HYPERGRAPH MATRIX MODELS

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ABSTRACT. The classical GUE matrix model of $N \times N$ Hermitian matrices equipped with the Gaussian measure can be used to count the orientable topological surfaces by genus obtained through gluing the edges of a polygon. We introduce a variation of the GUE matrix model that that enumerates certain edge-ramified CW complexes obtained from polygon gluings. We do this by replacing the Gaussian measure with a formal analogue related to generating functions that enumerate uniform hypergraphs. Our main results are three different ways to compute expectations of traces of powers. In particular, we show that our matrix model has a topological expansion.

1. INTRODUCTION

1.1. We begin by recalling the classical Gaussian Unitary Ensemble (GUE) matrix model and its connection to counting orientable maps. Let V be the N^2-dimensional real vector space of $N \times N$ complex Hermitian matrices equipped with Lebesgue measure. For any polynomial function $f : V \to \mathbb{R}$, define

$$\langle f \rangle_0 = \int_V f(X) \exp(-\operatorname{Tr} X^2/2)\, dX, \tag{1}$$

where $\operatorname{Tr}(X) = \sum_i X_{ii}$ is the sum of the diagonal entries, and put

$$\langle f \rangle = \langle f \rangle_0 / \langle 1 \rangle_0. \tag{2}$$

Let $k \geq 0$ be an integer, and consider (2) evaluated on the polynomial given by taking the trace of the kth power:

$$P(N) = \langle \operatorname{Tr} X^k \rangle. \tag{3}$$

For k odd (3) clearly vanishes for all n. On the other hand, for $k = 2r$ even and N fixed, it turns out that $P(N)$ is an integer, and as a function of N is a polynomial of degree $r+1$ with integral coefficients.

Furthermore, the polynomial $P(N)$ has the following remarkable combinatorial interpretation. Let Π be a polygon with $2r$ sides. Any (oriented) pairing s of the sides of Π determines a closed orientable map M_s with one face. This means M_s is a closed orientable topological surface with an embedded graph (the images of the edges and vertices of Π),

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such that the complement of the embedded graph is a 2-cell. Let $v(M_s)$ be the number of vertices in this graph. Then we have

$$P(N) = \sum_s N^{v(M_s)}, \tag{4}$$

where the sum is taken over all pairings of the edges of Π such that the resulting surface M_s is orientable. For example, writing $P_k(N)$ for (3), we have

$$P_4(N) = 2N^3 + N, \quad P_6(N) = 5N^4 + 10N^2, \quad P_8(N) = 14N^5 + 70N^3 + 21N.$$

The three pairings yielding $P_4(N)$ are shown in Figure 1. For more information about the connection between matrix models and maps, we refer to Harer–Zagier [8], Etingof [2, §4], Lando–Zvonkin [11], and Eynard [5].

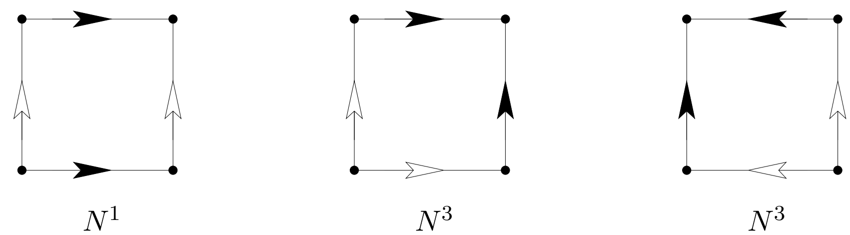


FIGURE 1. Computing $P_4(N) = 2N^3 + N$. The leftmost surface is a torus with embedded graph having one vertex. The next two are the 2-sphere with embedded graph having three vertices.

1.2. When one encounters these ideas for the first time, it seems surprising that the integral (3) should have anything to do with enumerating maps. There are two basic features that make this happen:

- First, the moments of the Gaussian measure

$$d\mu_2(x) = (2\pi)^{-1/2} e^{-x^2/2}\, dx$$

on $\mathbb{R}$ have a combinatorial interpretation. The kth moment

$$\langle x^k \rangle_2 := \int_{\mathbb{R}} x^k \, d\mu_2(x) \tag{5}$$

counts the number of ways to partition the finite set $[\![k]\!] := \{1, \dots, k\}$ into disjoint pairs. In particular (5) vanishes if k is odd, and otherwise equals the number $W_2(k) = (k-1)!! = k!/(2^{k/2}(k/2)!)$. For example

$$\langle x^2 \rangle_2 = 1, \quad \langle x^4 \rangle_2 = 3, \quad \langle x^6 \rangle_2 = 15.$$

This is the content of *Wick's theorem*, which plays a important role in the perturbative expansions of quantum field theory. From a combinatorial perspective, this interpretation of these moments leads to generating functions that enumerate graphs. More details are given in §2.1.

- Next, the polynomial $\operatorname{Tr} X^k$ has a combinatorial interpretation in terms of k-gons. Let $X_{u,v}$ be the variable in the (u,v)th entry of X. The monomials contributing to the trace polynomial have the form

$$X_{u_1,u_2} X_{u_2,u_3} \cdots X_{u_k,u_1}, \tag{6}$$

  where the subscripts can be freely taken from 1 to N. The monomial (6) can be represented by a labeled k-gon Π with a distinguished vertex. Each vertex gets a subscript in cyclic order, with the distinguished vertex getting u_1. The variable $X_{u_i,u_{i+1}}$ then corresponds to the ith edge in cyclic order (Figure 2).

With these facts in hand, the connection becomes more plausible. The variables in the monomial (6) correspond to edges of Π, and Wick's theorem implies that this monomial gives a nonzero expectation after appropriately pairing the edges. Carefully putting everything together one obtains the result (4).

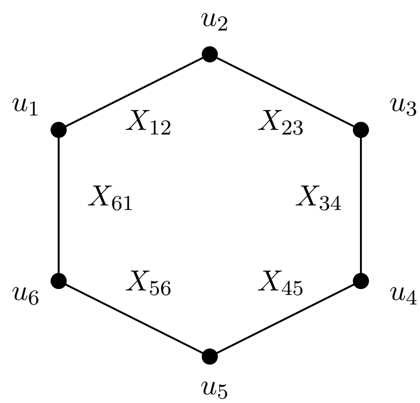


FIGURE 2. A polygon and variables. We have abbreviated X_{u_i,u_j} by X_{ij}.

1.3. The goal of this paper is to modify this basic model by replacing the Gaussian measure with a formal measure with combinatorially different moments. Choose a positive integer m. Then instead of counting the pairings of the finite set $[\![k]\!]$, the new moments count the number $W_{2m}(k)$ of partitions of $[\![k]\!]$ into disjoint subsets of order $2m$. Just as the Gaussian enables one to write generating functions enumerating graphs, these new "measures" lead to generating functions enumerating hypergraphs.

The main object of study is the analogue of (3). We define a linear form $\langle \cdot \rangle_{2m}$ on polynomial functions $f \colon V \to \mathbb{C}$, and put

$$P(N) = \left\langle \operatorname{Tr} X^k \right\rangle_{2m}, \tag{7}$$

where X is an $N \times N$ Hermitian matrix of variables. Our main results give different ways to understand (7) as a function of N. The first result (Theorem 3.6) shows that $P(N)$ vanishes unless $k = 2mr$ for some r, and when nonvanishing gives a explicit expression for (7) — which we write as $P_{2m,r}(N)$ — in terms of walks on directed graphs. In particular Theorem 3.6 shows that $P_{2m,r}(N)$ is a rational polynomial in N with coefficients having denominator at worst a power of 2.

The conclusion of Theorem 3.6 is quite different from (4), which expresses (3) in terms of orientable maps. The result (4) should be considered much more interesting, since it is the first step in many nontrivial applications of the GUE, such as computing the Euler characteristic of the mapping class group [8] and the topological recursion [3, 4]. Thus, in addition to Theorem 3.6, we provide two ways to view (7) geometrically.

First, in Theorem 4.7, we show that (7) can be written as a sum

$$P_{2m,r}(N) = \sum_s a_s(N),$$

where s is taken over all paritions of $\{1, \ldots, 2mr\}$ into blocks of order $2m$, and the polynomials $a_s(N) \in \mathbb{Z}[N]$ arise from counting points on certain subspace arrangements over a finite field. In the classical case $m = 1$ these arrangements always consist of one subspace, which gives another interpretation of the monomials in (4). For $m > 1$, on the other hand, more complicated arrangements can arise.

Finally, in Theorem 5.8, we give an analogue for $m > 1$ of (4). This expansion, or rather the normalized version

$$N^{1-r} P_{2,r}(N) = \sum_s N^{\chi(M_s)}, \tag{8}$$

where χ is the Euler characteristic, is called the *topological expansion*, since it expresses $P_{2,r}(N)$ in terms of the topology of the surface M_s. For $m > 1$, we prove

$$\left\langle \operatorname{Tr} X^{2mr} \right\rangle_{2m} = \sum_M (1 - 2^{2m-2})^{\ell(M)} N^{v(M)}. \tag{9}$$

In (9) the sum is taken over a generalization of maps that we call *CW maps with instructions.* The full definition is given in Definition 5.4, but the main points are (i) we must work with CW complexes with ramification along their edges instead of surfaces, and (ii) the terms in (9) contains a factor that remembers more about how the complex was assembled than just the final number of vertices. Corollary 5.9 gives another expression for (9) that shows it is the generalization to higher m of the topological expansion (8).

1.4. Here is a guide to the paper. In §2 we review the connections between Gaussian integrals and enumerating graphs. We also describe the new measure that will underlie our matrix model, and explain how this measure can be used to enumerate hypergraphs. In §3, we define our basic matrix model and explain a naive method of evaluating it in terms of walks on directed graphs with labeled vertices. Next in §4 we describe how to compute the integral by counting points on linear arrangements. As in (4), the contributions are indexed by gluings of polygons according to partitions of their edge sets. However, unlike the classical expression (4), the contributions are usually not monomials in N. Then in §5 we prove that our integral has a topological expansion similar to (4). This gives an expression for the integral as a sum of monomials, although again the contributions are more complicated than those in (4). Finally in §6 we give some directions for future work, and in Appendix A we give some tables of the polynomials $P_{2m,r}(N)$.

