SOME ELEMENTARY RAMANUJAN GRAPHS

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ABSTRACT. We give elementary constructions of two infinite families of Ramanujan graphs of unbounded degree. The first uses the geometry of buildings over finite fields, and the second uses triangulations of modular curves.

1. INTRODUCTION

Ramanujan graphs were defined by Lubotzky, Phillips, and Sarnak as graphs that have optimal spectra of their adjacency matrices; we refer to §2 for the definition. In [7–9] they gave explicit constructions of infinite families of fixed-degree Ramanujan graphs by taking arithmetic quotients of the SL_2 Bruhat-Tits tree. The eigenvalues of the adjacency matrices are the eigenvalues of Hecke operators acting on certain automorphic forms, and the condition on spectra becomes the Ramanujan-Petersson bound on the eigenvalues of cuspidal Hecke eigenforms. Hence the construction of these graphs uses many heavy number-theoretic machines, in particular the Jacquet-Langlands correspondence, and Deligne's proof of the Petersson conjecture for holomorphic cusp forms.

In this article we address a related but much simpler problem than the one solved by Lubotzy, Phillips, and Sarnak. Namely, we construct infinite families of Ramanujan graphs of *unbounded* degree. Families of such graphs have been constructed by a variety of authors, see for example [5, 10, 16]. We construct two basic families of such graphs. The first, in §3, uses the geomety of finite vector spaces. The second, in §4, uses certain triangulations of modular curves. In both cases we can directly compute the spectra of our graphs by exhibiting eigenvectors of the adjacency matrices, and can immediately verify the Ramanujan property. No deep estimates of Hecke eigenvalues or exponential sums are needed as in [5, 10, 16]; instead, the basic tools are geometry of linear spaces and some complex representation theory of finite groups. Finally, in §5, we show how to use our graphs and a simple "splitting" construction to construct four more families of Ramanujan graphs.

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2. Definitions and notation

We begin by fixing notation. Let $G = (V, E)$ be a connected undirected graph with vertex set V and edge set $E \subset V \times V$. We write $v \sim w$ if either (v, w) or $(w, v) \in E$, and assume that G has no loops or multiple edges. We also assume that G is k-regular for some integer $k \geq 2$; that is, any vertex is connected by an edge to exactly k vertices.

Let S_V be the complex vector space of functions $f\colon V \to \mathbb{C}$ equipped with the canonical basis $\{e_v \mid v \in V\}$, where $e_v(v) = 1$ and $e_v(w) = 0$ if $w \neq v$. For any subset $W \subset V$, we identify S_W with the subspace of S_V generated by $\{e_v \mid v \in W\}$.

The *adjacency operator* $A_G\colon S_V \to S_V$ is the linear transformation defined by

$$(A_G f)(v) = \sum_{w\sim v} f(w).$$

The *spectrum* of G is the set of eigenvalues of A_G, and is denoted Λ_G. Note that $\Lambda_G \subset \mathbb{R}$ since A_G is represented by a symmetric matrix on the basis $\{e_v\}$. With our assumptions on G, we have the following well known facts about Λ_G:

(1) $\Lambda_G \subset [-k, k]$.
(2) The eigenvalue k occurs with multiplicity 1.
(3) The eigenvalue $-k$ occurs if and only if G is bipartite, and if $-k$ occurs it does so with multiplicity 1.

Definition 2.1. A k-regular graph is called *Ramanujan* if $|\lambda| \leq 2\sqrt{k-1}$ for all $\lambda \in \Lambda_G$ with $|\lambda| \neq k$.

For an explanation of the terminology of Definition 2.1, as well as constructions and practical applications of these graphs, we refer to [6–9, 11].

3. Ramanujan graphs from finite vector spaces

Let p be a prime, let $q = p^f$, and let $\mathbb{F}_q$ be the finite field with q elements. We will work with the finite vector space $\mathbb{F}_q^n$, $n \geq 2$. For any integer k with $1 \leq k \leq n-1$, let $G_{k,n}$ be the cardinality of the finite Grassmannian $\mathrm{Gr}(k, n)(\mathbb{F}_q)$ of dimension k subspaces in $\mathbb{F}_q^n$. It is well known that

$$G_{k,n} = \frac{[n]_q!}{[k]_q![n-k]_q!},$$

where $[n]_q = (q^n-1)/(q-1)$ and $[n]_q! = \prod_{i=1}^n [i]_q$. In particular, $G_{k,n}$ is a polynomial in q of degree $k(n-k)$, the dimension of $\mathrm{Gr}(k, n)$. It is convenient to put $G_{0,n} = G_{n,n} = 1$.

Definition 3.1. For each integer k with $1 \leq k < n-k \leq n-1$, define a graph $B_q(k, n-k)$ as follows. The vertex set V is $V_k \cup V_{n-k}$, where V_i denotes the set of subspaces of $\mathbb{F}_q^n$ of dimension i. The only edges are between vertices in V_k and V_{n-k}, and $v \in V_k$ is joined to $w \in V_{n-k}$ if and only if v is a subspace of w.

