathbf{v})$ are difficult to compute in practice. In our second technique, we circumvent this problem by using the *Voronoĭ reduction algorithm* and a *sufficiently fine decomposition* of $\sigma(\mathbf{v})$ (§5). The former is an algorithm that computes which Voronoĭ cone contains a given point of C, and the latter is decomposition of $\sigma(\mathbf{v})$ into cones that are small enough to construct the homology (Theorem 5.12). We show how a sufficiently fine decomposition may be used to transform a generic cycle into a cycle built of Voronoĭ cones (Theorem 5.15).

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2. Background

In this section we recall facts about self-adjoint homogeneous cones and Hecke correspondences. For more details, the reader may consult [1, 11] for cones and [16, 20] for Hecke correspondences. §§2.1–2.6 closely follow Ash [1].

2.1. Let V be an $\mathbb{R}$-vector space defined over $\mathbb{Q}$, and let $C \subset V$ be an open cone. That is, C is open, contains no straight line, and C is closed under homotheties: if $x \in C$ and $\lambda \in \mathbb{R}^{>0}$, then $\lambda x \in C$. The cone C is called *self-adjoint* if there exists a scalar product $\langle\, ,\, \rangle$ on C such that

$$C = \left\{ x \in V \;\middle|\; \langle y, x \rangle > 0 \quad \text{for } y \in \bar{C} \smallsetminus \{0\} \right\}.$$

Such a cone is necessarily convex, as is $\bar{C}$.

Let G denote the connected component of the identity of the linear automorphism group of C, i.e. $G = \{\, g \in GL(V) \mid gC = C \,\}^0$. The cone C is called *homogeneous* if G acts transitively on C. If K denotes the isotropy group of a given point in C, then we may identify C with G/K. The self-adjointness of C implies that G is reductive and that C modulo homotheties is a riemannian symmetric space. We denote this symmetric space throughout by X, and always write $N = \dim X$.

2.2. We assume that all the above notions are compatible with the $\mathbb{Q}$-structure on V. That is, as a subgroup of $GL(V)$, G is defined by rational equations, and the scalar product is defined over $\mathbb{Q}$. These rationality conditions place the following restrictions on V.

Recall that a *Jordan algebra* is a finite-dimensional algebra J satisfying

1. $ab = ba$,
2. $a^2(ba) = (a^2b)a$,

for all $a, b \in J$. In general J is not associative. If J is defined over $\mathbb{R}$, we say that J is *euclidean* if $a^2 + b^2 = 0$ implies $a = b = 0$.

Fix a basepoint $p \in C(\mathbb{Q}) := C \cap V(\mathbb{Q})$. Then the rationality conditions on G hold if and only if V can be given the structure of a euclidean Jordan algebra defined over $\mathbb{Q}$ with identity p such that C is the set of invertible squares in V. This implies that the group of real points $G(\mathbb{R})$ must be isomorphic to a product of the following groups [11, p. 97]:

1. $GL_n(\mathbb{R})$.
2. $GL_n(\mathbb{C})$.
3. $GL_n(\mathbb{H})$.
4. $O(1, n-1) \times \mathbb{R}^\times$.
5. The noncompact Lie group with Lie algebra $\mathfrak{e}_{6(-26)} \oplus \mathbb{R}$ (This is the group of collineations of the projective plane over the Cayley numbers [22, p. 46].)

In each case V is a set of hermitian symmetric matrices. In other words, V is the set of $n \times n$ matrices A over an appropriate $\mathbb{R}$-algebra with involution τ, such that $A^t = A^\tau$. The cone C is then the subset of "positive-definite" matrices in an appropriate sense. For details we refer to [11, Ch. V].

2.3. Let H be a hyperplane in V. We say that H is a *supporting hyperplane* of C if H is rational and $H \cap C = \varnothing$ but $H \cap \bar{C} \neq \varnothing$. Since $\bar{C}$ is convex, these conditions imply that $\bar{C}$ lies entirely in one of the two closed half-spaces determined by H.

Given a rational supporting hyperplane H of C, let $C' = \mathrm{Int}(H \cap \bar{C})$. (Throughout, $\mathrm{Int}(A)$ is the interior of A in its linear span.) Then C' is called a *rational boundary component*, and is a self-adjoint homogeneous cone of smaller dimension than C. By convention, we also say that C is a (improper) rational boundary component. Let $\tilde{C}$ be the union of C and all its proper rational boundary components.

Here is characterization of the rational boundary components in terms of the Jordan algebra structure on V. If $e \in V(\mathbb{Q})$ satisfies $e^2 = e$, then we call e a *rational idempotent*, and write $V(e) = \{x \in V \mid xe = x\}$. The subspace $V(e)$ is a euclidean sub Jordan algebra defined over $\mathbb{Q}$. Then any rational boundary component arises as the subset of invertible squares $C(e) \subset V(e)$ for some choice of e.

Two idempotents e and f are called *orthogonal* if $ef = 0$. A rational idempotent is called *minimal* if it cannot be written as the sum of two orthogonal nonzero idempotents. Any rational idempotent e can be written as a sum of mutually orthogonal minimal idempotents, and the number needed is an invariant of e. This number is called the $\mathbb{Q}$*-rank* of e. By definition, the $\mathbb{Q}$-rank of a rational boundary component $C(e)$ is the $\mathbb{Q}$-rank of e. For any nonnegative integer k, let $C(k)$ denote the union of all rational boundary components of $\mathbb{Q}$-rank $\leq k$.

2.4. Definition. The *cusps* of C are the rank-one rational boundary components of C. The set of cusps is denoted $\Xi(C)$.

We will always use n to denote the $\mathbb{Q}$-rank of C itself. Note that $C(1) = \Xi(C)$, and that $C(k)$ is disjoint from C if $k < n$. If $k \geq n$, then $C(k) = \tilde{C}$.

2.5. *Remark.* Because of the homotheties, the $\mathbb{Q}$-rank of C is *not* the same as the $\mathbb{Q}$-rank of the algebraic group G. If n is the $\mathbb{Q}$-rank of C, then $n-1$ is the $\mathbb{Q}$-rank of G. For example, when $G = SL_n$, the $\mathbb{Q}$-rank of C is n.

2.6. Let $L \subset V(\mathbb{Q})$ be a lattice, i.e. a discrete subgroup of $V(\mathbb{Q})$ such that $L \otimes \mathbb{Q} = V(\mathbb{Q})$. Let Γ_L denote the subgroup of $G(\mathbb{Q})$ carrying L onto itself. An *arithmetic subgroup* of G is a discrete subgroup Γ commensurable with Γ_L for some L. Recall that Γ is *neat* if for any $\gamma \in \Gamma$, the multiplicative subgroup of $\bar{\mathbb{Q}}$ generated by the eigenvalues of γ is torsion-free [8, p. 117]. Any neat arithmetic group $\Gamma \subset \Gamma_L$ of finite index acts properly discontinuously and freely on C. Thus the quotient $\Gamma\backslash C$ is an Eilenberg-Mac Lane space for Γ, and the group cohomology $H^*(\Gamma)$ is $H^*(\Gamma\backslash C)$. In fact, since homotheties commute with the action of Γ, we may pass to the symmetric space X, and compute $H^*(\Gamma\backslash X)$ instead.

2.7. Fix a neat arithmetic group Γ. Given $g \in G(\mathbb{R})$, let $\Gamma^g = g^{-1}\Gamma g$ and $\Gamma' = \Gamma \cap \Gamma^g$. Let $\mathrm{Comm}(\Gamma)$ be the commensurator of Γ. This is the subgroup of $G(\mathbb{R})$ defined by

$$\mathrm{Comm}(\Gamma) := \Big\{ g \in G(\mathbb{R}) \;\Big|\; [\Gamma : \Gamma'] \quad \text{and} \quad [\Gamma^g : \Gamma'] < \infty \Big\}.$$

For any $g \in \mathrm{Comm}(\Gamma)$, the inclusions $\Gamma' \to \Gamma$ and $\Gamma' \to \Gamma^g$ determine a diagram

$$\begin{tikzcd} & \Gamma'\backslash X \arrow[dl, "s"'] \arrow[dr, "t"] & \\ \Gamma\backslash X & & \Gamma\backslash X \end{tikzcd}$$

Here $s(\Gamma' x) = \Gamma x$ and t is the composition of $\Gamma' x \mapsto \Gamma^g x$ with left multiplication by g. This diagram is called the *Hecke correspondence* associated to g. It can be shown that, up to isomorphism, the Hecke correspondence depends only on the double coset $\Gamma g \Gamma$. Furthermore, it can also be shown that Hecke correspondences extend naturally to the rational boundary components, inducing a correspondence[2]

$$\begin{tikzcd} & \Gamma'\backslash \tilde{X} \arrow[dl, "\tilde{s}"'] \arrow[dr, "\tilde{t}"] & \\ \Gamma\backslash \tilde{X} & & \Gamma\backslash \tilde{X} \end{tikzcd}$$

Because the maps s and t are proper, we obtain a map on cohomology

$$T_g := t_* s^* \colon H^*(\Gamma\backslash X; \mathbb{Z}) \to H^*(\Gamma\backslash X; \mathbb{Z}).$$

This is called the *Hecke operator* associated to g. We let $\mathscr{H}_\Gamma$ be the $\mathbb{Z}$-algebra generated by the Hecke operators, with product given by composition.