2. Counting graphs and uniform hypergraphs

2.1. In this section we define the measure underlying our matrix model. We begin with a discussion about how the integral (5) leads to generating functions of graphs. For more information, we refer to [2, 11]

As in §1.2, let $d\mu_2(x)$ be the Gaussian measure on polynomial functions on $\mathbb{R}$ with moments

$$\left\langle x^k \right\rangle_2 := \int_{\mathbb{R}} x^k \, d\mu_2(x) = W_2(k),$$

where $W_2(k)$ is the number of pairings on $[\![k]\!]$. Let $t, \xi_1, \xi_2, \ldots$ be indeterminates, and let $S(x)$ be the formal power series $\sum_{k\geq 1} \xi_k x^k/k!$. Using (5), we can evaluate the integral $\left\langle \exp(S(tx)) \right\rangle_2$ as a formal power series in t with coefficients in the polynomial ring $\mathbb{Q}[\xi_1, \xi_2, \ldots]$. We have

$$\left\langle \exp S(tx) \right\rangle_2 = 1 + A_2 t^2/2 + A_4 t^4/8 + A_6 t^6/48 + \cdots \tag{10}$$

where the first few coefficients are given by

$$A_2 = \xi_1^2 + \xi_2, \quad A_4 = \xi_1^4 + 6\xi_1^2\xi_2 + 4\xi_1\xi_3 + 3\xi_2^2 + \xi_4,$$
$$A_6 = \xi_1^6 + 15\xi_1^4\xi_2 + 20\xi_1^3\xi_3 + 45\xi_1^2\xi_2^2 + 15\xi_1^2\xi_4 + 60\xi_1\xi_2\xi_3 + 6\xi_1\xi_5 + 15\xi_2^3 + 15\xi_2\xi_4 + 10\xi_3^2 + \xi_6.$$

2.2. We claim that the series (10) can be interpreted as a generating function for graphs weighted by the inverses of the orders of their automorphism groups. Let $\mathbf{n} = (n_1, n_2, \ldots)$ be a vector of nonnegative integers, with n_k nonzero only for finitely many i. Let $|\mathbf{n}| = \sum n_k$. We say a graph g has *profile* $\mathbf{n}$ if it has n_k vertices of degree i. Let $G(\mathbf{n})$ be the set of all graphs of profile $\mathbf{n}$, up to isomorphism (we allow graphs to have loops and multiple edges). By an automorphism of a graph, we mean any self-map that permutes edges and vertices. In particular, automorphisms include flipping loops and permuting multiedges. For any $g \in G(\mathbf{n})$, let $\Gamma(g)$ be its automorphism group. Then we have the following theorem, which is a special case of [2, Theorem 3.3]:

2.3. Theorem. *We have*

$$\left\langle \exp S(tx) \right\rangle_2 = \sum_{\mathbf{n}} t^{|\mathbf{n}|} \sum_{g\in G(\mathbf{n})} \frac{\prod_k \xi_k^{n_k}}{|\Gamma(g)|}. \tag{11}$$

Proof. We refer to the proof of [2, Theorem 3.3] for full details, and here only give the main ideas for the convenience of the reader. First, the coefficient of t^M in (11) is given by

$$\sum_{\mathbf{n}} \frac{\xi_1^{n_1}\xi_2^{n_2}\cdots}{(1!)^{n_1}(2!)^{n_2}\cdots n_1! n_2! \cdots} x^{w(\mathbf{n})}$$

where the sum is taken over all profiles $\mathbf{n}$ with $|\mathbf{n}| = M$, and $w(\mathbf{n}) = n_1 + 2n_2 + 3n_3 + \cdots$. After integrating against $d\mu_2$, we obtain

$$\sum_{\mathbf{n}} \frac{\xi_1^{n_1}\xi_2^{n_2}\cdots}{(1!)^{n_1}(2!)^{n_2}\cdots n_1! n_2! \cdots} W_2(w(\mathbf{n})). \tag{12}$$

We can interpret the terms in (12) combinatorially as follows. For each k let F_k be a vertex with k attached 1/2-edges. We call F_k a *k-flower* and the 1/2-edges its *petals* (see Figure 3). Let $\mathscr{F} = \mathscr{F}(\mathbf{n})$ be the collection of $|\mathbf{n}|$ flowers with n_k k-flowers for $k \geq 1$. Since each k-flower has k petals, there are $\sum kn_k = w(\mathbf{n})$ petals altogether in $\mathscr{F}$. Let $P(\mathscr{F})$ be the graphs obtained by pairing the petals in $\mathscr{F}$ in all possible ways (see Figure 4). Clearly

$$|P(\mathscr{F})| = \left\langle x^{w(\mathbf{n})} \right\rangle_2 = W_2(w(\mathbf{n})),$$

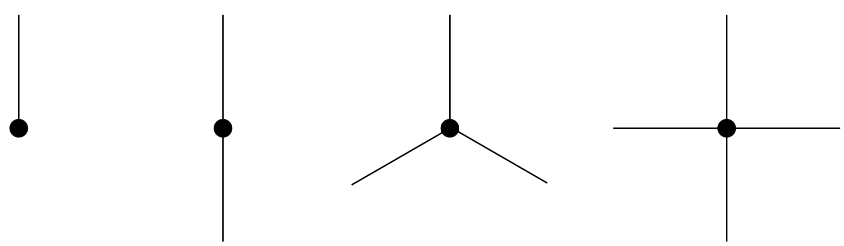
and any graph $g \in G(\mathbf{n})$ appears in $P(\mathscr{F})$. We have a group $G(\mathscr{F})$ of order

$$|G(\mathscr{F})| = \prod_{k\geq 1} (n_k)!(k!)^{n_k}$$

acting on $P(\mathscr{F})$ by permuting the flowers and their petals. The group $G(\mathscr{F})$ does not act transitively on $P(\mathscr{F})$, but its orbits are in bijection with $G(\mathbf{n})$. Moreover, all the automorphisms of any $g \in G(\mathbf{n})$ are induced from the action of $G(\mathscr{F})$. By the orbit-stabilizer theorem, we have

$$\frac{|P(\mathscr{F})|}{|G(\mathscr{F})|} = \sum_{g\in G(\mathbf{n})} |\Gamma(g)|^{-1}.$$

If we incorporate the weights ξ_k for each of the k-flowers in $P(\mathscr{F})$, we recover the monomials in ξ_k in (11). This completes the proof. □



F_1 F_2 F_3 F_4

FIGURE 3. Some flowers.

2.4. Example. Consider the term $5\xi_3^2/24$, which appears in the coefficient of t^6 in (10). There are two graphs with this profile, shown in Figure 5. The left has $2\cdot2\cdot2$ automorphisms, and the right has $2 \cdot 3!$, which gives $1/8 + 1/12 = 5/24$.

2.5. Now we want to replace the Gaussian measure, which is connected to counting pairings of a set, with something that is connected to counting partitions into higher order subsets. For any $m \geq 1$, let $W_{2m}(k)$ be the number of partitions of $[\![k]\!]$ into subsets of order $2m$. Then $W_{2m}(k) = 0$ unless $2m|k$, and in this case we have

$$W_{2m}(k) = \frac{k!}{(2m)!^{k/2m}(k/2m)!}. \tag{13}$$

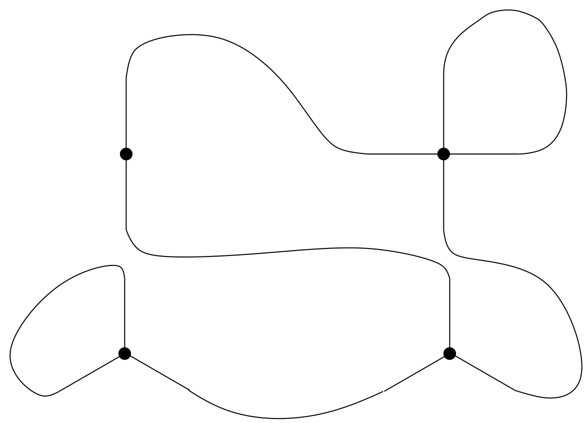


FIGURE 4. Gluing flowers into a graph with profile $\mathbf{n} = (0, 1, 2, 1, 0, \dots)$.

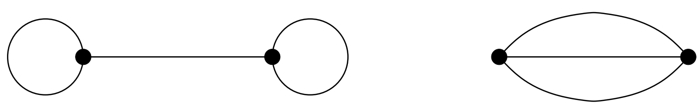


FIGURE 5. The two graphs with profile ξ_3^2.

Let $d\mu_{2m}(x)$ be the "measure" on polynomial functions on $\mathbb{R}$ that gives the monomial x^r the expectation $W_{2m}(r)$. More precisely, we consider the linear form on polynomials that takes x^r to $W_{2m}(r)$. Of course, this is not a measure in the usual sense, but we can formally regard it as such on any polynomial function. The "expectation" $\langle \exp S(tx) \rangle_{2m}$ is then a well-defined power series in t, again with coefficients in the polynomial ring $\mathbb{Q}[\xi_1, \xi_2, \dots]$, and has a combinatorial interpretation via hypergraphs.

2.6. Recall [1] that a *hypergraph* on a vertex set V is a collection of subsets of V, called the *hyperedges*. The *degree* of a vertex is the number of hyperedges it belongs to, and a hypergraph is *regular* if these numbers are the same for all vertices. The *order* of a hyperedge is its number of vertices. If all hyperedges have the same order d, we say that the hypergraph is *d-uniform*.

We extend the notion of a hypergraph by allowing V to be a multiset, in other words a set with a multiplicity map $V \to \mathbb{Z}_{\geq 1}$. We can extend the notions of regularity and uniformity above by incorporating the multiplicity in an obvious way: the order of a subset of a multiset is sum of the multiplicities of its elements. The resulting objects, originally considered by Ouvrard–Le Goff–Marchand-Maillet [14], are known as *hyperbaggraphs*.[1] However, we will continue use the simpler term *hypergraph*, with the understanding that our hypergraphs can have *hyperloops* (hyperedges consisting of a single vertex with multiplicity) and *hypermultiedges* (hyperedges with at least two different vertices and with at least one vertex with nontrivial multiplicity).

[1]In the CS literature, multisets are sometimes called *bags*.