By construction $B_q(k, n-k)$ is bipartite, and it is not hard to verify that $B_q(k, n-k)$ is connected. The duality on $\mathbb{F}_q^n$ exchanging subspaces of dimension and codimension k implies that $B_q(k, n-k)$ is regular, with degree $G_{k,n-k}$. For $n = 3$, $k = 1$, the graph $B_q(1, 2)$ is the rank 2 spherical building of type A over $\mathbb{F}_q$, whereas for $n > 2$ the graph is a subcomplex of this building. We have $\#V_k = \#V_{n-k} = G_{k,n}$.

Theorem 3.2. *The graph $B_q(1, n-1)$ has the following spectrum:*

(1) $\pm(q^{n-1} - 1)/(q - 1)$*, each with multiplicity* 1*;*
(2) $\pm\sqrt{q^{n-2}}$*, each with multiplicity*

$$(q^n - 1)/(q - 1) - 1.$$

Hence, since the graph $B_q(1, n-1)$ has degree $(q^{n-1} - 1)/(q - 1)$, it is Ramanujan for all q. For $k \geq 2$, the spectrum of $B_q(k, n-k)$ contains

$$\lambda = (G_{n-2k,n-k} - G_{n-2k-1,n-k-1})q^{-(n-2k)/2},$$

and thus if q is sufficiently large $B_q(k, n-k)$ cannot be Ramanujan.

Proof. We verify the claims about the spectra by constructing some explicit eigenvectors of the adjacency matrix $A = A_{B_q(k,n-k)}$.

Let $a, b \in V_1$ be two distinct lines, and for any k let $H_k(a)$ (respectively $H_k(a, b)$) be the set of subspaces of dimension k containing a (resp. containing a, b). Define the function $\Phi^+_{a,b} \in S_V$ by

(1) $\Phi^+_{a,b}(v) = 0$ if $v \in H_k(a, b) \cup H_{n-k}(a, b)$,
(2) $\Phi^+_{a,b}(v) = 1$ if $v \in H_k(a) \smallsetminus H_k(a, b)$,
(3) $\Phi^+_{a,b}(v) = -1$ $v \in H_k(b) \smallsetminus H_k(a, b)$,
(4) $\Phi^+_{a,b}(w) = q^{-(n-2k)/2}$ if $w \in H_{n-k}(a) \smallsetminus H_{n-k}(a, b)$,
(5) $\Phi^+_{a,b}w) = -q^{-(n-2k)/2}$ if $w \in H_{n-k}(b) \smallsetminus H_{n-k}(a, b)$,
(6) $\Phi^+_{a,b}(x) = 0$ for all other vertices x.

We can similarly define a vector $\Phi^-_{a,b}$ by changing the signs on $q^{-(n-2k)/2}$ in (4) and (5).

Figure 1 shows such a vector $\Phi^+_{a,b}$ for $B_2(1, 2)$; we picture $\Phi^+_{a,b}$ by labeling points and lines in $\mathbb{P}^2(\mathbb{F}_2)$. The black points correspond to a, b and are mapped respectively to 1 and -1 under $\Phi^+_{a,b}$. The two lines through a and not b are taken to $1/\sqrt{2}$, and the two lines through b and not a are taken to $-1/\sqrt{2}$. The function $\Phi^+_{a,b}$ vanishes on any unlabeled subspace.

Now fix a, b and write Φ^+ for $\Phi^+_{a,b}$. We claim Φ^+ is an eigenvector of A with eigenvalue λ. To ease notation, in what follows we write μ for $G_{n-2k,n-k} - G_{n-2k-1,n-k-1}$ and ν for $q^{-(n-2k)/2}$.

First we consider $A\Phi^+$ on dimension k subspaces. Let $v \in H_k(a) \smallsetminus H_k(a, b)$. Then v sits on $G_{n-2k,n-k}$ subspaces of codimension k. On $G_{n-2k-1,n-k-1}$ of these Φ^+ vanishes,

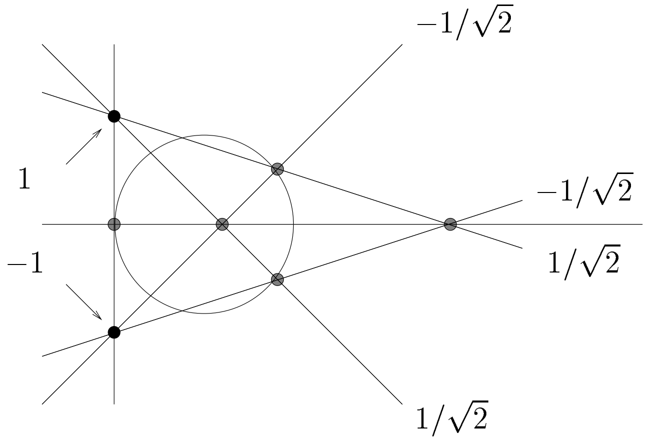

FIGURE 1. An eigenvector in $B_2(1,2)$

and on the remaining μ of them Φ^+ takes the value ν. Hence $(A\Phi^+)(v) = \mu\nu = \lambda$. Similarly, if $v \in H_k(b) \smallsetminus H_k(a,b)$, then $(A\Phi^+)(v) = -\lambda$.