3. Rational Polyhedral Cycles

In this section we develop some tools to compute the cohomology of Γ. The main result is a correspondence between classes in $H^*(\Gamma; \mathbb{Z})$ and homology classes in a chain complex built from rational polyhedral cones. Our construction relies on the reduction theory for self-adjoint homogeneous cones (§3.3) due to Ash [1].

3.1. Let $A \subset V(\mathbb{Q})$ be a finite set of nonzero points. The closed convex hull σ of the rays $\left\{\mathbb{R}^{\geq 0} x \;\middle|\; x \in A\right\}$ is called a *rational polyhedral cone.* We say that σ is a d-cone if it has dimension d. The rays through the vertices of the convex hull of A are called the *spanning rays* of σ. We denote the set of spanning rays by $R(\sigma)$. The group $G(\mathbb{Q})$ acts naturally on the set of rational polyhedral cones, and we denote the action by a dot: $\sigma \mapsto g \cdot \sigma$.

Let $\mathbf{x}$ be a finite tuple of nonzero points in $V(\mathbb{Q})$. Then a *pointed rational polyhedral cone* $\sigma(\mathbf{x})$ is the data of the cone generated by the $x \in \mathbf{x}$ and the tuple $\mathbf{x}$. There is a natural $G(\mathbb{Q})$-action on pointed rational polyhedral cones given by the action on tuples, and this action agrees with the action on the underlying cones. We will sometimes suppress the tuple $\mathbf{x}$ from the notation.

[2] Here one takes the Satake topology for the quotient (cf. §3.9).

We say that $\sigma(\mathbf{x})$ is *simplicial* if every subset of $R(\sigma(\mathbf{x}))$ spans a face of $\sigma(\mathbf{x})$. This means that $\sigma(\mathbf{x})$ is equivalent to a d-simplex (except it is missing a facet at infinity in V), where $d = \#R(\sigma(\mathbf{x}))$.

3.2. A collection of cones F is called a *fan* if it satisfies the following:

1. Any face τ of $\sigma \in F$ is also a member of F.
2. If $\sigma, \tau \in F$, then $\sigma \cap \tau$ is a common face of σ and τ.

The subsets $F_d := \{\sigma \in F \mid \dim \sigma \leq d\}$ provide a filtration of F by subfans. We do not require that F be either finite or locally finite.

3.3. Now we want to partition C into convex subsets using a rational polyhedral fan, in a manner compatible with the Γ_L-action. This requires some care, as C is open and any rational polyhedral cone is closed.

Let $\Gamma \subset G$ be an arithmetic group. A fan F is called a Γ-*admissible decomposition* of C when the following hold [1, p. 72]:

1. Each $\sigma \in F$ is the span of a finite number of rational vertices, and is a subset of $\bar{C}$.
2. For any $\sigma \in F$ and any $\gamma \in \Gamma$, we have $\gamma \cdot \sigma \in F$.
3. The set $\Gamma \backslash F$ is finite.
4. $C = \bigcup_{\sigma \in F} (\sigma \cap C)$.

Note that a Γ-admissible decomposition descends mod homotheties to a decomposition of X into open cells.

Here is a technique to construct Γ-admissible decompositions. It originated with Voronoĭ[24], and was generalized by Ash to all self-adjoint homogeneous cones [5, Ch. II]. Let L' be $L \smallsetminus \{0\}$.

3.4. Definition. The *Voronoĭ polyhedron* Π is the closed convex hull of $L' \cap \Xi(C)$.

According to the Lemma of [1, p. 75] and its proof, the polyhedron Π has vertices only in $\Xi(C)$, and each face of Π is the convex hull of a finite set of points. Clearly $\Pi \subset \bar{C}$, since $\Xi(C) \subset \bar{C}$.

3.5. Theorem. [5, p. 143] *The cones over the faces of* Π *form a* Γ-*admissible decomposition of* C.

3.6. Proposition. [1, cf. item (c), p. 75] *Suppose that* Γ *is neat. Then the fan* F *of cones over the faces of* Π *may be refined, without adding new one-dimensional fans, to a* Γ-*invariant simplicial fan* F'.

Proof. We present the proof, since it is the prototype for similar arguments later. Every cone can be subdivided into simplicial cones without adding new rays; we must prove that this can be done over all of F in a Γ-invariant way. We construct refinements F'_k of F_k by induction on the dimension. Our F' will be F'_N, where N is the dimension of V.

To begin, any 1-cone is simplicial. Hence we set $F'_1 = F_1$.

Now suppose that F_k has been Γ-equivariantly refined to F'_k. The set of cones $F_{k+1} \smallsetminus F_k$ is finite modulo Γ, and we may choose a set T of representatives of these orbits. Let $\sigma \in T$. Let $\operatorname{Nor}(\sigma)$ be the subgroup of Γ preserving σ as a set, and let $\operatorname{Stab}(\sigma)$ be the subgroup of Γ preserving each spanning ray of σ. Elements of $\operatorname{Stab}(\sigma)$ actually fix σ pointwise, since they must fix the primitive generator of each spanning ray in L'.

Hence elements of $\operatorname{Stab}(\sigma)$ fix pointwise the linear hull V_σ of σ in V. The quotient group $\operatorname{Nor}(\sigma)/\operatorname{Stab}(\sigma)$ is finite, since it is a permutation group on the primitive generators of the spanning rays of σ. For $\gamma \in \operatorname{Nor}(\sigma)$, this means the restriction of γ to V_σ is semisimple and has all its eigenvalues equal to roots of unity. These values occur among the eigenvalues of γ itself, and so must all be 1 by the neatness of Γ. Hence the restriction of $\operatorname{Nor}(\sigma)$ to V_σ acts trivially, meaning $\operatorname{Nor}(\sigma) = \operatorname{Stab}(\sigma)$.

Now choose any subdivision of σ into simplicial cones without adding new rays. Since $\operatorname{Stab}(\sigma)$ preserves this subdivision, so does $\operatorname{Nor}(\sigma)$. Hence we may use the Γ-action to carry the subdivisions for the $\sigma \in T$ to all of F_{k+1} in a well-defined way. $\square$

For the remainder of this paper, we assume Γ is neat, and we fix a Γ-invariant simplicial refinement of the cones over the faces of Π. We call the resulting simplicial cones the *Voronoĭ cones*, and denote the fan of Voronoĭ cones by $\mathscr{V}$.

We can make any Voronoĭ cone σ into a pointed cone $\sigma(\mathbf{x})$ by choosing an ordering of $R(\sigma)$. The rational points comprising $\mathbf{x}$ are the corresponding vertices of Π.

3.7. As in §2.3, any rational boundary component C' of C is itself a self-adjoint homogeneous cone with $\mathbb{Q}$-rank less than that of C. Let $\Xi(C') = \Xi(C) \cap C'$, the set of cusps in C'. The Voronoĭ fan $\mathscr{V}$ is compatible with the rational boundary components in the following sense.

3.8. Proposition. *Let C' be a rational boundary component of C and let Π' be $C' \cap \Pi$. Then Π' is a face of Π, and is the convex hull of $L' \cap \Xi(C')$.*

Proof. Let H be a supporting hyperplane of $\bar{C}$ cutting out C'. Section 2.3 implies H meets $\Xi(C')$, so $H \cap \Pi \neq \varnothing$. Since $\Pi \subseteq \bar{C}$, H is a supporting hyperplane of Π. We have $\Pi' = C' \cap \Pi = H \cap \bar{C} \cap \Pi = H \cap \Pi$. Thus Π' is cut out from Π by a supporting hyperplane; by definition, this means Π' is a face of Π.