2.7. With these definitions, we can formulate an analogue of Theorem 2.3: the expectation $\langle \exp S(tx) \rangle_{2m}$ now enumerates $2m$-uniform hypergraphs of all profiles weighted by the inverses of their automorphism groups. More precisely, as before let $\mathbf{n} = (n_1, n_2, \dots)$ be a profile vector with $|\mathbf{n}| = \sum n_i$. We now say a d-uniform hypergraph h has profile $\mathbf{n}$ if it has n_i vertices of degree i. Let $HG(\mathbf{n})$ be the set of all hypergraphs of profile $\mathbf{n}$, up to isomorphism, and let $HG^{(2m)}(\mathbf{n}) \subset HG(\mathbf{n})$ be the subset of those that are $2m$-uniform. We define an automorphism of a hypergraph to be any self-map that permutes hyperedges and vertices. In particular, a hypergraph can have automorphisms that permute the vertices within a hyperedge, including flipping hyperloops and permuting hypermultiedges. For any $h \in HG(\mathbf{n})$, let $\Gamma(h)$ be its automorphism group.

2.8. Theorem. *We have*

$$\langle \exp S(tx) \rangle_{2m} = \sum_{\mathbf{n}} t^{|\mathbf{n}|} \sum_{h \in HG^{(2m)}(\mathbf{n})} \frac{\prod_k \xi_k^{n_k}}{|\Gamma(h)|}.$$

Proof. The proof is a simple modification of that of Theorem 2.3. We again have flowers F_k with k-petals, but now we interpret each petal as being a $1/2m$ th hyperedge. A profile $\mathbf{n}$ determines a collection of flowers $P(\mathscr{F})$. To assemble $P(\mathscr{F})$ into an element of $HG^{(2m)}(\mathbf{n})$, we must choose a grouping of the petals into sets of order $2m$ (see Figure 6). This is accomplished by integrating $x^{w(\mathbf{n})}$ against the measure $d\mu_{2m}$. The rest of the proof is the same as before. □

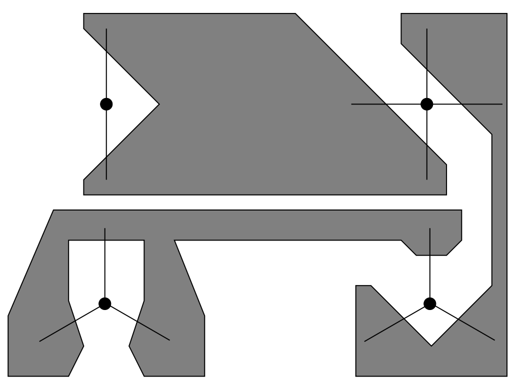


FIGURE 6. A 4-uniform hypergraph with profile $\mathbf{n} = (0, 1, 2, 1, 0, \dots)$.

2.9. Example. Suppose $2m = 4$. We have

$$\langle \exp S(tx) \rangle_4 = 1 + B_4 t^4/24 + B_8 t^8/1152 + \cdots \tag{14}$$

where

$$B_4 = \xi_1^4 + 6\xi_1^2\xi_2 + 4\xi_1\xi_3 + 3\xi_2^2 + \xi_4, \quad B_8 = \xi_1^8 + 28\xi_1^6\xi_2 + 56\xi_1^5\xi_3 + \cdots + 35\xi_4^2 + \xi_8.$$

The computation of the contribution $35\xi_4^2/1152$, which appears in the coefficient of t^8, is as follows. There are three hypergraphs of this profile, each with two hyperedges. The underlying set of vertices has 2 elements a, b, and we represent a hyperedge by a monomial in these variables. The profile ξ_4^2 means that each vertex has degree 4, and since $2m = 4$ we must have uniformity 4. Any such hypergraph can be represented by a pair of monomials in variables a, b of total degree 4. There are three hypergraphs altogether:

$$\{a^4, b^4\}, \quad \{a^3b, ab^3\}, \quad \{a^2b^2, a^2b^2\}. \tag{15}$$

The orders of the automorphism groups are

$$2 \cdot (4!)^2, \quad 2 \cdot (3!)^2, \quad 2 \cdot 2 \cdot (2!)^2(2!)^2. \tag{16}$$

For example, the automorphisms of the rightmost hypergraph come from interchanging the vertices, interchanging the two hyperedges, and the internal flips within the hyperedges; the last type of automorphism cannot occur for graphs. Adding the inverses of these orders, one finds $1/1152 + 1/72 + 1/64 = 35/1152$, which agrees with B_8 above.

3. The matrix model

3.1. In this section we define our matrix model. The idea is simply to replace the measure in (1), which is a product of Gaussians, with a product of the measures $d\mu_{2m}$.

We begin by explicitly choosing coordinates in V, the real vector space of $N \times N$ complex Hermitian matrices. The space V has real dimension N^2. First we take real variables $x_{u,u}$ $(u = 1, \dots, N)$ and $x_{u,v}, y_{u,v}$ $(1 \le u < v \le N)$. Let $X = (X_{u,v})$ be an $N \times N$ matrix of Hermitian variables, with diagonal entries $X_{v,v} = x_{v,v}$ and off-diagonal entries $X_{u,v} = x_{u,v} + iy_{u,v}$ for $u < v$ and $X_{v,u} = \bar{X}_{u,v}$.

3.2. Definition. Given any polynomial function $f \colon V \to \mathbb{C}$, we define

$$\langle f \rangle_{2m} \in \mathbb{C}$$

as follows. First put

$$\langle x_{u,v}^{2mr} \rangle_{2m} = \langle y_{u,v}^{2mr} \rangle_{2m} = \frac{W_{2m}(2mr)}{2^r}, \quad u \neq v,$$

and

$$\langle x_{u,u}^{2mr} \rangle_{2m} = W_{2m}(2mr),$$

where $W_{2m}(k)$ is given by (13); and

$$\langle x_{u,v}^{k} \rangle_{2m} = \langle y_{u,v}^{k} \rangle_{2m} = \langle x_{u,u}^{k} \rangle_{2m} = 0$$

for $k > 0$ not divisible by $2m$. We then extend $\langle \cdot \rangle_{2m}$ linearly to all polynomials.

3.3. Definition. For an r-tuple $(a_1, \dots, a_r)$ of non-negative integers with $a_l \le a_{l+1}$, let $S(a_1, \dots, a_r)$ denote the set of set partitions of $[\![a]\!]$ where

$$a = \sum_{l=1}^{r} a_l$$

into subsets of orders $a_1, \dots, a_r$. Also let m_i denote the number of values k such that there are exactly i indices $l, 1 \le l \le r$, with $a_l = k$. Thus

$$|S(a_1, \dots, a_r)| = \frac{1}{\prod_{i=1}^{r} i!^{m_i}} \binom{a}{a_1, \dots, a_r}.$$

We also let $S(2mr;2m)$ denote the set of set partitions of $[\![2mr]\!]$ into subsets each of order $2m$. Given $s \in S(2mr;2m)$, we identify s with a set of r functions

$$s_i : [\![2m]\!] \to [\![2mr]\!]$$

with $1 \le i \le r$ whose images are disjoint. We refer to these as "blocks" or "subsets" in the set partition.

We have the following generalization of Wick's theorem:

3.4. Lemma. (i) *Let $f_1, f_2, \dots, f_{2mr}$ be a set of (not necessarily distinct) linear functions of $x_{u,v}, y_{u,v}$. Then*

$$\langle \prod_{i=1}^{2mr} f_i \rangle_{2m} = \sum_{s \in S(2mr;2m)} \prod_{l=1}^{r} \langle \prod_{j=1}^{2m} f_{s_l(j)} \rangle_{2m}. \tag{17}$$

(ii) *For integers $a, b \ge 0$ and indices $u < v$, we have*

$$\langle X_{u,v}^a \bar{X}_{u,v}^b \rangle_{2m} = \langle (x_{u,v} + iy_{u,v})^a (x_{u,v} - iy_{u,v})^b) \rangle_{2m} \neq 0$$

if and only if (a, b) is a nonnegative integral linear combination of the lattice points

$$(0, 2m), (2, 2m-2), \dots, (m, m), \dots, (2m, 0)$$

for m even and

$$(1, 2m-1), (3, 2m-3), \dots, (m, m), \dots, (2m-1, 0)$$

for m odd.

Proof. We first prove (i). Our proof is a direct generalization of the proof of Wick's theorem presented in Lando–Zvonkin [11, Theorem 3.2.5]. First recall that for $u < v$

$$\langle x_{u,u}^{2mn} \rangle_{2m} = C_0^m W_{2m}(2mn)$$

$$\langle x_{u,v}^{2mn} \rangle_{2m} = \langle y_{u,v}^{2mn} \rangle_{2m} = C_1^n W_{2m}(2mn)$$

where we have defined the normalizations $C_0 = 1$ and $C_1 = \frac{1}{2}$. The theorem still holds if we use any non-zero numbers for C_0 and C_1, and we prove it in that generality.

Both sides of (17) are linear in each f_i. It is therefore sufficient to prove the formula for products of coordinate functions, i.e., when

$$f_i = x_{u,v}, \ \ y_{u,v}, \ \text{ or } x_{u,u}$$

for each i. We simplify notation by letting the set $\{x_k\}$, $1 \le k \le N^2$, denote the set of all variables $x_{u,u}, x_{u,v}$ and $y_{u,v}$.

We claim that both sides equation (17) are non-zero if and only if for each x_k, the number of f_i that equal x_k is a multiple of $2m$. We write the left side of equation (17) as

$$\prod_{k=1}^{N^2} x_k^{m_k}$$

where

$$\sum_{k=1}^{N^2} m_k = 2mr.$$

The independence of the x_k implies

$$\langle \prod_{k=1}^{N^2} x_k^{m_k} \rangle_{2m} = \prod_{k=1}^{N^2} \langle x_k^{m_k} \rangle_{2m}. \tag{18}$$

By definition of $\langle \cdot \rangle_{2m}$,

$$\langle x_k^{m_k} \rangle_{2m} \neq 0$$

if and only if

$$m_k = 2mn_k$$

for some integer $n_k \geq 0$. And on the right side of (17), each expression

$$\langle \prod_{j=1}^{2m} f_{s_l(j)} \rangle_{2m}$$

becomes an expression of the form

$$\langle \prod_{k=1}^{N^2} x_k^{\mu_k} \rangle_{2m} \tag{19}$$

for integers $\mu_k \geq 0$ that depend on s and satisfy

$$\sum_{k=1}^{N^2} \mu_k = 2m.$$

Thus, by definition of $\langle \cdot \rangle_{2m}$, expression (19) is non-zero if and only there is one index k such that $\mu_k = 2m$. This implies the claim.