If $v \in H_k(a,b)$, then clearly $(A\Phi^+)(v) = 0$. The only other dimension k subspaces possibly receiving a nonzero contribution are those $v \notin H_k(a) \cup H_k(b)$ that lie in a codimension k subspace $w \in H_{n-k}(a) \cup H_{n-k}(b)$. However any such v is contained in the same number of w from $H_{n-k}(a)$ as from $H_{n-k}(b)$, and thus $(A\Phi^+)(v) = 0$.

Next we consider the action of $A\Phi^+$ on codimenion k subspaces. Each $w \in H_{n-k}(a) \smallsetminus H_{n-k}(a,b)$ contains μ dimension k subspaces on which Φ^+ is nonzero and equals 1, and thus $(A\Phi^+)(w) = \mu$. Similarly if $w \in H_{n-k}(b) \smallsetminus H_{n-k}(a,b)$ then $(A\Phi^+)(w) = -\mu$. The only other codimension k subspaces possibly receiving a nonzero contribution from Φ^+ are those in $H_{n-k}(a,b)$, but each of these contains the same number of dimension k subspaces from $H_k(a)\smallsetminus H_k(a,b)$ as from $H_k(b)\smallsetminus H_k(a,b)$; thus $A\Phi^+$ vanishes on them. Hence Φ^+ is an eigenvector with eigenvalue λ. A similar discussion shows Φ^- is an eigenvector with eigenvalue $-\lambda$.

If $k \geq 2$, then the regularity of $B_q(k, n-k)$ is a polynomial in q of degree $k(n-2k)$, whereas $\lambda(q) = O(q^{(2k-1)(n-2k)/2})$. Hence in this case $B_q(k, n-k)$ cannot be Ramanujan if q is sufficiently large.

On the other hand if $k = 1$, then $\lambda = \sqrt{q^{n-2}}$, which satisfies the Ramanujan bound. To finish the proof, we must show that $\pm\lambda$ are the only interesting eigenvalues of A. This follows from the fact that if $\{x_i\}$ is any linearly indepedent subset of a vector space, then the span of the vectors $\{x_i - x_j\}$ has codimension one in the span of the x_i. If we let $W^\pm$ be the spans of $\{\Phi^\pm_{a,b}\}$ as a, b range over all pairs of distinct lines in $\mathbb{F}_q^n$, we see that $\dim W^+ = \dim W^- = (q^n - 1)/(q-1) - 1$. But $W^+ \cap W^- = \{0\}$, and by a dimension count the proof is complete. $\square$

Remark 3.3. It can happen that some $B_q(k, n-k)$ are Ramanujan if $k \geq 2$. For example, if $k = 2$ and $n = 5$, the spectrum consists of $\{\pm 1, \pm\lambda, \pm q\}$, and in fact $B_2(2,3)$ is Ramanujan.

Remark 3.4. There are other (nonbipartite) graphs one can define using incidence of linear spaces over finite fields. For example, given $1 \leq k < l \leq n-1$, one can let $I_q(k,l,n)$ be the graph with vertex set V_l and with two vertices v, v' joined by an edge if $\dim v \cap v' = k$. Since we have not studied such graphs in detail, we will only discuss the graph $I_q(1,2,n)$. In this case the regularity is $(q+1)(q^{n-2}+q^{n-3}+\cdots+q)$. Choose two distinct lines a, b, and define $\Phi = \Phi_{a,b}$ by $\Phi(H_2(a) \smallsetminus H_2(a,b)) = -\Phi(H_2(b) \smallsetminus H_2(a,b)) = 1$, and $\Phi = 0$ otherwise. Then Φ is an eigenvector with eigenvalue $q^{n-2} + q^{n-3} + \cdots + q^2 - 1$, and thus $I_q(1,2,n)$ cannot be Ramanujan if $q \gg 0$.

Remark 3.5. For $n = 2$, the graph $B_q(1,2)$ was originally studied by Tanner [15], who computed the spectra of all generalized N-gons. In fact, the results of [15] and Theorem 3.2 suggest a more general collection of graphs that may yield some bipartite Ramanujan graphs.

Let $\mathscr{D}$ be a connected Coxeter diagram associated to a finite Weyl group, that is, a diagram of type $A_n, \ldots, G_2$. Associated to $\mathscr{D}$ and $\mathbb{F}_q$ is the spherical building $\mathscr{B}(\mathbb{F}_q)$. This is a finite simplicial complex with a surjective map from its vertices to the vertices of $\mathscr{D}$ such that the vertices of any top-dimensional simplex of $\mathscr{B}(\mathbb{F}_q)$ have distinct images.