Let $\operatorname{hull}(S)$ denote the convex hull of a set S. We have $\Pi' = H \cap \Pi = H \cap \operatorname{hull}(L' \cap \Xi(C))$. Since $\Xi(C)$ lies entirely in one of the closed half-spaces determined by H, the latter set equals $\operatorname{hull}(H \cap L' \cap \Xi(C))$. In turn, this is $\operatorname{hull}(L' \cap H \cap \bar{C} \cap \Xi(C)) = \operatorname{hull}(L' \cap C' \cap \Xi(C)) = \operatorname{hull}(L' \cap \Xi(C'))$. $\square$

3.9. Let $\bar{X}$ be the bordification of the symmetric space X constructed by Borel-Serre [9]. Let $Y = \Gamma \backslash X$ and $\bar{Y} = \Gamma \backslash \bar{X}$. Let $\partial \bar{Y} = \bar{Y} \smallsetminus Y$. Since Γ is neat, $\bar{Y}$ is a compact manifold with corners of dimension N that is homotopy equivalent to Y. Thus

$$H^*(\Gamma; \mathbb{Z}) = H^*(Y; \mathbb{Z}) = H^*(\bar{Y}; \mathbb{Z})$$

and by Lefschetz duality

$$H^*(\bar{Y};\mathbb{Z}) = H_{N-*}(\bar{Y},\partial\bar{Y};\mathbb{Z}).$$

Rather than use $\bar{Y}$ to compute the cohomology of Γ, we will use a certain singular compactification of Y, which was originally constructed by Satake [19, 18]. Recall that $\tilde{C} \subset \bar{C}$ is the union of C with its rational boundary components. We equip $\tilde{C}$ with the *Satake topology* [19]. Let $\tilde{X}$ be the quotient of $\tilde{C}$ by homotheties, and let $\tilde{Y} = \Gamma\backslash\tilde{X}$. Then $\tilde{Y}$ is a compact hausdorff space called a *Satake compactification* of Y, and we write $\partial\tilde{Y} = \tilde{Y} \smallsetminus Y$.

3.10. Proposition. $H_*(\bar{Y},\partial\bar{Y};\mathbb{Z}) = H_*(\tilde{Y},\partial\tilde{Y};\mathbb{Z})$.

Proof. According to Zucker [25], there is a quotient map $q : \bar{Y} \to \tilde{Y}$ that is the identity on Y. Let $U \subset \bar{Y}$ be any collared neighborhood of $\partial\bar{Y}$; this admits a deformation retraction to $\partial\bar{Y}$. Let V be $q(U)$, which is a neighborhood of $\partial\tilde{Y}$. The composition of q with the retraction yields a retraction of V onto $\partial\tilde{Y}$.

The result then follows from the sequence of isomorphisms

$$H_*(\bar{Y},\partial\bar{Y}) \xrightarrow{r} H_*(\bar{Y},U) \xrightarrow{e} H_*(Y,U\cap Y) \xrightarrow{\mathrm{id}} H_*(Y,V\cap Y) \xrightarrow{e} H_*(\tilde{Y},V) \xrightarrow{r} H_*(\tilde{Y},\partial\tilde{Y}),$$

where the maps r are induced by the appropriate retractions, and the maps e are induced by excision. ☐

3.11. Proposition. *Under quotienting by homotheties $\pi\colon \tilde{C} \to \tilde{Y}$, the Voronoĭ cones descend to a finite triangulation of $\tilde{Y}$.*

Proof. The Voronoĭ cones descend modulo homotheties to give a decomposition of $\tilde{X}$ into closed simplices; this has the same formal properties as in §3.2. By Theorem 1 of [5, p. 113], the restriction of the Satake topology to each closed simplex $\tilde{\sigma}$ coincides with the ordinary topology on the simplex. By a neatness argument like that in Proposition 3.6, $\tilde{\sigma} \cap \gamma\tilde{\sigma} \neq \varnothing$ for $\gamma \in \Gamma$ implies γ fixes $\tilde{\sigma}$ pointwise. This implies the map $\tilde{X} \to \tilde{Y}$ restricts to a bijection—hence an embedding—on $\tilde{\sigma}$. Thus $\tilde{Y}$ has a decomposition into closed simplices with the same formal properties as in §3.2—that is, a triangulation of $\tilde{Y}$. By (3) of §3.3, the triangulation is finite. ☐

3.12. Let Δ_k be the standard k-simplex, that is

$$\Delta_k := \Big\{(r_0,\dots,r_k) \subset \mathbb{R}^{k+1} \;\Big|\; \sum r_i = 1, \quad \text{and} \quad r_i \geq 0\Big\}.$$

Let e_i be the ith vertex of Δ_k. Let $C_k(\tilde{Y})$ be the group of integral singular k-chains. In other words, $C_k(\tilde{Y})$ is the free abelian group generated by all continuous maps $s\colon \Delta_k \to \tilde{Y}$.

Let $\mathbf{C}^R_k$ be the free abelian group generated by the set of all pointed rational polyhedral cones $\sigma(\mathbf{x})$, as $\mathbf{x}$ varies over all $(k+1)$-tuples of nonzero points in $\tilde{C}(\mathbb{Q})$. In general, we shall use boldface to denote complexes built from polyhedral cones, before taking the quotient by homotheties. Notice the shift in degree: $\mathbf{C}^R_k$ is built from $(k+1)$-dimensional cones. Cycles in $\mathbf{C}^R_k$ will push forward to the "correct" degree k in $H_k(\tilde{Y}, \partial\tilde{Y})$ and related groups.

The obvious boundary map $\partial\colon \mathbf{C}^R_k \to \mathbf{C}^R_{k-1}$ makes $\mathbf{C}^R_*$ into a chain complex. By abuse of notation, we let $\sigma(\mathbf{x})$ denote both the pointed rational polyhedral cone determined by $\mathbf{x}$ as well as the class of this cone in $\mathbf{C}^R_k$.

Given a chain $\xi = \sum n(\mathbf{x})\sigma(\mathbf{x}) \in \mathbf{C}^R_k$, where $n(\mathbf{x}) \in \mathbb{Z}$, we define the *support* of ξ by

$$\operatorname{supp}(\xi) := \left\{\sigma(\mathbf{x}) \mid n(\mathbf{x}) \neq 0\right\}.$$

Any chain ξ determines a singular chain $[\xi] \in C_k(\tilde{Y})$ via the homotheties and mod Γ. If $\mathbf{x} = (x_0, \dots, x_k)$, the cone $\sigma(\mathbf{x})$ induces $s\colon \Delta_k \to \tilde{Y}$ by $s(e_i) = \pi(x_i)$, linear extension, and taking the quotient modulo Γ. The map $[\ ]\colon \mathbf{C}^R_* \to C_*(\tilde{Y})$ is a morphism of complexes.

3.13. Proposition. *Any class in $H_*(\tilde{Y}, \partial\tilde{Y})$ can be represented by a chain in $\mathbf{C}^R_*$.*

Proof. This follows from Proposition 3.11. Any homology class can be written as the image of a $\mathbb{Z}$-linear combination of Voronoĭ cones, and such cones have a rational pointed structure. □

3.14. Definition. Given $u \in H_*(\tilde{Y}, \partial\tilde{Y})$, a chain $\xi \in \mathbf{C}^R_*$ with $[\xi] = u$ is called a *lift* of u.

3.15. Let $\mathbf{Z}^R_*$ be the *relative* cycle group

$$\mathbf{Z}^R_* := \left\{\xi \in \mathbf{C}^R_* \,\middle|\, [\partial\xi] \text{ is supported on } \partial\tilde{Y}.\right\}$$

In general, elements of $\mathbf{Z}^R_*$ will not be cycles with respect to the boundary map in $\mathbf{C}^R_*$: they will have boundaries mapping to $\partial\tilde{Y}$, and will only be cycles modulo the Γ-action.

Recall that n is the $\mathbb{Q}$-rank of C.

3.16. Proposition. *Let $\xi \in \mathbf{C}^R_k$ be supported on cones spanned by cusps.*

1. *If $k = n-1$, then $\xi \in \mathbf{Z}^R_k$.*
2. *If $k < n-1$, then the class of ξ is zero in $H_*(\tilde{Y}, \partial\tilde{Y})$.*

Proof. Recall that $C(i)$ is the union of all rational boundary components of C of $\mathbb{Q}$-rank $\leq i$. Let $D \subset C(i)$ and $E \subset C(j)$ be rational boundary components. According to [1, Lemma 4], we have $D + E \subset C(i+j)$. Hence if $\mathbf{x}$ is a k-tuple with $x \in \Xi(C)$ for all $x \in \mathbf{x}$, then $\sigma(\mathbf{x}) \subset C(k)$.