We thus write the left side of (17) as

$$\langle \prod_{k=1}^{N^2} x_k^{2mn_k} \rangle_{2m} = \prod_{k=1}^{N^2} \langle x_k^{2mn_k} \rangle_{2m}$$

and apply, for any integer $n \geq 0$,

$$\langle x_{u,u}^{2mn} \rangle_{2m} = C_0^n W_{2m}(2mn) = W_{2m}(2mn) \langle x_{u,u}^{2m} \rangle_{2m}^n,$$
$$\langle x_{u,v}^{2mn} \rangle_{2m} = C_1^n W_{2m}(2mn) = W_{2m}(2mn) \langle x_{u,v}^{2m} \rangle_{2m}^n,$$

and the same equation for $y_{u,v}$ to obtain

$$\langle \prod_{k=1}^{N^2} x_k^{2mn_k} \rangle_{2m} = \prod_{k=1}^{N^2} (W_{2m}(2mn_k) \langle x_k^{2m} \rangle_{2m}^{n_k}) \tag{20}$$

$$= \prod_{k=1}^{N^2} (\sum \langle x_k^{2m} \rangle_{2m}^{n_k}), \tag{21}$$

where the k-th sum is over all $W_{2m}(2mn_k)$ $2m$-groupings of the $2mn_k$ copies of the monomial x_k. Expanding the product yields a sum over all the set partitions s such that for each l there exists a k such that

$$f_{s_l(j)} = x_k$$

for $1 \leq j \leq 2m$. These are exactly the set partitions described in the claim that give a non-zero contribution. This completes the proof of item (i).

We now prove item (ii). For integers $a, b \geq 0$ with $a + b = 2m$, we claim

$$\langle X_{u,v}^a \bar{X}_{u,v}^b \rangle_{2m} = \begin{cases} 2C_1 \text{ if } a - b \equiv 0 \mod 4 \\ 0 \text{ if otherwise.} \end{cases}$$

We expand the left side into monomials and then note that the only non-zero contribution can come from the two monomials

$$x_{u,v}^{a+b} + (-1)^b i^{a+b} y_{u,v}^{a+b}.$$

We have for any integers a and b

$$(-1)^b i^{a+b} = i^{a-b}.$$

We then use $a + b = 2m$ and apply

$$\langle x_{u,v}^{2m} \rangle_{2m} = \langle y_{u,v}^{2m} \rangle_{2m} = C_1.$$

This proves the claim.

Now allowing $a + b = 2mr$, we apply item (i) to obtain

(22)
$$\langle X_{u,v}^a \bar{X}_{u,v}^b \rangle_{2m} = \sum (\prod_{i=1}^{r} i!^{m_i}) |S(a_1, \dots, a_l)| |S(2m - a_1, \dots, 2m - a_r)| \prod_{l=1}^{r} \langle X_{u,v}^{a_l} \bar{X}_{u,v}^{2m-a_l} \rangle_{2m},$$

where the sum is over all r-tuples

$$(a_1, \dots, a_r)$$

with $0 \leq a_l \leq a_{l+1} \leq 2m$ such that

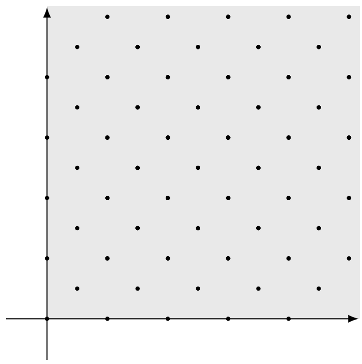
$$a = \sum_{l=1}^{r} a_l$$

and with m_i denoting the number of values k such that there are exactly i indices $l, 1 \leq l \leq r$, with $a_l = k$. Thus the right side of equation (22) is non-zero if and only if there exists at least one such r-tuple satisfying

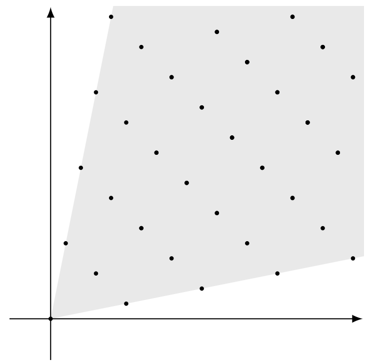
$$a_l - (2m - a_l) \equiv 0 \mod 4$$

for each l. This means that each a_l is even if m is even, and each a_l is odd if m is odd. This proves item (ii). □

Figure 7 shows two examples of the exponents (a, b) that satisfy the second statement of Lemma 3.4. When m is even, the cone σ generated by the exponents is the first quadrant. When m is odd, the cone σ is a proper subcone. In the classical case $m = 1$ all exponents satisfy $a = b$ and σ degenerates to a line.



(a) $2m = 4$



(b) $2m = 6$

FIGURE 7. The cones of exponents giving nontrivial expectations $\langle X_{u,v}^a \bar{X}_{u,v}^b \rangle_{2m}$ for $2m = 4$ and $2m = 6$.

3.5. Now we are ready to define the hypergraph matrix model. For any $k \geq 0$, we set

$$P(N) = \langle \operatorname{Tr} X^k \rangle_{2m},$$

where $X = (X_{u,v})$ is $N \times N$. This can be interpreted as computing the integral

$$P(N) = \int_V \operatorname{Tr} X^k \, d\mu_{2m}(X),$$

where $d\mu_{2m}(X)$ means the "measure" that is the product over the real coordinates of the measure $d\mu_{2m}$ from §2.5, where in addition the measures for the off-diagonal coordinates have been weighted by $1/\sqrt{2}$. The first main result of the paper is the following:

3.6. Theorem. *We have $P(N) = 0$ unless $k = 2mr$ for some integer $r \geq 0$. If $k = 2mr$, then $P(N) = P_{2m,r}(N)$ is a rational polynomial in N of degree $r + 1$ with coefficients in $\mathbb{Z}[1/2]$.*

Later (Theorems 4.7 and 5.8) we will see that $P_{2m,r}(N)$ is actually an *integral* polynomial in N.

Proof. By the generalized Wick theorem (Lemma 3.4), it is clear that $P_k(N)$ must vanish unless $k = 2mr$. We will show that $P_{2m,r}(N)$ is a polynomial by explicitly evaluating it in terms of counting walks on labeled graphs.

Let γ_N be the complete directed graph on N vertices with loops. Thus γ_N has a loop attached to each of its vertices, and between distinct vertices u and v there is a directed edge from u to v and one from v to u. We attach the matrix variable $X_{u,u}$ to the loop at vertex u and the variables $X_{u,v}, X_{v,u} = \bar{X}_{u,v}$ to the edges between u and v. The terms in the trace polynomial

$$\operatorname{Tr} X^k = \sum_{1 \leq u_1, \ldots, u_k \leq N} X_{u_1,u_2} X_{u_2,u_3} \cdots X_{u_k,u_1} \tag{23}$$

can then be interpreted as encoding walks on γ_N that start and stop at the vertex u_1, as u_1 ranges over all vertices of γ_N.

Now consider computing the expectation $\left\langle \operatorname{Tr} X^k \right\rangle_{2m}$. A term on the right of (23) can give a nonzero contribution if and only if the following conditions are met:

(i) We have $k = 2mr$, as above.
(ii) There is a k-step walk on γ_N starting and stopping at vertex u_1.
(iii) Any edge or loop appearing in the walk must appear in parallel batches of order a multiple of $2m$. In other words,
  (a) if a loop appears in the walk, the number of times it appears is a multiple of $2m$;
  (b) if a nonloop edge appears in the walk, it and its opposite together must appear a multiple of $2m$ times.
(iv) The edges between vertices $u \neq v$ must appear in the walk as described in item (ii) of Lemma 3.4. In other words, the corresponding submonomial must have the form $X_{u,v}^a \bar{X}_{u,v}^b$ for a, b in the lemma.

Any walk satisfying the conditions above determines a unique connected subgraph $\gamma \subseteq \gamma_N$, namely the subgraph containing all the edges in the walk. It is clear that γ must satisfy two conditions:

(B) the graph γ must be a balanced digraph (the in-degree of any vertex equals its out-degree); and
(E) the graph γ arises in computation of the trace polynomial only if it has Eulerian tours starting at any of its vertices.

We will call a balanced digraph on $\leq N$ vertices *admissible* if it meets conditions (B) and (E).

In light of item (iii), we can use an admissible γ to determine a unique connected *undirected* graph $g = g(\gamma)$, called the *reduction* of γ, that has r edges and loops. More precisely, the vertices of g are the same as those of γ. If γ has $2ms$ loops at vertex u, then g has s loops at the same vertex. If γ has $2ms$ edges going between vertices u and v, with any orientations, then g has s undirected edges between u and v. It is clear that any graph with r edges can be the reduction of some γ as above.

Now let G_r be the set of connected graphs with r edges. We will compute $P_{2m,r}(N)$ as a sum over G_r. Given $g \in G_r$, there is a canonical (undirected) thickening $\tilde{g}$ of g that replaces each edge with $2m$ edges. We let $D(g)$ be the set of admissible digraph structures on $\tilde{g}$. For any $\gamma \in D(g)$, we put

(i) $W(\gamma)$ to be the total number of Eulerian tours of γ beginning and ending at any vertex, and where parallel edges in the same direction are considered indistinguishable;
(ii) $E(\gamma)$ to be the total contribution from groupings of the edges of γ according to Lemma 3.4.

Finally we define $N(g)$ to be the number of vertex labelings of g, where the labels are taken from $[\![N]\!]$, and where each vertex gets a distinct label. In particular, let $\Gamma_v(g)$ be the quotient of the automorphism group $\Gamma(g)$ of g by the subgroup that fixes the vertices.