Suppose that $\mathscr{D}$ admits an involution $\theta\colon \mathscr{D} \to \mathscr{D}$. This means $\mathscr{D}$ is of type A_n, $B_2 = C_2$, D_n, E_6, F_4, or G_2. Then if two vertices x, y of $\mathscr{D}$ are interchanged by θ, we can form a graph $\mathscr{B}_{x,y}(\mathbb{F}_q)$ by taking the 1-dimensional subcomplex of $\mathscr{B}(\mathbb{F}_q)$ whose vertices are the inverse images of x and y. Since x and y are exchanged by θ, the graph $\mathscr{B}_{x,y}(\mathbb{F}_q)$ will be regular and bipartite. For example, the graphs $B_q(k, n-k)$ arise by taking x and y to be the kth vertices from the left and right of $\mathscr{D} = A_{n-1}$. The cases $\mathscr{D} = A_2, B_2, G_2$ were investigated by Tanner.

One can ask which $\mathscr{B}_{x,y}(\mathbb{F}_q)$ are Ramanujan for all q. Theorem 3.2 contains the complete answer for type A. By explicit computation, one can show that for D_4 if one takes x, y to be any pair of the three degree one vertices, then $\mathscr{B}_{x,y}(\mathbb{F}_2)$ is Ramanujan. However we do not know the spectrum of this graph as $q \to \infty$.

4. Ramanujan graphs from modular curves

Let $\mathfrak{H} \subset \mathbb{C}$ be the upper halfplane, and let $\Gamma = PSL_2(\mathbb{Z})$. Then Γ acts on $\mathfrak{H}^* = \mathfrak{H} \cup \mathbb{Q} \cup \{\infty\}$ by linear fractional transformations. We define $G(\infty)$ to be the infinite graph with vertex set $\mathfrak{H}^* \smallsetminus \mathfrak{H}$, and with an edge between two vertices v, w if and only if there exists $\gamma \in \Gamma$ such that $\gamma \cdot 0 = v$, $\gamma \cdot \infty = w$. Note that the graph $G(\infty)$ has unbounded degree.

Definition 4.1. Let $N > 1$ be an integer, and let $\Gamma(N) \subset PSL_2(\mathbb{Z})$ be the subgroup of matrices congruent to the identity modulo N. The graph $G(N)$ is defined to be the quotient $\Gamma(N)\backslash G(\infty)$.

For example, if $N = 3, 4, 5$, the graph $G(N)$ is the 1-skeleton of respectively the tetrahedron, octahedron, and icosahedron. The graph $G(7)$ is the 1-skeleton of a beautiful triangulation of a surface of genus three investigated by Klein [4]. In general $G(N)$ is the 1-skeleton of a triangulation of the modular curve $X(N)$, and as such has appeared in the work of Brooks [1,2] and (essentially) Shabat-Voevodsky [13].

The graphs $G(N)$ can be described more explicitly as follows. Let $V(N) = \{(a,b) \in (\mathbb{Z}/N\mathbb{Z})^2 - (0,0) \mid gcd(a,b,N) = 1\}/\pm 1$, and let $E(N)$ be the set of all pairs $((a,b),(c,d)) \in V(N) \times V(N)$ such that $ad - bc = \pm 1$. Then $G(N)$ is isomorphic to the graph $(V(N), E(N))$. This description also shows that the finite group $L_N = PSL_2(\mathbb{Z}/N\mathbb{Z})$ acts on $G(N)$ through right multiplication on the vertex set $V(N)$. This action is transitive on $V(N)$, and the stabilizer of $(0,1)$ is the upper unipotent subgroup U. Hence $V(N)$ can be identified with $U\backslash L_N$, and the neighbors of any vertex Ux are the vertices $\{Ux\sigma \mid \sigma \in \Sigma\}$, where $\Sigma \subset L_N$ is any subset of N matrices with bottom rows $(1,j)$, $j = 0, \dots, N-1$. In other words, $G(N)$ is the Schreier graph of L_N with respect to the finite index subgroup U. From this it follows that $G(N)$ is N-regular and is connected.

We now focus on the case $N = p$ a prime. In the sequel it will be useful to have the following additional description of $G(p)$.

The group $\mathbb{F}_p^\times$ acts on the set $(\mathbb{Z}/p\mathbb{Z})^2 - (0,0)$ diagonally: if $\varepsilon \in \mathbb{F}_p^\times$ and $(a,b) \in (\mathbb{Z}/p\mathbb{Z})^2 - (0,0)$, then $\varepsilon \cdot (a,b) = (\varepsilon a, \varepsilon b)$. Hence this gives an action of $\mathbb{F}_p^\times$ on the vertex set $V(p)$, and we say that two vertices in the same orbit are *associates*.[1] Since -1 acts trivially on $V(p)$, the set of associates of a given vector v contains $(p-1)/2$ elements. Now fix a vertex v; the set of vertices adjacent to v naturally forms a p-gon. Any vertex not associate to v is adjacent to exactly one associate of v. Hence $G(p)$ can be decomposed into $(p-1)/2$ "caps," each containing $p+1$ vertices and each with a different associate of v at its center. Moreover, one can verify that if w is a vertex not associate to v, then w is incident to exactly two vertices on each cap not containing it. This accounts for all vertices and edges of $G(p)$.

We now state the main theorem of this section.