Both statements in the Proposition follow from this containment. In (1), $\operatorname{supp}(\partial\xi)$ consists of cones that lie in $C(n-1)$, which under homotheties and Γ map to a singular chain supported on $\partial\tilde{Y}$. This implies that ξ is a relative cycle. In (2), $\operatorname{supp}(\xi)$ is a subset of $C(n-1)$, and itself maps to a singular chain supported on $\partial\tilde{Y}$. □

3.17. Example. Suppose that G is the $\mathbb{Q}$-group with $G(\mathbb{Q}) = SL_n(\mathbb{Q})$, and let $\mathbf{x}$ be a rational n-tuple with $x \in \Xi(C)$ for all $x \in \mathbf{x}$. Then the class

$$[\sigma(\mathbf{x})] \in H_{n-1}(\tilde{Y}, \partial\tilde{Y}) = H^{N-n+1}(\Gamma)$$

is a *minimal modular symbol* as in Ash-Rudolph [7].

3.18. *Remark.* Since cycles supported on Voronoĭ cones map surjectively onto $H_*(\tilde{Y}, \partial\tilde{Y})$, we obtain that $H_k(\tilde{Y}, \partial\tilde{Y})$ vanishes for $k < n-1$. Equivalently, $H^k(\Gamma;\mathbb{Z}) = 0$ if $k > N-n+1$. This is a special case of a much more general result from [9]: the cohomological dimension of Γ is $N-n+1$.

3.19. Now we describe the action of $\mathscr{H}_\Gamma$ on $H^{N-*}(\Gamma) = H_*(\tilde{Y}, \partial\tilde{Y})$ in the setting of §§3.12–3.15. Choose a Hecke operator T_g. Then we may decompose the double coset $\Gamma g \Gamma$ as

$$\Gamma g \Gamma = \coprod_{s \in S} \Gamma s \tag{1}$$

for some set $S \subset G(\mathbb{Q})$, which is finite since $g \in \mathrm{Comm}(\Gamma)$. The Hecke correspondence carries the point $\Gamma x \in \Gamma\backslash X$ to the finite set of points $\{\Gamma s x\}_{s\in S}$ in $\Gamma\backslash X$.

Given a class $u \in H_*(\tilde{Y}, \partial\tilde{Y})$ with lift $\xi = \sum_{\mathbf{x}\in A} n(\mathbf{x})\sigma(\mathbf{x})$, the Hecke operator acts by

$$u \longmapsto \Bigg[\sum_{\substack{\mathbf{x}\in A\\ s\in S}} n(\mathbf{x})\sigma(s\cdot\mathbf{x})\Bigg]. \tag{2}$$

One can easily show that the image of the map in (2) is a well-defined homology class.

4. First Algorithm

As before, let N be the dimension of the symmetric space X, and let n be the $\mathbb{Q}$-rank of C. We want to compute the action of the Hecke operators on $H^*(\Gamma)$, which we identify with the relative homology $H_{N-*}(\tilde{Y}, \partial\tilde{Y})$. Concretely, we must identify a finite basis[3] of $H_*(\tilde{Y}, \partial\tilde{Y})$, and compute the transformation matrix of a given Hecke operator in terms of this basis.

Let $\mathbf{Z}^V_* \subset \mathbf{Z}^R_*$ be the polyhedral cycles supported on Voronoĭ cones. Let $\mathbf{Z}^{\mathscr{H}}_* \subset \mathbf{Z}^R_*$ be the subgroup of *Hecke images*; that is, $\xi \in \mathbf{Z}^{\mathscr{H}}_*$ if and only if $\xi = T(\eta)$ for some $T \in \mathscr{H}_\Gamma$ and $\eta \in \mathbf{Z}^V_*$. By Proposition 3.11, $\mathbf{Z}^V_*$ contains a finite set of cycles whose homology classes form a basis of $H_*(\tilde{Y}, \partial\tilde{Y})$. Hence to compute the action of a Hecke operator, it suffices to construct an algorithm that transforms a cycle in $\mathbf{Z}^{\mathscr{H}}_*$ to cycle in $\mathbf{Z}^V_*$ that generates the same homology class in $H_*(\tilde{Y}, \partial\tilde{Y})$. In Theorems 4.4 and 4.11 we describe an algorithm that accomplishes this.

[3] If $H_*(\tilde{Y}, \partial\tilde{Y})$ has torsion, then by "basis" we mean a minimal set of elements generating $H_*(\tilde{Y}, \partial\tilde{Y})$ as an abelian group.

4.1. We begin by establishing some constructions appearing in the algorithm. Recall that $R(\sigma)$ denotes the set of spanning rays for a cone σ.

Let $\sigma \subset \tilde{C}$ be a rational polyhedral cone satisfying $R(\sigma) \subset \Xi(C)$. Then $\sigma \cap \mathscr{V}$ denotes the fan obtained by intersecting σ with the Voronoĭ fan $\mathscr{V}$. That is, writing $\{\sigma_\alpha\}_{\alpha \in A}$ for $\sigma \cap \mathscr{V}$, then $\bigcup_{\alpha \in A} \sigma_\alpha = \sigma$, and each σ_α is the intersection of a Voronoĭ cone and some (not necessarily proper) face of σ. We call $\sigma \cap \mathscr{V}$ the *canonical fan* associated to σ.

Given a cone $\sigma_\alpha \in \sigma \cap \mathscr{V}$, let V_α denote the Voronoĭ cone inducing σ_α. That is, V_α is the smallest Voronoĭ cone containing σ_α.

4.2. Proposition. *If σ is a rational polyhedral cone, then $\sigma \cap \mathscr{V}$ is a finite fan.*

Proof. According to the proof of the main theorem in [1], the intersection of any rational polyhedral cone with the Voronoĭ polyhedron Π is cut out by the faces of σ and finitely many supporting hyperplanes of Π. Thus σ can meet at most a finite number of Voronoĭ cones, and the result follows. □

Let $\mathbf{x}$ be a rational tuple. Although $\sigma(\mathbf{x})$ and the cones in $\mathscr{V}$ are simplicial, the canonical fan $\sigma(\mathbf{x}) \cap \mathscr{V}$ need not be. Let $F(\mathbf{x})$ denote a simplicial refinement of the canonical fan that does not add any new 1-cones.

4.3. We are now ready to present our first algorithm. In Theorem 4.4, we describe the algorithm on cycles in $\mathbf{Z}^{\mathscr{H}}_{n-1}$. Theorem 4.11 describes the algorithm on cycles in $\mathbf{Z}^{\mathscr{H}}_{k}$ for $k > n-1$.

Let $\xi \in \mathbf{Z}^{\mathscr{H}}_{n-1}$ be a Hecke image. Write ξ as $\sum n(\mathbf{x})\sigma(\mathbf{x})$, where $n(\mathbf{x}) \in \mathbb{Z}$. Since $k = n-1$, Proposition 3.16 implies that each $\sigma(\mathbf{x})$ is already a cycle. Hence we may assume without loss of generality that $\xi = \sigma(\mathbf{x})$.

4.4. Theorem. *Let $\xi = \sigma(\mathbf{x}) \in \mathbf{Z}^{\mathscr{H}}_{n-1}$. The following algorithm constructs a cycle $\xi^V \in \mathbf{Z}^V_{n-1}$ such that $[\xi] = [\xi^V]$.*

1. *Construct the canonical fan $\sigma(\mathbf{x}) \cap \mathscr{V}$, and construct a simplicial refinement $F(\mathbf{x})$ that does not add any new 1-cones.*
2. *For each 1-cone $\rho_\beta \in F(\mathbf{x})$, select a cusp $v_\beta \in R(V_\beta) \cap \Xi(C')$. Here V_β is the Voronoĭ cone inducing ρ_β, and C' is the smallest (not necessarily proper) rational boundary component containing ρ_β. Let y_β denote the vertex of Π generating v_β.*
3. *For each n-cone $\sigma_\alpha \in F(\mathbf{x})$, construct a pointed simplicial n-cone $\tau(\mathbf{y}_\alpha)$ by taking the n points $\{y_\beta \mid \rho_\beta \subset \sigma_\alpha\}$ and ordering them so that the induced orientation on $\tau(\mathbf{y}_\alpha)$ matches the orientation σ_α inherits from σ.*

Then $\tau(\mathbf{y}_\alpha) \in \mathbf{Z}^V_{n-1}$, and the desired cycle is

$$\xi^V = \sum_{\sigma_\alpha \in F(\mathbf{x})_n} \tau(\mathbf{y}_\alpha).$$

4.5. We begin the proof by showing that the choices in Step 2 of Theorem 4.4 are possible.

4.6. Lemma. *Let $\rho_\beta \in F(\mathbf{x})$ be a 1-cone. Then the set $R(V_\beta) \cap \Xi(C')$ is nonempty.*

Proof. By Proposition 3.8, the intersection $\Pi' := C' \cap \Pi$ is a face of Π. The Voronoĭ cone V_β is the cone over a face A_β of Π. So $V_\beta \cap C'$ is the cone over the face $A_\beta \cap \Pi'$ of Π, which is nonempty since $V_\beta \cap C' \neq \varnothing$. The set $R(V_\beta) \cap \Xi(C')$ consists of the cusps generated by the vertices of $A_\beta \cap \Pi'$. □

Now we construct a cycle that serves as a step between $\sigma(\mathbf{x})$ and ξ^V.