Then if $v(g)$ is the number of vertices of g, we have

$$N(g) = N(N-1)\cdots(N-v(g)+1)/|\Gamma_v(g)|.$$

Putting everything together, we find

$$P_{2m,r}(N) = \left\langle \operatorname{Tr} X^{2mr} \right\rangle_{2m} = \sum_{g \in G_r} N(g) \sum_{\gamma \in D(g)} W(\gamma)E(\gamma). \tag{24}$$

Equation (24) shows that $P_{2m,r}(N)$ is a rational polynomial in N. Indeed, it is clear that $W(\gamma)$ and $N(g)$ are integers, and that $E(\gamma)$ is rational with denominator at worst a power of 2. Hence $P_{2m,r}(N) \in \mathbb{Z}[1/2][N]$. Furthermore, the maximal number of vertices for a graph $g \in G_r$ occurs when g is a tree, and is $r+1$. This shows the degree of $P_{2m,r}(N)$ is $r+1$, which completes the proof. □

3.7. Example. Let $m = 2$. We show how one computes $P_8(N) = 6N^3 + 21N^2 + 8N$. In this case $r = 2$, and there are 4 graphs in G_2, shown in the top of Figure 8. The canonical thickenings replace each edge with 4 new edges. Each graph has only one admissible digraph structure, namely with half the edges oriented one direction and half oriented in the opposite direction. The bottom of Figure 8 shows these digraphs. For each digraph $\gamma_1, \ldots, \gamma_4$ we have labeled the vertices with the number of Eulerian tours that start and end there. The contributions $W(\gamma_i)$ are therefore

$$1, \quad 2, \quad 20, \quad 12.$$

The contributions $E(\gamma_i)$ are given by

$$35, \quad 19, \quad 1, \quad 1.$$

Finally we have the contributions $N(g)$. They equal

$$N, \quad N(N-1)/2, \quad N(N-1), \quad N(N-1)(N-2)/2.$$

Altogether, we find $P_8(N) = 6N^3 + 21N^2 + 8N$.

More examples of the polynomials $P_{2m,r}(N)$ for small values of m and r are in Appendix A. These were computed using Theorem 3.6. For example, determining $P_{2m,4}$ used 30 different graphs with 4 edges. We used `nauty` [12] to help compute these graphs.

3.8. Remark. In Example 3.7, each thickening has a unique admissible digraph structure: for any two distinct vertices u, v, the edges running between u and v are divided into two subsets of the same size, one of which has all edges oriented from u to v the other with edges oriented from v to u. It is clear that this can be done for the thickening of any graph g, and thus for any m the there is at least one nonzero contribution all $g \in G_r$ to (24). Typically, though, this digraph structure is not the only admissible one for g. For instance, if $m = 2$ and $r = 3$, then if g is a 3-cycle there are admissible digraphs on $\tilde{g}$ that have all edges oriented in the same direction. It is easy to see that the admissible digraph structures on $\tilde{g}$ correspond to the lattice points in a rational polytope determined by g and m.

3.9. Remark. One can show directly that the leading coefficient of $P_{2m,r}(N)$ is a positive integer $C_r^{(m)}$. For $m = 1$ this coefficient is the rth Catalan number, which counts (among

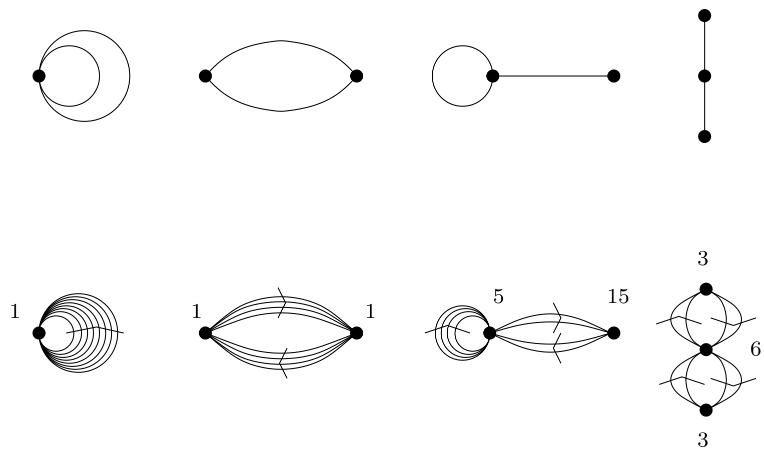


FIGURE 8. Graphs in G_2 and admissible digraph structures on their thickenings.

many things) the number of plane trees with $r+1$ vertices. For $m > 1$ we call these numbers *hypergraph Catalan numbers*; some of their properties, including other combinatorial interpretations, can be found in [6].

4. Computing the N-contribution I: arrangements

4.1. We recall from the introduction that $P_{2r}(N)$ has a geometric interpretation when $m = 1$. Let $\Pi = \Pi_{2r}$ be a polygon with $2r$ vertices. Any pairing $s \in S(2r; 2)$ of the edges of Π determines a compact orientable surface M_s with an embedded graph, in other words an *orientable map*. We have

$$P_{2r}(N) = \sum_{s \in S(2r;2)} N^{v(M_s)}, \tag{25}$$

where $v(M_s)$ is the number of vertices of the map.

4.2. The goal of this section is to generalize (25) to $m > 1$. As a first step, we see that the sum over pairings $S(2r; 2)$ must be replaced by a sum over partitions of $[\![2mr]\!]$ into blocks of order $2m$. To each $s \in S(2mr; 2m)$ we will associate a polynomial $a_s(N) \in \mathbb{Z}[N]$ that we call the *N-contribution* of s. We will prove in Theorem 4.7 that

$$P_{2m,r}(N) = \sum_{s \in S(2mr;2m)} a_s(N). \tag{26}$$

We remark that, unlike the classical case $m = 1$, the N-contributions need not be monomials in N, and moreover need not have positive coefficients.

4.3. We begin by recalling notation from the introduction. The trace polynomial is

$$\operatorname{Tr} X^k = \sum_{1 \le u_1, \ldots, u_k \le N} X_{u_1,u_2} X_{u_2,u_3} \cdots X_{u_k,u_1}.$$

We fix a polygon Π with k vertices and with one vertex distinguished. The vertices of Π are labeled in cyclic order with the u_i, with the distinguished vertex receiving u_1. The edge between u_i and u_{i+1} is labeled with the variable $X_{u_i,u_{i+1}}$.

Let $s \in S(2mr; 2m)$ be a partition. When $m = 1$, each block has two edges E, E' that must be glued with opposite orientations to make the corresponding product of variables $X_E X_{E'}$ give a nonzero contribution to $P_{2m,r}(N)$. In particular, if E appears first when walking counterclockwise around the perimeter of Π starting at the distinguished vertex, and is directed positively (pointing counterclockwise), then E' must be oriented in the opposite direction (pointing clockwise).

For $m > 1$, there are more possibilities for identifying edges. Blocks now consist of $2m$ edges; if the first is positively directed, then there are 2^{2m-1} choices of possible directions for the other. Not all of these will arise in the computation of $P_{2m,r}(N)$, and as a first step we consider which are necessary:

4.4. Definition. Let $s \in S(2mr; 2m)$ and let $E_1, \ldots, E_{2m}$ be the edges corresponding to a fixed block of s. We say that an assignment of directions to the E_i is *orientable* if (i) the first edge E_1 is positively oriented, and (ii) if there are a positively directed and b negatively directed edges, then (a, b) is a lattice point arising in item (ii) in Lemma 3.4. Otherwise we say the edge directions are *nonorientable*.

For example, for $m = 1$ the only orientable assignment is $+\,-$; the other assignment $+\,+$ is nonorientable. This explains the terminology, since if one uses edge directions $+\,+$ for any block of a pairing when gluing a polygon, the resulting map is nonorientable. For $m = 2$ the orientable assignments are

$$+ + + +, \quad + + - -, \quad + - + -, \quad + - - +;$$

all others are nonorientable. The orientable assignments correspond to the three lattice points $(4, 0), (2, 2), (0, 4)$ in Figure 7(a). The assignment $+ + + +$ arises when $(a, b) = (4, 0)$ or $(0, 4)$. The other three mixed assignments arise when $(a, b) = (2, 2)$.

It is clear that for general m one has 2^{2m-2} orientable assignments, and the same number of nonorientable assignments. We denote by $\mathscr{O}$ the set of orientable assignments for $2m$ edges.

4.5. Now let $s \in S(2mr; 2m)$ be a partition of $[\![2mr]\!]$ into r blocks of order $2m$. Let $\mathscr{B}$ be the blocks of s. Let $\mathbb{F}_q$ be a finite field of order $q \gg 0$, and let W be a $2mr$-dimensional vector space over $\mathbb{F}_q$. Choose a basis of W, and let $z_1, \ldots, z_{2mr}$ be the coordinate functions.

Each function $\omega\colon \mathscr{B} \to \mathscr{O}$ determines a subspace $W_\omega \subset W$ as follows. Using ω we can identify the edges of Π to make a CW-complex M_ω. The map $\Pi \to M_\omega$ determines a set of linear equations: if two vertices u_i, u_j of Π become identified in M_ω, then the set contains $z_i - z_j = 0$. The set of these equations determines the subspace $W_\omega \subset W$.

4.6. Definition. The *arrangement* $\mathscr{A}_s \subset W$ attached to the set partition s is

$$\mathscr{A}_s = \{W_\omega \mid \omega\colon \mathscr{B} \longrightarrow \mathscr{O}\}.$$

Let $a_s(q) \in \mathbb{Z}[q]$ be the number of points in $\mathscr{A}_s$.

4.7. Theorem. *We have*

$$P_{2m,r}(N) = \sum_{s \in S(2mr;2m)} a_s(N).$$

Proof. The theorem follows from the preceding discussion and Lemma 3.4. For each partition the only monomials in a block that can give a nonzero expectation are those for which the edges are directed orientably. Thus we must consider all such choices, which corresponds to considering all functions $\omega\colon \mathscr{B} \to \mathscr{O}$. Given any ω we can freely assign the subscripts of the edge variables $X_{u_i,u_{i+1}}$, subject to the equivalence relation imposed by ω. The number of assignments is N^d, where $d = \dim_{\mathbb{F}_q}(W_\omega)$. Since the number of $\mathbb{F}_q$ points of W_ω is q^d, the polynomial $a_s(N)$ exactly counts the number of assignments. This completes the proof. $\square$

4.8. Example. We consider $m = 2$ and $r = 2$. There are 35 partitions of $[\![8]\!]$ into blocks of size 4. The polygon Π is an octagon. The cyclic group C_8 of order 8 acts on Π by rotation, and if two partitions s, s' are equivalent under rotation then $a_s(N) = a_{s'}(N)$. Thus it suffices to count the sizes of the C_8 orbits and for each orbit determine $a_s(N)$ for a representative.