Theorem 4.2. *Let $p \geq 3$ be a prime. Then the graph $G(p)$ is Ramanujan. More precisely, $G(p)$ has the following spectrum:*

(1) *p with multiplicity one,*
(2) *-1 with multiplicity p, and*
(3) *$\pm\sqrt{p}$, each with multiplicity $(p-1)^2/4 - 1$.*

[1] This is not an action on the graph, since it does not preserve edges.

We prove Theorem 4.2 by explicitly constructing eigenvectors for the adjacency matrix A. Since the induced L_p-action on $S_{V(p)}$ commutes with that of A, first we describe $S_{V(p)}$ as an L_p-module. For the convenience of the reader we recall some facts about the complex representation theory of L_p [3, 12].

Let ε be a generator of $\mathbb{F}_p^\times$, let μ be $\left(\begin{smallmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{smallmatrix}\right)$, and let $\lambda \in L_p$ be an element of order $(p+1)/2$. The group L_p has $(p+5)/2$ conjugacy classes that can naturally be grouped into five types:

(1) The class I of the identity.
(2) The class B (resp. B') of $\left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$ (resp. $\left(\begin{smallmatrix} 1 & 0 \\ \varepsilon & 1 \end{smallmatrix}\right)$). We have $|B| = |B'| = (p^2-1)/2$.
(3) The classes C_k represented by μ^k, where $1 \le k \le (p-5)/4$ if $p = 1 \bmod 4$ and $1 \le k \le (p-3)/4$ if $p = 3 \bmod 4$.
(4) The classes D_l represented by λ^l, where $1 \le l \le (p-1)/4$ if $p = 1 \bmod 4$ and $1 \le l \le (p-3)/4$ if $p = 3 \bmod 4$.
(5) A class E, represented by $\mu^{(p-1)/4}$ (resp. $\lambda^{(p+1)/4}$) if $p = 1 \bmod 4$ (resp. $3 \bmod 4$).

There are five types of representations:

(1) The trivial representation $\mathbf{1}$.
(2) The Steinberg representation St of degree p. This is the quotient by the trivial representation of the permutation representation of L_p on the space of complex-valued functions on $\mathbb{P}^1(\mathbb{F}_p)$.
(3) The principal series representations PS_θ, each of degree $p+1$. These are indexed by characters $\theta\colon \mathbb{F}_p^\times \to \mathbb{C}$ satisfying $\theta(-1) = 1$. We have $PS_\theta \simeq PS_{\bar{\theta}}$. There are $(p-5)/4$ isomorphism classes if $p = 1 \bmod 4$ (the quadratic character is excluded), and $(p-3)/4$ if $p = 3 \bmod 4$.
(4) The discrete series representations, each of degree $p-1$. These are associated to additive characters of the subgroup U. There are $(p-1)/4$ isomorphism classes if $p = 1 \bmod 4$, and $(p-3)/4$ if $p = 3 \bmod 4$.
(5) There are two split principal series (resp. discrete series), each of degree $(p+1)/2$ (resp. $(p-1)/2$) if $p = 1 \bmod 4$ (resp. $3 \bmod 4$). We denote the split principal series by $SPS_\pm$.

We will need the following values of the irreducible characters of L_p:

	I	B	B'	C_k	D_l
St	p	0	0	1	-1
PS_θ	$p+1$	1	1	$\theta(\varepsilon^k)+\theta(\varepsilon^{-k})$	0
$SPS_\pm$	$(p+1)/2$	$(1\pm\sqrt{p})/2$	$(1\mp\sqrt{p})/2$	$(-1)^k$	0

Lemma 4.3. *As an L_p-module, the vector space $S_{V(p)}$ is isomorphic to the direct sum of the following irreducibles:*

(1) $\mathbf{1}$*, with multiplicity one.*
(2) St*, with multiplicity one.*

(3) *PS_θ, each with multiplicity two.*
(4) *If $p = 1 \bmod 4$, the two split principal series representations $SPS_\pm$, each with multiplicity one.*

Proof. The description of $G(p)$ as a Schreier graph shows that $S_{V(p)}$ cannot contain any discrete series representations. It is easy to see that the character χ of L_p acting on $S_{V(p)}$ is

$$\chi(I) = (p^2 - 1)/2, \quad \chi(B) = \chi(B') = (p - 1)/2, \quad \chi = 0 \text{ otherwise,}$$

and a standard character computation finishes the proof. □

Now we describe some subspaces of $S_{V(p)}$.

(1) For any vertex v, let $S_v \in S_{V(p)}$ be the function defined by
   (a) $S_v(\varepsilon v) = p$, for all $\varepsilon \in \mathbb{F}_p^\times$.
   (b) $S_v(w) = -1$ for all other vertices w.
   Let S be the subspace generated by all S_v, as v ranges over $V(p)$.
(2) Let $\theta\colon \mathbb{F}_p^\times \to \mathbb{C}$ be a nontrivial character with $\theta(-1) = 1$. For any vertex v, let $P^+_{\theta,v}$ be the function defined by
   (a) $P^+_{\theta,v}(\varepsilon v) = \theta(\varepsilon)\sqrt{p}$, for all $\varepsilon \in \mathbb{F}_p^\times$.
   (b) $P^+_{\theta,v}(\varepsilon w) = \theta(\varepsilon)$ if $v \sim w$.
   Let P^+_θ be the subspace generated by all $P^+_{\theta,v}$, as v ranges over $V(p)$.
(3) Let θ be as above, and define $P^-_{\theta,v}$ by replacing $\sqrt{p}$ with $-\sqrt{p}$ in the definition of $P^+_{\theta,v}$. Let P^-_θ be the corresponding subspace.