4.7. Lemma. *The fan $F(\mathbf{x})$ induces a cycle $\xi' \in \mathbf{Z}^R_{n-1}$ with $[\sigma(\mathbf{x})] = [\xi']$ in $H_{n-1}(\tilde{Y}, \partial\tilde{Y})$.*

Proof. Essentially, we use $F(\mathbf{x})$ to refine $\sigma(\mathbf{x})$. From each 1-cone $\rho \in F(\mathbf{x})$, choose a nonzero $x_\rho \in V(\mathbb{Q})$. Then for each n-cone $\sigma_\alpha \in F(\mathbf{x})_n \smallsetminus F(\mathbf{x})_{n-1}$, use the points $\{x_\rho \mid \rho \subset \sigma_\alpha\}$ to construct n-tuples $\mathbf{x}_\alpha$ as in Step 3 of Theorem 4.4. These tuples generate pointed simplicial cones $\sigma(\mathbf{x}_\alpha)$, and we then take

$$\xi' = \sum_{\sigma_\alpha \in F(\mathbf{x})_n} \sigma(\mathbf{x}_\alpha).$$

□

For the cycle ξ^V to lie in $\mathbf{Z}^V_{n-1}$, we must show that the cones constructed in Step 3 are Voronoĭ cones.

4.8. Lemma. *Each $\tau(\mathbf{y}_\alpha)$ is a Voronoĭ cone.*

Proof. Let $\sigma_\alpha \in F(\mathbf{x})_n$ and $\tau(\mathbf{y}_\alpha)$ be as in Step 3. Each cusp v_β used to construct $\tau(\mathbf{y}_\alpha)$ is the spanning ray of a Voronoĭ cone V_β that induced a 1-cone $\rho_\beta \subset \sigma_\alpha$. Since ρ_β ranges over the 1-cones with $\rho_\beta \subset \sigma_\alpha$, it follows that $V_\beta \subseteq V_\alpha$. Thus all the v_β lie in $R(V_\alpha)$. Since V_α is a simplicial cone, the result follows. □

4.9. Lemma. *The class in $H_{n-1}(\tilde{Y}, \partial\tilde{Y})$ generated by*

$$\xi^V = \sum_{\sigma_\alpha \in F(\mathbf{x})_n} \tau(\mathbf{y}_\alpha)$$

is equal to $[\sigma(\mathbf{x})]$.

Proof. We will construct a chain $\eta \in \mathbf{C}^R_n$ so that $\partial\eta = \sigma(\mathbf{x}) - \sum \tau(\mathbf{y}_\alpha) + \mu$, where $\operatorname{supp}(\mu)$ consists of cones lying in $\tilde{C} \smallsetminus C$.

Consider the fan of simplicial cones $F(\mathbf{x})$. By taking the intersection of a generic affine hyperplane H with $F(\mathbf{x})$, we obtain a convex union of $(n-1)$-simplices in H, which we denote by P.

Now consider the product $P \times \Delta_1$. This can be realized as the convex union of a set of simplicial prisms in some affine space $H' \supset H$. (See Figure 2.) We subdivide $P \times \Delta_1$ into simplices without adding new vertices, and call the resulting union P'.

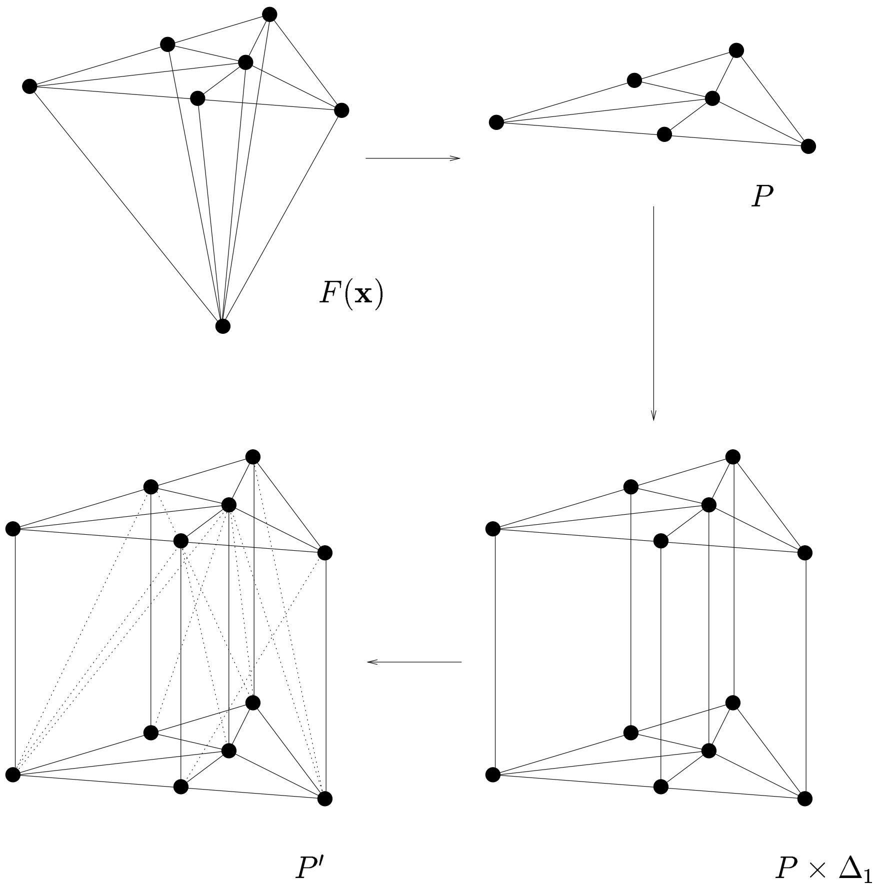

FIGURE 2.

Next, we linearly map P' to $\tilde{C}$ as follows. On the upper face, we take the vertices of P' to the vertices of Π that give the pointed structure to $\sum \tau(\mathbf{y}_\alpha)$. On the lower face of P', we take the vertices to the points $\{x_\rho\}$ in Lemma 4.7 that give the pointed structure to the cycle $\sum \sigma(\mathbf{x}_\alpha)$. This defines a collection of pointed $(n+1)$-cones in $\tilde{C}$, and hence defines a chain η.

From the construction, we have $\partial\eta = \sigma(\mathbf{x}) - \sum \tau(\mathbf{y}_\alpha) + \mu$, where μ is a chain induced by the outer sides of P'. To finish the proof, we must show that $\operatorname{supp}(\mu) \subset \tilde{C} \smallsetminus C$.

To see this, let σ' be a maximal proper face of $\sigma(\mathbf{x})$, and let $P(\sigma')$ be the subset of P' corresponding to $\sigma' \times \Delta_1$. By Proposition 3.16, we have $\sigma' \subset \tilde{C} \smallsetminus C$. In fact, σ' determines a unique proper rational boundary component C' such that $\sigma' \subset C'$.

Any 1-cone in $F(\mathbf{x})$ that meets σ' is a subset of σ', and hence also lies in C'. This means that the intersection of $P(\sigma')$ with the lower face of P' determines a collection of cones lying in C'.

Now consider the portion of $P(\sigma')$ lying in the upper face of P'. The 1-cones determined by this subset include the spanning rays of σ' as before, as well some cusps used in the construction of the $\mathbf{y}_\alpha$. By Proposition 3.8 and the restrictions placed on these cusps in Step 2 of Theorem 4.4, these cusps also lie in C'. Hence all the cones in the image of $P(\sigma')$ lie in C'. Applying this argument to all maximal proper faces of $\sigma(\mathbf{x})$ completes the proof.

□

Finally we complete the proof of the theorem.

Proof of Theorem 4.4. By Lemma 4.6, we may choose the cusps in Step 1 without obstruction. Lemmas 4.7 and 4.9 imply that the cycle ξ^V from Step 3 satisfies $[\xi] = [\xi^V]$, and Lemma 4.8 implies that ξ^V is supported on Voronoĭ cones. □

4.10. Now we modify our algorithm to work on cycles $\xi \in \mathbf{Z}_k^{\mathscr{H}}$, where $k > n-1$. The basic construction is the same, but there is one crucial difference. When $k = n-1$, the boundary of any $\xi = \sum n(\mathbf{x})\sigma(\mathbf{x})$ automatically lies in $\tilde{C}$. But for ξ to be a relative cycle when $k > n-1$, its boundary can meet $\tilde{C} \smallsetminus C$; this part must vanish mod Γ. This means that we must be vigilant in our construction to ensure that the choices in Theorem 4.4 and Lemmas 4.6–4.9 can be made equivariantly over ξ.