Figure 9 shows representatives for the 7 orbits, along with the size of the orbit (to the upper left) and the N-contribution of each orbit. It turns out that all the N-contributions monomials, except for the third in the top row, which is $2N^2 - N$. For this partition, the subspace arrangement $\mathscr{A}_s \subset \mathbb{F}_q^8$ consists of two 2-planes intersecting in a line. The orientation assignments that give the two 2-planes are shown in Figure 10.

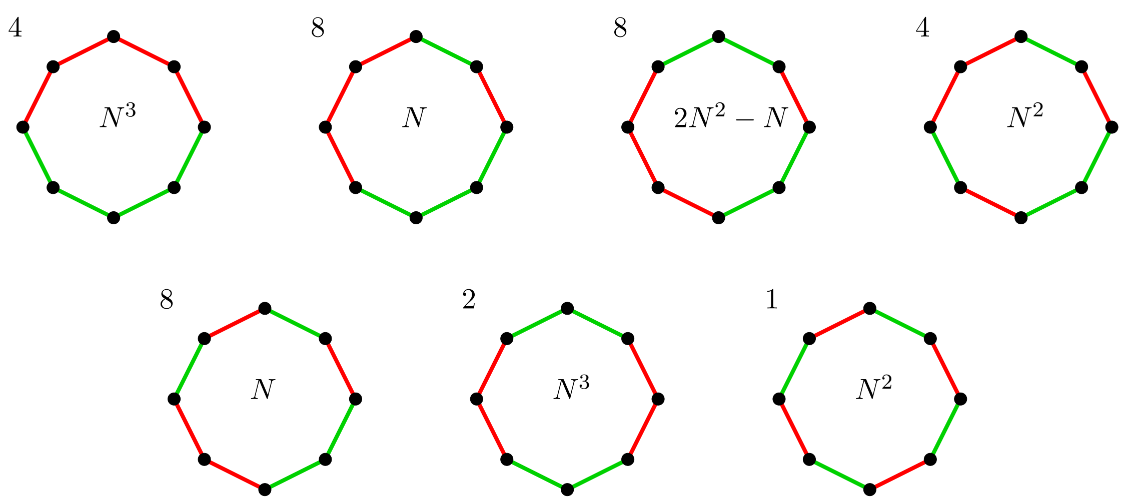


FIGURE 9. Computing $P_8(N) = 6N^3 + 21N^2 + 8N$. Each octagon has a partition s into two order 4 blocks. The N-contributions $a_s(N)$ appear in the center. Above and to the left of each octagon is the number of partitions equivalent to it modulo rotation.

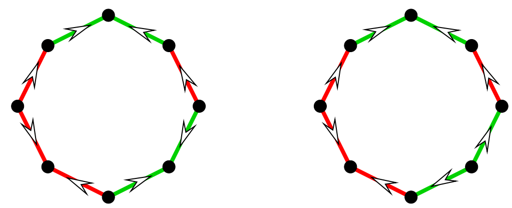


FIGURE 10. Two different orientation assignments ω_1, ω_2 leading to $2N^2 - N$. In both octagons, the leftmost vertex is labeled 1, and the labels increase counterclockwise. The left assignment ω_1 gives the equations $z_1 = z_2 = z_4 = z_5 = z_7 = z_8$, $z_3 = z_6$. The right assignment ω_2 gives $z_1 = z_3 = z_5 = z_7$, $z_2 = z_4 = z_6 = z_8$. Thus the two subspaces $W_{\omega_1}, W_{\omega_2} \subset \mathbb{F}_q^8$ are 2-planes intersecting in the line $z_1 = \cdots = z_8$. Counting the points in this arrangement gives $2q^2 - q$.

5. Computing the N-contribution II: topological expansion

We now express $P_{2m,r}(N)$ as a sum of monomials in N. Instead of a sum over maps, we have a weighted sum over what we call *CW maps with instructions*. Recall that $P_{2m,r}(N)$ is defined as the expectation

$$\left\langle \mathrm{Tr} X^{2mr} \right\rangle_{2m}.$$

In Theorem 5.8 we express the expectation of more general functions $f(X, \gamma)$ corresponding to an arbitrary directed graph γ, and then Corollary 5.9 gives the result when that function is the trace.

Let K and r be positive integers. We let $V(1), \ldots, V(K)$ be a set of indices, where an index $V(i)$ may take any integer value from 1 to N. Let γ be a directed graph with ordered vertex set

$$V_\gamma = \{V(1), V(2), \ldots, V(K)\}$$

and ordered edge set

$$E_\gamma = \{E(1), E(2), \ldots, E(2mr)\}$$

where

$$E(i) = (E(i,1), E(i,2))$$

and $E(i,1)$ and $E(i,2)$ denote elements of V_γ. When the index $V(i)$ takes on an integer value h, we think of the vertex $V(i)$ in γ as being labeled by the integer h.

Recall that X denotes an $N \times N$ Hermitian matrix. For $E(i) \in E_\gamma$, let $X_{E(i)}$ denote an entry of X given

$$X_{E(i)} = X_{E(i,1),E(i,2)}.$$

5.1. Definition. Let $f(X, \gamma)$ denote the function

$$f(X, \gamma) = \sum_{\substack{1 \le V(j) \le N \\ 1 \le j \le |V_\gamma|}} \prod_{i=1}^{|E_\gamma|} X_{E(i)}.$$

Recall that $S(2mr; 2m)$ denotes the set of all set partitions of the set

$$[\![2mr]\!] = \{1, 2, \ldots, 2mr\}$$

into r subsets each of order $2m$. Let s denote an element of $S(2mr; 2m)$. Such an s determines r functions s_i

$$s_i\colon [\![2m]\!] \longrightarrow [\![2mr]\!], \quad 1 \le i \le r, \tag{27}$$

with disjoint images. We view a set partition of $[\![2mr]\!]$ as a set partition of the ordered edge set E_γ of γ. Then we have the following result.

5.2. Lemma.

$$\big\langle f(X,\gamma)\big\rangle_{2m} = \sum_{s \in S(2mr;2m)} \sum_{\substack{1 \le V(j) \le N \\ 1 \le j \le |V_\gamma|}} \prod_{i=1}^{r} \big\langle \prod_{h=1}^{2m} X_{E(s_i(h))} \big\rangle_{2m}.$$

Proof. This follows directly from the definitions and item (i) in the generalization of Wick's theorem (Lemma 3.4). □

We next show how Lemma 3.4 is used to calculate $f(X, \gamma)$.

5.3. Lemma. *Given a set of values for $V(1), \ldots, V(K)$ and a function s_i of the form (27), the expression*

$$\big\langle \prod_{j=1}^{2m} X_{E(s_i(j))} \big\rangle_{2m} \tag{28}$$

is equal to 1 if and only if one of the following is true:

(i) *There are two distinct integers c and d such that*

$$(E(s_i(j), 1), E(s_i(j), 2)) = (c, d) \text{ or } (d, c) \text{ for } 1 \le j \le 2m,$$

and the number of edges that equal (c, d) differs from the number of edges that equal (d, c) by a multiple of 4.

(ii) *There is some integer c such that*

$$E(s_i(j), 1) = E(s_i(j), 2) = c \text{ for } 1 \le j \le 2m.$$

Otherwise (28) *is equal to 0.*

Proof. This follows directly from item (ii) of Lemma 3.4. □

We next define a CW map with instructions.

5.4. Definition. We define an *l-edge gluing instruction* to be either:

(i) a partition of $[\![2l]\!]$ into two subsets of order l such that $2i - 1$ and $2i$ are not in the same subset for $1 \le i \le l$; or

(ii) The whole set $[\![2l]\!]$.

We call case (i) a *non-loop gluing instruction*, and we call case (ii) the *loop gluing instruction*. We define a *CW map with instructions* to be the following triple (γ, s, I) consisting of:

(i) a directed graph γ;
(ii) a set partition s of the edge set E_γ of γ; and
(iii) a correspondence I that takes each subset in s of order l to an l-edge gluing instruction.

5.5. Lemma. *Given two distinct integers c and d with $1 \leq c, d \leq N$, there are exactly 2^{2m-2} possible choices for u_i and $v_i \in \{c, d\}$ with $u_i \neq v_i$ such that*

$$\langle X_{c,d} \prod_{i=2}^{2m} X_{u_i,v_i} \rangle_{2m} \neq 0.$$

Proof. Using condition 1 of Lemma 5.3, there are

$$\frac{1}{2}\binom{2m}{m} + \sum_{i=1}^{\lfloor \frac{m}{2} \rfloor} \binom{2m}{m-2i}$$

ways for the condition to be satisfied. Applying the identity

$$\sum_{i=-\lfloor \frac{m}{2} \rfloor}^{\lfloor \frac{m}{2} \rfloor} \binom{2m}{m-2i} = 2^{2m-1}$$

completes the proof. □

Each of the 2^{2m-2} possibilities corresponds to a unique $2m$-edge non-loop gluing instruction. We call this set of 2^{2m-2} non-loop gluing instructions the *orientable* non-loop gluing instructions and denote it by $\mathscr{O}$ as in Section 4.

5.6. Definition. Let M be a CW map with instructions on a directed graph γ. Given an ordered set of l edges of γ

$$\{(v_1, v_2), (v_3, v_4), \ldots, (v_{2l-1}, v_{2l})\}, \tag{29}$$

we let v_i correspond to the integer i; then a set partition of $[\![2l]\!]$ as in Definition 5.4 corresponds to a set partition of the vertices in the list (29). We say that two vertices in the same subset are in the same equivalence class. Thus all the different gluing instructions given by M together determine an equivalence relation on the vertices of γ. We call each equivalence class a *map vertex* and let $v(M)$ denote the number of map vertices of M. We let $\ell(M)$ denote the number of instructions of M that are loop-gluing instructions.