Lemma 4.4. *The vector S_v is an eigenvector of A with eigenvalue -1, and $P^\pm_{\theta,v}$ is an eigenvector with eigenvalue $\pm\sqrt{p}$.*

Proof. Consider the cap decomposition of $G(p)$ with centers the associates of v, and let w be any vertex not associate to v. Then $(AS_v)(\varepsilon v) = -p$, whereas $(AS_v)(w) = p + (-1)(p - 1) = 1$. Hence $(AS_v)(x) = -(AS_v)(x)$ for all $x \in V(p)$.

For $P^+_{\theta,v}$, we have $(AP^+_{\theta,v})(\varepsilon v) = \theta(\varepsilon)p$. If w is adjacent to εv, then we have $(AP^+_{\theta,v})(w) = \theta(\varepsilon)\sqrt{p} - \sum_{\alpha\in\mathbb{F}_p^\times}\theta(\alpha)$, since there are two edges from w to each cap not containing it, and since $\theta(-1) = 1$. Thus $(AP^+_{\theta,v})(w) = \theta(\varepsilon)\sqrt{p}$, and $P^+_{\theta,v}$ is an eigenvector with eigenvalue $\sqrt{p}$. A similar argument works for $P^-_{\theta,v}$. □

To compute the dimensions of the spaces S, $P^\pm_\theta$, we will need a lemma about Gauss sums. We are indebted to F. Hajir for the proof.

Lemma 4.5. *Let $g_a(\theta)$ be the Gauss sum $\sum_{t \bmod p}\theta(t)\omega^{at}$, where $\omega = e^{2\pi i/p}$, and $a \neq 0$. Then $g_a(\theta)$ is real if and only if $p = 1 \bmod 4$ and θ is the quadratic character.*

Proof. The sum $g_a(\theta)$ is real if and only if $g_a(\theta) = \theta(-1)g_a(\overline{\theta})$. Write $\theta(-1) = e \in \{\pm 1\}$. Suppose

$$g_a(\theta) - eg_a(\overline{\theta}) = \sum_{t \bmod p} h(t)\omega^{at} = 0, \tag{1}$$

where $h(t) = \theta(t) - e\overline{\theta}(t)$ is in the field $F = \mathbb{Q}(e^{2\pi \mathrm{i}/(p-1)})$. Dividing (1) by ω and putting $H(x) = \sum_{t=1}^{p-1} h(t)x^{t-1}$, we see $H(\omega^a) = 0$. So ω^a satisfies a degree $p-2$ equation over F. If a single $h(t)$ were nonzero, a consequence would be that the extension $F(\omega^a)/F$ would have degree at most $p-2$. But $F(\omega^a)/F$ has degree $p-1$ because $F/\mathbb{Q}$ and $\mathbb{Q}(\omega^a)/\mathbb{Q}$ are disjoint, as p is totally ramified in one and unramified in the other.

Hence all the $h(t)$ must vanish. Now consider the possibilities for e. If $e = 1$, then $\theta(t) = \overline{\theta}(t)$, so θ is quadratic and $p = 1 \bmod 4$. On the other hand, if $e = -1$, then $\theta(t) = -\overline{\theta}(t)$ for all t. But $\theta(1)$ must be 1 giving $1 = -1$, which is absurd. This completes the proof. □

Lemma 4.6. *As L_p-modules, we have the following isomorphisms:*

1. $S \simeq St$;
2. $P_\theta^+ \simeq P_\theta^- \simeq PS_\theta$ *if* $\theta^2 \neq 1$;
3. $P_\theta^\pm \simeq SPS_\pm$ *if* $\theta^2 = 1$.

Proof. It is easy to see that L_p acts on the spaces $S, P_\theta^\pm$ with the correct characters; our main task is to compute the dimensions of these spaces. It is not hard to see $\dim S = p$, so we focus on $P_\theta^\pm$. By the construction, it is clear that $\dim P_\theta^\pm \leq p+1$, so we show that $\dim P_\theta^\pm = p+1$ if θ is not quadratic, and is $(p+1)/2$ if θ is quadratic.