4.11. Theorem. *Let $\xi \in \mathbf{Z}_k^{\mathscr{H}}$ with $k > n-1$. Then the choices in Theorem 4.4 can be made Γ-equivariantly. Specifically,*

1. *The fans $\{F(\mathbf{x}) \mid \sigma(\mathbf{x}) \in \operatorname{supp}(\xi)\}$ can be constructed Γ-equivariantly, yielding a fan $F(\xi)$.*
2. *For any two 1-cones $\rho, \rho' \in F(\xi)$ satisfying $\gamma\cdot\rho = \rho'$ with $\gamma \in \Gamma$, the corresponding cusps v and v' can be chosen in Step 2 to satisfy $\gamma \cdot v = v'$.*

Furthermore, the construction of P' from $P \times \Delta_1$ in Lemma 4.9 can be performed Γ-equivariantly over all of $\operatorname{supp}(\xi)$. Thus, the algorithm in Theorem 4.4, with the modifications above, can be used to construct a cycle $\xi^V \in \mathbf{Z}_k^V$ satisfying $[\xi] = [\xi^V]$.

Proof. The first assertion is the key, for if the fans $F(\mathbf{x})$ can be constructed Γ-equivariantly, then any construction using them can be done Γ-equivariantly by choosing representatives for the Γ-orbits.

Now the intersection of $\operatorname{supp}(\xi)$ with $\mathscr{V}$ consists of a finite set of simplicial cones, so a fortiori there are only a finite number of such cones modulo Γ. This means that we may adapt the proof of Proposition 3.6 and construct the refinements using induction on the dimension. □

5. Second Algorithm

In this section we show how to transform cycles in $\mathbf{Z}_*^{\mathscr{H}}$ to cycles in $\mathbf{Z}_*^V$ without explicitly constructing the fans $F(\mathbf{x})$ from §4.1. Instead, we assume only the existence of an "oracle" that answers the following question:

Given a point $x \in \tilde{C}$, which Voronoĭ cone contains x?

This oracle for $x \in C$ is the *Voronoĭ reduction algorithm* [24][13]. Since the rational boundary components are self-adjoint homogeneous cones of lower rank, and they receive a Voronoĭ decomposition by Π (Proposition 3.8), we are justified in assuming the existence of this oracle on both C and its boundary components. Furthermore, in practice the oracle is not difficult to implement, and requires only slightly more information than that necessary for cohomology computations with $\mathscr{V}$.[4] Accordingly, we define

5.1. Definition. Given a point $x \in \tilde{C}$, let $S(x)$ be the set of cusps that span the unique smallest Voronoĭ cone containing x. By abuse of notation, we also write $S(\rho)$ for a 1-cone ρ.

The idea behind our second algorithm is this. Although it is expensive to construct the fans $F(\mathbf{x})$, it is easy to subdivide a lift $\xi \in \mathbf{Z}_*^{\mathscr{H}}$ into a cycle ξ' supported on smaller pointed simplicial cones. If the top-dimensional cones in ξ' are small enough, then any two nearby 1-cones $\rho_1, \rho_2 \in \operatorname{supp}(\xi')$ will lie in the same or adjacent cones of $F(\mathbf{x})$. Thus $S(\rho_1) \cap S(\rho_2)$ will be nonempty. Hence we can hope to choose cusps $v_i \in S(\rho_i)$ and assemble them into Voronoĭ cones as in Theorems 4.4 and 4.11 to build a cycle in $\mathbf{Z}_*^V$. This is the approach we take, but our task is complicated by two issues:

- For efficiency, we want to subdivide as little as possible.
- All choices must be made Γ-equivariantly.

To address these issues, we introduce *sufficiently fine decompositions* of a fan with respect to $\mathscr{V}$ (§5.4) and the *constrained barycentric subdivision* of a fan with respect to a subfan (§5.2). Theorem 5.12 describes an algorithm that constructs sufficiently fine decompositions, and Theorem 5.15 shows how to use such a decomposition to transform a cycle in $\mathbf{Z}_*^{\mathscr{H}}$ to a cycle in $\mathbf{Z}_*^V$ in the same homology class.

5.2. Let $\Delta = \Delta_k$ be the standard k-simplex (§3.12), and let $[\![k]\!]$ be the finite set $\{0, 1, \ldots, k\}$. There is a bijection between faces of Δ and subsets of $[\![k]\!]$: the subset $I \subset [\![k]\!]$ corresponds to the convex hull of $\{e_i \mid i \in I\}$.

Recall that the *barycentric subdivision* of Δ is the simplicial complex with vertices corresponding to nonempty subsets of $[\![k]\!]$, and with i-faces corresponding to proper flags of length $i+1$:

$$\varnothing \subsetneqq I_0 \subsetneqq \cdots \subsetneqq I_i \subsetneqq [\![k]\!].$$

We denote the barycentric subdivision by $\mathscr{B}(\Delta)$. Using the concrete description of Δ given in §3.12, we may realize the vertex of $\mathscr{B}(\Delta)$ corresponding to $I \subset [\![k]\!]$ by the point $(\sum_{i \in I} e_i)/\#I$.

There is another approach to the barycentric subdivision using the *stellar subdivision* $\mathscr{S}(\Delta)$. As an abstract simplicial complex, $\mathscr{S}(\Delta)$ is isomorphic to the $(k+1)$-simplex. In the embedding of Δ given in §3.12, $\mathscr{S}(\Delta)$ is the image of the affine map

[4] See [15] for implementation in the case of $SL_2(\mathbb{Z})$.

$\Delta_{k+1} \to \Delta_k$ that takes $e_{k+1} \mapsto (\sum_{i \in [\![k]\!]} e_i)/(k+1)$ and is the identity on the other vertices (Figure 3). The image of e_{k+1} is called the *star point*.

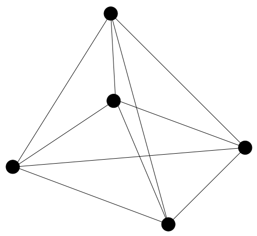

FIGURE 3. The stellar subdivision of Δ_3.

The barycentric subdivision can now be constructed as follows. First construct $\mathscr{S}(\Delta)$. Then stellar subdivide the original $(k-1)$-faces of Δ, and take the cone of these faces using the star point of $\mathscr{S}(\Delta)$ as the cone point. Continue subdividing the lower-dimensional faces of the original Δ, each time coning with the previously constructed star points. After the 1-faces have been subdivided, the result is $\mathscr{B}(\Delta)$ (Figure 4).

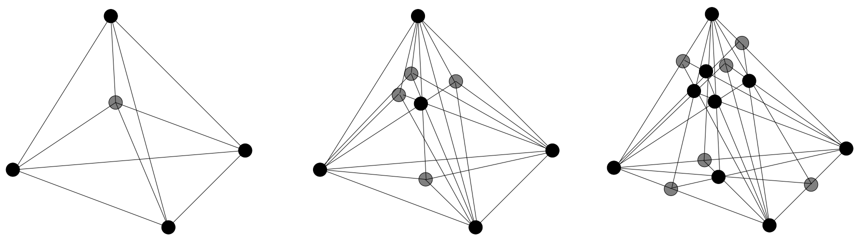

FIGURE 4. Using the stellar subdivision to construct $\mathscr{B}(\Delta)$. The grey vertices are added at each step.

Now let $\cup\Delta_\alpha$ be a closed union of proper faces of Δ. We define the *relative barycentric subdivision* $\mathscr{B}(\Delta, \cup\Delta_\alpha)$ as follows. We construct the sequence of stellar subdivisions above, but do not subdivide any simplices in $\cup\Delta_\alpha$ (Figure 5).

Both the barycentric subdivision and the relative barycentric subdivision can be iterated, and we denote the resulting simplicial complexes by $\mathscr{B}^i(\Delta)$ respectively $\mathscr{B}^i(\Delta, \cup\Delta_\alpha)$.

Let $\sigma(\mathbf{x})$ be a pointed simplicial cone. Then the barycentric subdivision of $\sigma(\mathbf{x})$ is the fan $\mathscr{B}(\sigma(\mathbf{x}))$ constructed by barycentrically subdividing the simplex Δ spanned by the tuple $\mathbf{x}$. We denote the 1-cone on the barycenter of Δ by β_σ. We may similarly construct the relative barycentric subdivision of $\sigma(\mathbf{x})$ with respect to a union of proper faces. Finally, we may apply either construction to a fan of pointed simplicial cones by applying it to each simplex separately.