A *labeling of* M is an labeling of the vertices of γ with integers in $[\![N]\!]$ such that if two vertices of γ are in the same equivalence class determined by M, then they have the same label.

Therefore the number of possible labelings of a CW map M with instructions is

$$N^{v(M)}.$$

5.7. Remark. Two different CW maps M_1 and M_2 with instructions may have the same labeling of γ as follows. Suppose M_1 and M_2 have the same set partition s and the same gluing instructions I, except for some subset s_i for which M_1 has a non-loop gluing instruction and M_2 has the loop gluing instruction. Any labeling of γ for M_2 is then also labeling for M_1.

Likewise suppose we are given a set partition s of E_γ and a labeling of γ which comes from some CW map M with instructions. Then each subset s_i has its vertices labeled either all with the same number, or with exactly two different numbers such that the vertices in each edge are labeled distinctly. If there two distinct labels in an s_i, then this determines uniquely the gluing instruction that M has for that s_i. If there is only one label, then M may have the loop-gluing instruction or any of the 2^{2m-2} non-loop gluing instructions for s_i.

In the next theorem, we show how to account for the common labelings by using inclusion-exclusion.

5.8. Theorem.

$$\big\langle f(X,\gamma)\big\rangle_{2m} = \sum_M (1-2^{2m-2})^{\ell(M)} N^{v(M)}$$

where the sum is over CW maps M with instructions such that the directed graph for M is γ; each subset in a set partition of E_γ is of order $2m$; and each gluing instruction is either the loop gluing instruction or one of the 2^{2m-2} orientable non-loop gluing instructions.

Proof. In Lemma 5.2, we fix a set partition $s \in S(2mr;2m)$ and consider

$$\sum_{\substack{1\le v_j\le N\\ 1\le j\le |V|}} \prod_{i=1}^{r} \big\langle \prod_{j=1}^{2m} X_{E(s_i(j))}\big\rangle_{2m}.$$

This expression is equal to the number of labelings of γ that, for each subset of $2m$ edges determined by s, satisfy the criteria in Lemma 5.3. That is, it is the number of labelings of γ that can arise as labelings of CW maps M with instructions that have the set partition s. Let A denote the set of these labelings. To such a labeling $a \in A$ we associate an r-tuple of integers $a(i)$ from 0 to 2^{2m-2}: if the i-th subset s_i has two distinct labels, then we set $a(i)$ to be the number from 1 to 2^{2m-2} that corresponds to that non-loop gluing instruction, where we have fixed an arbitrary ordering on the orientable non-loop gluing instructions; and if the i-th subset s_i has only one label, then we set $a(i)$ to be 0.

For integers $0 \le g_i \le 2^{2m-2}$, let $A(g_1, g_2, \ldots, g_r) \subset A$ denote the set of labelings a such that

$$a(i) = g_i.$$

Since the sets $A(g_1, g_2, \ldots, g_r)$ are disjoint for distinct r-tuples $(g_1, g_2, \ldots, g_r)$, we have

$$|A| = \sum_{0\le g_1,g_2,\ldots,g_r\le 2^{2m-2}} |A(g_1, g_2, \ldots, g_r)|.$$

We define $\tilde{A}(g_1, \ldots, g_r)$ to be

$$\tilde{A}(g_1, \ldots, g_r) = \bigcup_{\tilde{g}_i} A(\tilde{g}_1, \ldots, \tilde{g}_r)$$

where

$$\tilde{g}_i \in \begin{cases} \{0\} \text{ if } g_i = 0 \\ \{0, g_i\} \text{ if } g_i \neq 0. \end{cases}$$

We let $g_i = 0$ indicate the loop-gluing instruction and let M denote the CW map with set partition s and gluing instructions $(g_1, \ldots, g_r)$. Then

$$|\tilde{A}(g_1, \ldots, g_r)| = N^{v(M)}.$$

Using

$$A = \bigcup_{0 \leq g_1, g_2, \ldots, g_r \leq 2^{2m-2}} \tilde{A}(g_1, g_2, \ldots, g_r),$$

we compute $|A|$ by inclusion-exclusion. We claim

$$|A| = \sum_M (1 - 2^{2m-2})^{\ell(M)} N^{v(M)},$$

where the sum is over all maps CW M with instructions on γ using the set partition s. This is equivalent to

$$|A| = \sum_{0 \leq g_1, g_2, \ldots, g_r \leq 2^{2m-2}} (1 - 2^{2m-2})^{\#\text{ 0's in } (g_1, \ldots, g_r)} |\tilde{A}(g_1, \ldots, g_r)|.$$

Thus it is sufficient to prove for each labeling a that

$$1 = \sum_{0 \leq g_1, g_2, \ldots, g_r \leq 2^{2m-2}} (1 - 2^{2m-2})^{\#\text{ 0's in } (g_1, \ldots, g_r)} \mathbf{1}(a \in \tilde{A}(g_1, \ldots, g_r)) \tag{30}$$

where

$$\mathbf{1}(a \in \tilde{A}(g_1, \ldots, g_r)) = \begin{cases} 1 \text{ if } a \in \tilde{A}(g_1, \ldots, g_r) \\ 0 \text{ if } a \notin \tilde{A}(g_1, \ldots, g_r) \end{cases}.$$

Suppose in an r-tuple $(g_1, \ldots, g_r)$ that $g_{i_1}, \ldots, g_{i_k} = 0$ for some $i_1, \ldots, i_k$ and $g_i \neq 0$ for all other i. We then say that another r-tuple $(g'_1, \ldots, g'_r)$ contains $(g_1, \ldots, g_r)$ if $g'_{i_j}, 1 \leq j \leq k$, are any integers from 0 to 2^{2m-2} and $g_i = g'_i$ for all other i. Then

$$A(g_1, \ldots, g_r) \subset \tilde{A}(g'_1, \ldots, g'_r)$$

exactly when $(g'_1, \ldots, g'_r)$ contains $(g_1, \ldots, g_r)$. There are

$$(2^{2m-2})^{k-j} \binom{k}{k-j}$$

r-tuples that have exactly j zeros and that contain $(g_1, \ldots, g_r)$. Thus the right side of equation (30) becomes

$$\sum_{j=0}^{k} (1 - 2^{2m-2})^j (2^{2m-2})^{k-j} \binom{k}{k-j}.$$

By the binomial theorem this number is equal to 1. Summing over all $s \in S(2mr; 2m)$ completes the proof. $\square$

5.9. Corollary. *Let $\langle \cdot \rangle_{2m}$ now denote the linear form with normalizations $C_0 = \frac{1}{N}$ and $C_1 = \frac{1}{2N}$ as in Lemma 3.4. We have the topological expansion*

$$N\big\langle \mathrm{Tr}(X^{2mr})\big\rangle_{2m} = \sum_M (1-2^{2m-2})^{\ell(M)} N^{\chi(M)}.$$

where the sum is over CW maps M with instruction as in Theorem 5.8 and γ is a $2mr$-gon with directed edges all in the same direction; and where $\chi(M)$ denotes the Euler characteristic

$$\chi(M) = v(M) - r + 1.$$

Proof. If γ is a $2mr$-gon with directed edges all in the same direction, along with an ordering on the edges, then

$$f(X,\gamma) = \mathrm{Tr}(X^{2mr}).$$

We have

$$\Big\langle \prod_{j=1}^{2m} X_{E(s_i(j))}\Big\rangle_{2m} = \begin{cases} C_0 \text{ if } s_i \text{ has the loop gluing instruction} \\ 2C_1 \text{ if } s_i \text{ has an orientable non-loop gluing instruction} \end{cases}.$$

The normalizations $C_0 = \frac{1}{N}$ and $C_1 = \frac{1}{2N}$ amount to dividing the expression on the right of the theorem statement by N^r. Multiplying by N makes the exponent of each monomial

$$v(M) - r + 1$$

which we interpret as

$$\#\ \text{vertices} - \#\ \text{edges} + \#\ \text{faces}$$

since after gluing we have r edges and the polygon has exactly one face. This completes the proof. □

We note that in Corollary 5.9, when $m = 1$ we obtain formula (8) for $P_2(N)$ as follows. When $m = 1$, there is exactly one orientable non-loop gluing instruction. This instruction pairs an edge (c, d) with an edge (d, c) for integer labels $c \neq d$. And since

$$(1 - 2^{2\cdot 1-2})^{\ell(M)} = \begin{cases} 0 \text{ if } \ell(M) > 0 \\ 1 \text{ if } \ell(M) = 0, \end{cases}$$

the sum over CW maps with instructions reduces to a sum over classical maps.

For a directed graph γ with $2mr$ edges, there are $W_{2m}(2mr)(2^{2m-2}+1)^r$ CW maps with instructions on γ, using the loop and orientable non-loop gluing instructions.

5.10. Example. We compute $\big\langle f(X,\gamma)\big\rangle_4$, using the normalizations of Definition 3.2, for γ an octagon with all its edges directed in the same direction. Thus $m = 2$ and $r = 2$, giving 875 CW maps with instructions (which we just abbreviate to maps in the following). There are 35 maps that have two loop-gluing instructions and each of these maps has exactly 1 map vertex. Their contribution is thus

$$(-3)^2 35N.$$

There are $8 \cdot 35 = 280$ maps that have exactly one loop-gluing instruction; 260 of these have exactly 1 map vertex and 20 have exactly 2 map vertices. Their contribution is thus

$$(-3)(260N + 20N^2).$$

There are $16 \cdot 35 = 560$ maps with no loop-gluing instructions; 473 of these have exactly one map vertex, 81 have exactly 2 map vertices, and 6 have exactly three map vertices. Their contribution is thus

$$473N + 81N^2 + 6N^3.$$

Adding up these contributions gives

$$8N + 21N^2 + 6N^3$$

which agrees with Example 4.8.

6. Complements

We conclude with some complements and directions for further work.

6.1. First, in all known examples the polynomials $P_{2m,r}(N)$ have positive integral coefficients. This does not follow from the results in this paper, since all expressions we have for $P_{2m,r}(N)$ involve sums of polynomials with coefficients that can be negative. We do not know if the coefficients are always positive, and if so if they naturally count something (see §6.3 below for a suggestion). We remark that when one considers similar polynomials for products of traces of powers — i.e. for CW maps with instructions built from more than one 2-cell — negative coefficients *do* occur. (Such geometric objects will be considered in sequel to this paper.)