Fix a vertex v, and let $W = \{w \mid w \sim v\}$. Then P_θ^+ is the span of $P_{\theta,v}^+$ and $\{P_{\theta,w}^+ \mid w \in W\}$. There is an obvious action of $\mathbb{Z}/p\mathbb{Z}$ on W and hence on S_W given by rotation about v. For each $w \in W$, let ξ_w be the image of $P_{\theta,w}^+$ under the projection $S_{V(p)} \to S_W$, and let Ξ_W be the span of the ξ_w. The set of vectors $\{\xi_w\}$ also admits a $\mathbb{Z}/p\mathbb{Z}$-action, and hence we can think of them as giving a $\mathbb{Z}/p\mathbb{Z}$-equivariant map $S_W \to S_W$. The kernel of this map corresponds to relations among the ξ_w, and it is a direct sum of $\mathbb{Z}/p\mathbb{Z}$-eigenspaces. If one writes out these relations in coordinates, one sees that to determine $\dim \Xi_W$ it suffices to check the vanishing of expressions of the form

$$\sqrt{p} + \sum_{t \bmod p} \overline{\theta}(p)\omega^{at}, \quad a = 0, \ldots, p-1, \tag{2}$$

where $\omega = e^{2\pi \mathrm{i}/p}$.

The sum (2) is nonzero if $a = 0$. By Lemma 4.5 it is nonzero if $\theta^2 \neq 1$, and one can check (using $g_a(\theta) = \theta(a^{-1})g_1(\theta)$) that it vanishes for half of $a = 1, \ldots, p-1$ if $\theta^2 = 1$. (Note $\theta(-1) = 1$, hence if $\theta^2 = 1$ then $p = 1 \bmod 4$.) Thus if $\theta^2 \neq 1$, we have

$\dim \Xi_W = p$, and if $\theta^2 = 1$, we have $\dim \Xi_W = (p+1)/2$. These are lower bounds on the dimension of P_θ^+.

Now let $W' = W \cup \{v\}$, and for $w \in W'$ let ξ'_w be the image of $P^+_{\theta,w}$ under the projection $S_{V(p)} \to S_{W'}$. Let Ξ'_W be the span of the ξ'_w. We have $\dim \Xi'_W = \dim \Xi_W$, since the ξ'_w satisfy

$$\sqrt{p}\xi'_v + \sum_{w \in W} \xi_w = 0.$$

Thus the only possible new nontrivial relation among the $P^+_{\theta,w}$ is

$$\sqrt{p}P^+_{\theta,v} + \sum_{w \in W} P^+_{\theta,w}. \tag{3}$$

If $\theta^2 \neq 1$, then (3) does not hold, and $\dim P_\theta^+ = p+1$. If $\theta^2 = 1$, then (3) does hold, and $\dim P_\theta^+ = (p+1)/2$. Similar considerations apply to P_θ^-. $\square$

Proof of Theorem 4.2. We have $P_\theta^- \cap P_\theta^+ = \{0\}$, since they are distinct eigenspaces of A. Hence Lemmas 4.3 and 4.6 show that we have constructed all the eigenspaces of $S_{V(p)}$, and Lemma 4.4 shows that the spectrum of $G(p)$ is as indicated. $\square$

Now we consider the case of general N. Write $N = \prod_{p^e || N} p^e$, where the p are distinct primes. We have the following partial description of the spectrum of $G(N)$.

Proposition 4.7. *For each prime power p^e exactly dividing N, choose an $\alpha(p^e)$ in the spectrum of $G(p^e)$. Then the spectrum of $G(N)$ contains $\alpha(N) = \prod_{p^e||N} \alpha(p^e)$.*

Proof. For any prime power p^e, let $\widetilde{V}(p^e) = V(p^e) \times \mathbb{F}_2$ if $p^e \neq 2$, and let $\widetilde{V}(2) = V(2)$. The vertex set $V(N)$ can be identified with $(\prod_{p^e||N} \widetilde{V}(p^e))/\pm 1$, where ± 1 acts on each $\widetilde{V}(p^e)$ through multiplication on $\mathbb{F}_2$. Two vertices $v = ((v_p, \varepsilon_p)), v' = ((v'_p, \varepsilon'_p)) \in V(N)$ are adjacent if and only if for each $p|N$ we have $v_p \sim v'_p$ and $\varepsilon_p = \varepsilon'_p$.

Let $\pi_p \colon V(N) \to V(p^e)$ be the projection induced from $\widetilde{V}(p^e) \to V(p^e)$. Then if $\xi(p^e) \in S_{V(p^e)}$ is an eigenclass on $G(p^e)$ with eigenvalue $\alpha(p^e)$, the function $\xi(N) = \prod_p \pi_p^*(\xi(p^e)) \in S_{V(N)}$ is an eigenvector with eigenvalue $\alpha(N)$. $\square$

Proposition 4.7 implies that most $G(N)$ cannot be Ramanujan. In fact, one can see that the spectrum of $G(p^e)$ contains $-p^{e-1}$. This implies that the spectrum of $G(N)$ contains $-N/p$ for each $p|N$; since the degree of $G(N)$ is N, if p is small compared to N the graph $G(N)$ will not be Ramanujan. On the other hand, there are some composite N such that $G(N)$ is Ramanujan, for example $G(2p)$ for $p = 2, 3, 5, 7$ (and no other p).