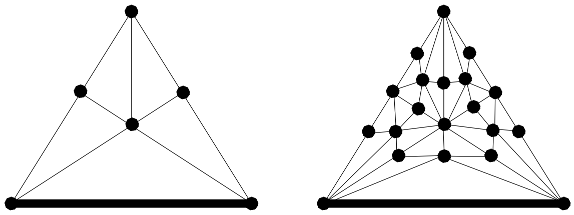

FIGURE 5. $\mathscr{B}^1(\Delta, \cup\Delta_\alpha)$ and $\mathscr{B}^2(\Delta, \cup\Delta_\alpha)$. The heavy line represents $\cup\Delta_\alpha$.

5.3. *Remark.* Elementary arguments show that the set of vertices in $\mathscr{B}^i(\Delta, \cup\Delta_\alpha)$ becomes dense in Δ as $i \to \infty$, no matter what proper subcomplex $\cup\Delta_\alpha$ we choose.

5.4. We want to formalize how much a given fan must be subdivided for our purposes. Recall that $R(\tau)$ denotes the set of spanning rays of a cone τ, and that if $\rho \subset \tilde{C}$ is a 1-cone, then $S(\rho)$ denotes the set of spanning rays of the unique smallest Voronoĭ cone containing ρ.

5.5. Definition. Let $\sigma \subset \tilde{C}$ be a d-dimensional simplicial cone, and let Σ be a finite simplicial fan with $\bigcup_{\tau\in\Sigma} \tau = \sigma$. Then the fan Σ is said to be a *sufficiently fine decomposition of* σ if the following is true: for every d-cone $\tau \in \Sigma$, we have

$$S(\tau) := S(\beta_\tau) \cap \bigcap_{\rho\subset R(\tau)} S(\rho) \neq \varnothing. \tag{3}$$

More generally, we can speak of a sufficiently fine decomposition of a set of d-cones. In this case, we mean that each cone in the set has been refined into a fan whose d-cones satisfy (3).

5.6. Example. Suppose that σ is a 2-cone, i.e. σ is a cone on an interval I. Then a partition of I into intervals $[x_i, x_{i+1}]$ induces a sufficiently fine decomposition of σ if x_i and x_{i+1} lie in adjacent or the same Voronoĭ cones. (The condition on barycenters is automatic in this case.) This is equivalent to the notion of sufficiently fine for modular symbols for $\mathbb{Q}$-rank one groups presented in [13].

5.7. We want to discuss the relationship between sufficiently fine decompositions and the canonical fans of §4.1. We begin by introducing an open covering of a fan.

5.8. Definition. Let F be a fan. Given any $\sigma \in F$, let $U_\sigma(F)$ be the open star of σ in F. In other words,

$$U_\sigma(F) = \bigcup_{\substack{\sigma\subseteq\sigma' \\ \sigma'\in F}} \operatorname{Int}\sigma'.$$

Let $\mathscr{U}(F)$ be $\{U_\sigma(F) \mid \sigma \in F\}$.

Notice that if $\sigma \subset \sigma'$, then $U_\sigma(F) \supset U_{\sigma'}(F)$. Also, if $F' \subset F$ is a subfan (a subset of F that is also a fan), then

$$U_\sigma(F') = U_\sigma(F) \cap F'.$$

In general the sets in $\mathscr{U}(F)$ are not convex.

Let $F(\sigma)$ be the canonical fan $\sigma \cap \mathscr{V}$, with no simplicial refinement.

5.9. Proposition. *Let σ be a simplicial d-cone, and let Σ be a simplicial fan that is a refinement of σ. Let $\tau \in \Sigma$, and let β_τ be the barycenter of τ. Then τ satisfies* (3) *if and only if there is a set $U \in \mathscr{U}(F(\sigma))$ such that $R(\tau) \cup \{\beta_\tau\} \subset U$.*

Proof. Assume τ satisfies (3). Then there is a Voronoĭ cone V_α such that $S(\tau) \subset R(V_\alpha)$. This Voronoĭ cone induces $\sigma_\alpha \in F(\sigma)$. The open star U_{σ_α} of σ_α in the canonical fan is the intersection of σ with the open star of V_α in $\mathscr{V}$. Then $R(\tau) \cup \{\beta_\tau\}$ is contained in U_{σ_α}. The converse is obtained by simply reversing this argument. □

A consequence of the preceding Proposition is that sufficiently fine decompositions exist for any rational simplicial cone in $\tilde{C}$.

5.10. Proposition. *Let σ be a rational simplicial d-cone. Then the fan $\mathscr{B}^i(\sigma)$ gives a sufficiently fine decomposition of σ for $i >> 0$.*

Proof. By Proposition 5.9, any fan subordinate to $\mathscr{U}(F(\sigma))$ gives a sufficiently fine decomposition of σ. Hence the statement follows immediately from the fact that $\mathscr{U}(F(\sigma))$ is finite. □

5.11. We now describe an algorithm that constructs sufficiently fine decompositions of a given set of cones. Recall that if Σ is a set of cones, then we let $\Sigma_d = \{\sigma \in \Sigma_d \mid \dim \sigma \leq d\}$.

5.12. Theorem. *Let $\xi \in \mathbf{Z}_k^{\mathscr{H}}$, and let $\Sigma = \operatorname{supp}(\xi)$. Then the following algorithm constructs a sufficiently fine decomposition Σ' of Σ:*

1. *Set $\Sigma'_1 = \Sigma_1$, and set $j = 1$.*
2. *Assume that Σ'_j has been constructed for $1 \leq j \leq d$, where $d < k$. Let $\bar{\Sigma}$ be any Γ-equivariant extension of Σ'_j to Σ_{j+1} without adding new 1-cones (see Figure 6). Then for some $i > 0$, the fan given by the relative barycentric subdivision $\mathscr{B}^i(\bar{\Sigma}, \Sigma'_j)$ is sufficiently fine, and can be taken for Σ'_{j+1}.*
3. *If $j + 1 = k$, terminate. Otherwise, increment j and return to Step 2.*

Proof. We must verify two facts. First we claim that $\bar{\Sigma}$ can be constructed Γ-equivariantly. This follows because each of the sets $\Sigma_{j+1} \smallsetminus \Sigma_j$ is finite modulo Γ, and so we may subdivide a set of representatives and translate by Γ, as in Proposition 3.6.

Now we claim that $\mathscr{B}^i(\bar{\Sigma}, \Sigma'_j)$ will be sufficiently fine for $i \gg 0$. If we were performing the usual barycentric subdivision, then by Proposition 5.10 we would succeed.

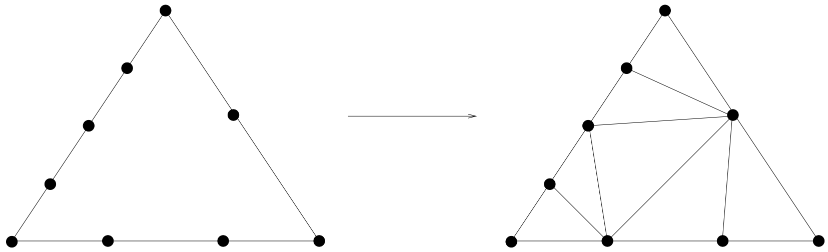

FIGURE 6. An affine slice of the extension of Σ_1' to $\bar{\Sigma}$.

We must show using the relative barycentric subdivision with respect to $\bar{\Sigma}_j$ poses no obstruction.

So let $\tau \in \Sigma_j'$ be a j-cone that appears in the refinement of a given $\sigma \in \Sigma_j$. Let β_τ be the barycenter of τ. Since Σ_j' is sufficiently fine, Proposition 5.9 implies that there exists a set $U_\alpha \in \mathscr{U}(F(\sigma))$, the open star of $\sigma_\alpha \subset F(\sigma)$, such that

$$R(\tau) \cup \{\beta_\tau\} \subset U_\alpha.$$

(See Figure 7. In this and the following figure, we depict a generic affine slice of these objects.)

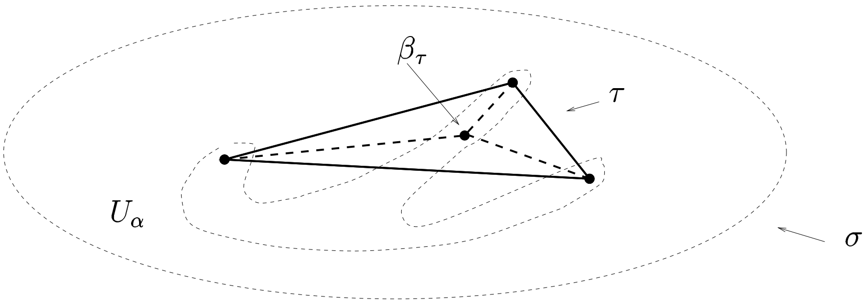

FIGURE 7. U_α and τ in σ.

Suppose that σ appears as a face of a simplex $\sigma' \in \Sigma_{j+1}$. Then the canonical fan $F(\sigma)$ is a subfan of $F(\sigma')$, which implies $\sigma_\alpha \subset F(\sigma')$. Moreover, U_α is the intersection of σ with the set $U_\alpha(F(\sigma')) \subset \mathscr{U}(F(\sigma'))$. We can picture $U_\alpha(F(\sigma'))$ as a "thickening" of U_α into σ'.

Now extend Σ_j' to $\bar{\Sigma}$, and begin iterating the constrained barycentric subdivision. Away from $\partial\sigma'$, the new 1-cones will fill σ' densely, and by Proposition 5.10 the decomposition will become sufficiently fine there. The only possible problem is near $\partial\sigma'$. Over each $\tau \in \partial\sigma'$, the new 1-cones of $\mathscr{B}^i(\bar{\Sigma}, \Sigma_j')$ will converge to β_τ. In Figure 8 we show the affine slices of these 1-cones as grey dots. By Remark 5.3, these new 1-cones come arbitrarily close to β_τ and eventually enter $U_\alpha(F(\sigma'))$. By Proposition 5.9, this implies $\mathscr{B}^i(\bar{\Sigma}, \Sigma_j')$ will be sufficiently fine for some $i >> 0$.

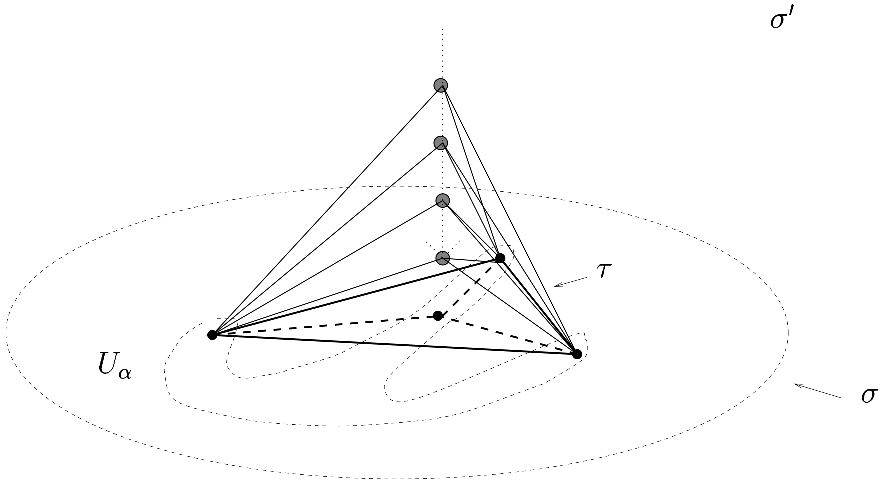

FIGURE 8. The 1-cones converge to β_τ and enter $U_\alpha(F(\sigma')) \supset U_\alpha$.

□

5.13. *Remark.* One might think that the set $S(\beta_\tau)$ could be omitted from (3) in the definition of sufficiently fine decomposition. However, since the $U_\alpha \in \mathscr{U}(F(\sigma))$ are in general non-convex, the fact that $R(\tau) \subset U_\alpha$ does not imply $\beta_\tau \subset U_\alpha$. We need $\beta_\tau \in U_\alpha$ in the proof above to ensure that sequence of 1-cones in Figure 8 enters $U_\alpha(F(\sigma'))$.

5.14. We conclude the paper by showing how a sufficiently fine decomposition as in Theorem 5.12 may be used to transform a cycle in $\mathbf{Z}_*^{\mathscr{H}}$ to a cycle in $\mathbf{Z}_*^V$ giving the same homology class in $H_*(\tilde{Y}, \partial\tilde{Y})$.

5.15. Theorem. *Let $\xi = \sum n(\mathbf{x})\sigma(\mathbf{x}) \in \mathbf{Z}_k^{\mathscr{H}}$ be a Hecke image, and let $F(\xi)$ be a Γ-equivariant sufficiently fine decomposition of* $\operatorname{supp}(\xi)$. *Let $\mathscr{B}(F(\xi))$ be the barycentric subdivision of $F(\xi)$. Then the following algorithm constructs a cycle $\xi^V \in \mathbf{Z}_k^V$ satisfying $[\xi] = [\xi^V]$.*

1. *Each 1-cone $\rho_I \in \operatorname{supp}(\mathscr{B}(\xi))$ is the barycenter of a set $\{\rho_i\}_{i\in I}$ of* 1*-cones of $F(\xi)$. For each ρ_I, choose a cusp $v_I \in \bigcap_{i\in I} S(\rho_i)$. Make these choices Γ-equivariantly over $\mathscr{B}(F(\xi))$. Let y_I be the vertex of* Π *generating v_I.*
2. *Each k-cone $\tau_\alpha \in \operatorname{supp}(\mathscr{B}(\xi))$ corresponds to a flag*

$$\{\rho_1\} \subset \{\rho_1, \rho_2\} \subset \cdots \subset \{\rho_1, \dots, \rho_k\}$$

where each ρ_i is a 1*-cone in* $\operatorname{supp}(F(\xi))$ *(cf. Figure 9). For each such flag, assemble the points*

$$y_{\{1\}}, y_{\{1,2\}}, \dots, y_{\{1,\dots,k\}}$$

into a k-tuple $\mathbf{y}_\alpha$, using the orientation on τ_α. Form rational pointed cones $\tau(\mathbf{y}_\alpha)$ using these tuples.

Then the desired cycle is

$$\xi^V = \sum_{\tau_\alpha \in \mathscr{B}(F(\xi))_k} \tau(\mathbf{y}_\alpha).$$

Proof. Let I be a set indexing a 1-cone $\rho_I \in \mathrm{supp}(\mathscr{B}(F(\xi)))$. Then the definition of sufficiently fine implies that the set $\bigcap_{i\in I} S(\rho_i)$ is nonempty. Hence we may select a cusp v_I. Also, the cusps v_I can be chosen Γ-equivariantly over all of $\mathrm{supp}(\mathscr{B}(F(\xi)))$ by applying a neatness argument similar to Proposition 3.6.

Next, that ξ^V is homologous to ξ follows from arguments identical to those presented in Lemmas 4.7 and 4.9.

What remains to be shown is that the cones $\tau(\mathbf{y}_\alpha)$ are Voronoĭ cones. This follows since the cusps

$$v_{\{1\}}, v_{\{1,2\}}, \dots, v_{\{1,\dots,k\}}$$

are all spanning rays of the Voronoĭ cone containing ρ_1. (See Figure 10 for an example when $k = 2$ and $\Gamma \subset SL_2(\mathbb{Z})$.) Since this Voronoĭ cone is simplicial, this completes the proof of the theorem. □

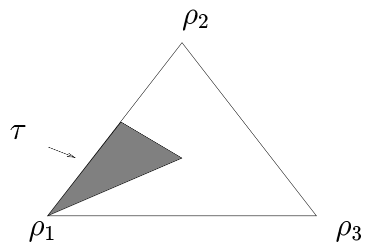

FIGURE 9. τ corresponds to the ordered triple (ρ_1, ρ_2, ρ_3).

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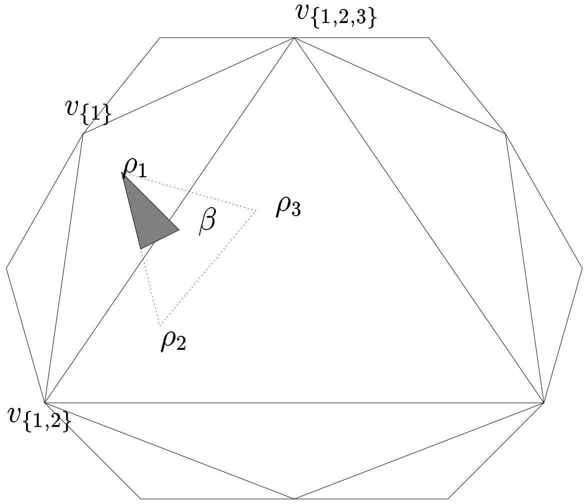

FIGURE 10. An affine slice of part of $\mathscr{V}$ in the case $\Gamma \subset SL_2(\mathbb{Z})$. The shaded triangle is $\tau(\mathbf{y}_\alpha)$.

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