Remark 4.8. Let $q = p^f$ be a prime power. Then one can define a graph $G^*(q)$ by replacing $\mathbb{Z}/p\mathbb{Z}$ with $\mathbb{F}_q$ in the description of $G(p)$ as a Schreier graph. (We use the notation $G^*(q)$ to distinguish this graph from the graph $G(q)$.) The group $L_q = PSL_2(\mathbb{F}_q)$ acts on $G^*(q)$, and presumably $G^*(q)$ is Ramanujan.

These graphs can also be realized as geodesic arcs connecting cusps in locally symmetric spaces. For example, let $\mathfrak{H}_3 = SL_2(\mathbb{C})/SU(2)$ be the three-dimensional hyperbolic upper halfspace, and let $\mathscr{O} = \mathbb{Z}[\sqrt{-1}]$ be the Gaussian integers. Let $\Gamma = SL_2(\mathscr{O})$, let p be a prime, and let $\Gamma(p) \subset \Gamma$ be the subgroup of matrices congruent to the identity modulo p. We can define a graph $G^*(\infty)$ by taking the Γ-translates of the ideal geodesic connecting 0 and ∞ in $\mathfrak{H}_3^* = \mathfrak{H}_3 \cup \mathbb{Q}(\sqrt{-1}) \cup \infty$. If $p = 3 \bmod 4$, then $\Gamma(p)\backslash G^*(\infty) \simeq G^*(p^2)$.

5. Splitting

In this final section we show how one can produce other families of Ramanujan graphs from $B_q(1, n-1)$ and $G(p)$ through a simple "splitting" construction.

Definition 5.1. Let $k \geq 2$ be an integer, and let $[\![k]\!]$ be the set $\{1, \dots, k\}$. Given a graph $G = (V, E)$, let G^k be the graph (V^k, E^k), where

(1) $V^k = V \times [\![k]\!]$, and
(2) $((v, i), (v', j)) \in E^k$ if and only if $(v, v') \in E$.

Note that if G has v vertices, e edges, and is r-regular, then G^k has kv vertices, k^2e edges, and is kr-regular. Figure 2 illustrates the passage from $G(3)$ to $G^2(3)$.

Proposition 5.2. *Suppose G has spectrum $\Lambda_G = \{\mu_0, \dots, \mu_{n-1}\}$. Then G^k has spectrum $\Lambda_{G^k} = \{k\mu_0, \dots, k\mu_{n-1}\} \cup \{0\}$. Moreover, the multiplicity $\mathrm{m}(\mu)$ of μ in Λ_G equals the multiplicity $\mathrm{m}(k\mu)$ of $k\mu$ in Λ_{G^k}, and $\mathrm{m}(0) = \#V \cdot (k-1)$.*

Proof. Let $\pi\colon G^k \to G$ be the projection that on vertices takes the form $(v, i) \mapsto v$. If f is an eigenvector for A_G with eigenvalue μ, then $\pi^* f$ is an eigenvector of A_{G^k} with eigenvalue $k\mu$. Furthermore, choose $v \in V$ and fix two vertices $(v, i), (v, j) \in \pi^{-1}(v)$. Then the function $g = g_{v,i,j} \in \mathbb{C}[V^k]$ defined by

- $g(v, i) = 1$,
- $g(v, j) = -1$, and
- $g(w) = 0$ for all other vertices of G^k

is an eigenvector of A_{G^k} with eigenvalue 0.

To prove the statement about multiplicities, first observe that $\mathrm{m}(k\mu) \geq \mathrm{m}(\mu)$ for any $\mu \in \Lambda_G$, since any relation among the eigenvectors $\{\pi^* f\}$ pushes forward to a relation among the eigenvectors $\{f\}$. Now if $v \in V$ is fixed, the space of functions $F_v = \{g_{v,i,j} \mid i \neq j\}$ has dimension $k-1$, and the 0-eigenspace of A_{G^k} is $\bigoplus_{v \in V} F_v$. Thus $\mathrm{m}(0) = \#V \cdot (k-1)$, and $\mathrm{m}(k\mu) = \mathrm{m}(\mu)$ for all $\mu \in \Lambda_G$. $\square$

Combining Proposition 5.2 with the computation of spectra in Theorems 3.2 and 4.2 yields the following:

Corollary 5.3. *The graphs $B_q^k(1, n-1)$ are Ramanujan for $k = 2, 3, 4$, and the graph $G^2(p)$ is Ramanujan.*

Remark 5.4. The graph $G^2(p)$ can also be constructed by taking the vertex set $V^2(p) = (\mathbb{Z}/p\mathbb{Z})^2 - (0,0)$, and then joining two vertices (a,b), $(c,d) \in V^2(p)$ by an edge if $ad - bc = 1$. Note that $|V^2(p)| = 2|V(p)|$, since we do not mod out by -1. Hence $G^2(p)$ admits an action of $SL_2(\mathbb{F}_p)$, and one should be able to compute its spectrum directly as in the proof of Theorem 4.2.

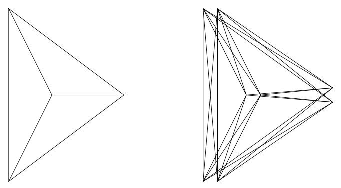

FIGURE 2. Splitting $G(3)$ to $G^2(3)$

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