6.2. Next, in the case $m = 1$ Harer–Zagier [8] proved a beautiful generating function for the polynomials $P_{2r}(N)$ (see [11, §3.1] for an exposition). Let

$$T(N, s) = 1 + 2Ns + 2s \sum_{r \geq 1} \frac{P_{2r}(N)}{(2r-1)!!} s^r.$$

Then we have

$$T(N, s) = \left(\frac{1+s}{1-s}\right)^N. \tag{31}$$

It would be very interesting to find the generalization of (31) to higher m. Generating functions for the leading coefficients of the $P_{2m,r}(N)$ for m fixed and $r \to \infty$ can be found in [6].

6.3. The leading coefficients $C_r^{(m)}$ of the $P_{2m,r}$, studied in [6], can be directly defined in terms of plane trees with additional data. In particular, let $\mathcal{T}_{mr}$ be the set of plane trees with $mr + 1$ vertices. Then

$$C_r^{(m)} = \sum_{T \in \mathcal{T}_{mr}} N_m(T)$$

where $N_m(T)$ is the number of *admissible m-labelings* of T (see [6, §4] for more details). Since a plane tree can be regarded as the 1-skeleton of an orientable map of genus 0, one can ask if there is an interpretation of the other coefficients of $P_{2m,r}(N)$ in terms of maps of higher genera with labeling data on their 1-skeletons. Note that, if so, then one expects that nonorientable maps should also be involved.

6.4. In addition to the GUE, classically one considers two other matrix models, the *Gaussian orthogonal ensemble* (GOE) and the *Gaussian symplectic ensemble* (GSE). The former replaces the vector space V with the vector space $V_{\mathbb{R}}$ of $N \times N$ real symmetric matrices, the latter with the space $V_{\mathbb{H}}$ of $N \times N$ quaternionic Hermitian matrices. The combinatorial interpretations of these models were thoroughly investigated by Mulase–Waldron [13]. In both cases the expectations $\langle \operatorname{Tr} X^k \rangle$ are computed in terms of topological surfaces, but in contrast to the GUE one also must include nonorientable surfaces. For example, for the GOE one has

$$P^{\mathbb{R}}_{2r}(N) = 2^{-r} \sum_s N^{v(M_s)},$$

where the sum is now taken over *all* possible pairings of the edges of Π, regardless of whether the resulting surface is orientable or not. The GSE case is similar, except that now one has

$$P^{\mathbb{H}}_{2r}(N) = 2^{-r} \sum_s \alpha(s) N^{v(M_s)},$$

where $\alpha(s) \in \{\pm 1\}$ depends on the topology of M_s.

One can define hypergraph versions of the GOE, GSE models in the obvious way. For instance for the GOE, one replaces the off-diagonal complex variables $X_{u,v} = x_{u,v} + iy_{u,v}$ with real variables satisfying $X_{u,v} = X_{v,u}$. The polynomials $P^{\mathbb{R}}_{2m,r}(N)$ can be computed using the techniques in §4 with some modifications. Let $\mathscr{O}^{\pm}$ be the set of all edge directions, both orientable and nonorientable (thus $|\mathscr{O}^{\pm}| = 2^{2m-1}$). Given $s \in S(2mr; 2m)$ with blocks $\mathscr{B}$, we now consider functions

$$\omega \colon \mathscr{B} \longrightarrow \mathscr{O}^{\pm}$$

and the associated subspace arrangements $\mathscr{A}^{\pm}_s$. We then have

$$P^{\mathbb{R}}_{2m,r}(N) = 2^{-r} \sum a^{\pm}_s(N),$$

where now $a^{\pm}_s(q)$ counts the points in $\mathscr{A}^{\pm}_s$.

The analogous computations for the GSE have not been investigated except in special cases.

Mulase–Waldron also prove a duality theorem for the GOE and the GSE. In particular, if

$$P^{\mathbb{R}}_{2r}(N) = a_{r+1}N^{r+1} + a_r N^r + \cdots + a_1 N,$$

then

$$P^{\mathbb{H}}_{2r}(N) = 2^{2r} a_{r+1} N^{r+1} - 2^{2r-1} a_r N^r + \cdots \pm 2^r a_1 N.$$

The same relation does not hold for the hypergraph versions. For example, for $m = 2$ and $r = 2$, we have

$$P^{\mathbb{R}}_8(N) = \frac{3N^3}{2} + \frac{57N^2}{4} + \frac{77N}{4},$$
$$P^{\mathbb{H}}_8(N) = 24N^3 + 12N^2 - N.$$

The failure of the duality is not surprising, since now the N-contributions are no longer monomials attached to surfaces. In the hypergraph case the N-contributions are more complicated polynomials attached to subspace arrangements; the arrangements themselves need not even be pure (maximal subspaces need not have the same dimension). However, one might expect a duality holds that takes into account the more complicated structure of the N-contributions.

We note in passing that Mulase–Waldron also suggest constructing Gaussian ($m = 1$) analogues of the classical matrix models using Clifford algebras over $\mathbb{R}$ in place of $\mathbb{C}, \mathbb{H}$. To the best of our knowledge such models have not been constructed. One could consider hypergraph versions as well.

6.5. The vector spaces underlying the classical matrix ensembles have another interpretation, namely as *simple formally real Jordan algebras over* $\mathbb{R}$ [10]. Briefly, a *Jordan algebra* over a field k is a nonassociative algebra over k whose multiplication satisfies $x \bullet y = y \bullet x$ and $(x \bullet x) \bullet (y \bullet x) = ((x \bullet x) \bullet y) \bullet x$; it is *simple* if it cannot be written as a direct sum. Although nonassociative, Jordan algebras are power-associative: if one puts $x^n := x \bullet x^{n-1}$ for $n > 1$, then x^n can be computed as $x \bullet \cdots \bullet x$ with any choice of bracketing. A Jordan algebra A over $\mathbb{R}$ is called *formally real* if $\sum_{i=1}^n x_i^2 = 0$ implies each $x_i = 0$. It is known that a real Jordan algebra being formally real is equivalent to it having a *positive definite trace form* $\mathrm{Tr}\colon A \to \mathbb{R}$. This is a linear map satisfying $\mathrm{Tr}(x^2) > 0$ for all $x \in A$, $x \neq 0$, and one has in addition that the trace pairing $\mathrm{Tr}(x \bullet y)$ is a positive definite quadratic form on A. For the matrix spaces above, the Jordan product is given by $x \bullet y = (xy + yx)/2$, and the trace is the usual one.

For a simple formally real Jordan algebra, one can make a Gaussian measure $e^{-\mathrm{Tr}\, X^2/2}\, dX$, and can thus define the expectations $\langle \mathrm{Tr}\, X^k \rangle$. Such algebras were classified by Jordan, von Neumann, and Wigner in 1934 [9]. Apart from the spaces of Hermitian matrices, there are two others:

- The *spin factor* $\mathbb{S} = \mathbb{S}_{1,N}$ of pairs $\mathbf{x} = (x_0, x) \in \mathbb{R} \times \mathbb{R}^N$ equipped with the Jordan product $\mathbf{x} \bullet \mathbf{y} = (x_0 y_0 + x \cdot y, x_0 y + y_0 x)$, where $\cdot$ denotes the usual dot product on $\mathbb{R}^N$. The trace form is $\mathrm{Tr}(\mathbf{x}) = x_0$.
- The *Albert algebra* $\mathbb{A}$ of 3×3 Hermitian matrices over the *octonions* $\mathbb{O}$, equipped with the same Jordan product as $V_{\mathbb{R}}$, $V_{\mathbb{C}}$, $V_{\mathbb{H}}$, and with the usual trace as trace form.

The matrix models associated to $\mathbb{S}$ and $\mathbb{A}$ were considered in [7], where the expectations were computed in terms of various combinatorial objects. One could also define hypergraph analogues of these models. The algebras $\mathbb{S}_{1,N}$ depend on a dimensional parameter, and so one could expect to determine a polynomial $P_k(N)$. The Albert algebra is exceptional (the $N \times N$ octonionic Hermitian matrices are not a Jordan algebra for $N \geq 4$). Nevertheless, as in [7] one might expect to compute $\langle \mathrm{Tr}\, X^k \rangle$ as a sum over powers of 3; such an expansion allows one define the expectation $\langle \mathrm{Tr}\, X^k \rangle$ for $N > 3$, even though the matrix model doesn't exist.

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Appendix A. Tables of $P_{2m,r}(N)$

m	$P_{2m,2}(N)$
1	$2N^3 + N$
2	$6N^3 + 21N^2 + 8N$
3	$20N^3 + 288N^2 + 154N$
4	$70N^3 + 3685N^2 + 2680N$
5	$252N^3 + 49500N^2 + 42626N$
6	$924N^3 + 698250N^2 + 652904N$
7	$3432N^3 + 10174752N^2 + 9880116N$
8	$12870N^3 + 151215333N^2 + 149311992N$
9	$48620N^3 + 2274899220N^2 + 2262619810N$
10	$184756N^3 + 34501184146N^2 + 34421895508N$
11	$705432N^3 + 526280849952N^2 + 525767926476N$
12	$2704156N^3 + 8063562387946N^2 + 8060236749448N$
13	$10400600N^3 + 123990434535968N^2 + 123968821537484N$
14	$40116600N^3 + 1912243020174900N^2 + 1912102240088720N$
15	$155117520N^3 + 29566604787876288N^2 + 29565685839436904N$
16	$601080390N^3 + 458159040047894757N^2 + 458153029822320120N$
17	$2333606220N^3 + 7113280057151509380N^2 + 7113240678135172770N$
18	$9075135300N^3 + 110628264279384010866N^2 + 110628005849959243436N$
19	$35345263800N^3 + 1723156011388074958800N^2 + 1723154312923210454700N$
20	$137846528820N^3 + 26876807772080973901810N^2 + 26876796594449267800180N$
21	$538257874440N^3 + 419727658377406433224500N^2 + 419727584728000854024720N$
22	$2104098963720N^3 + 6562126588326317167437420N^2 + 6562126102514004328079760N$
23	$8233430727600N^3 + 102698863964408113176128640N^2 + 102698860756613228059232280N$
24	$32247603683100N^3 + 1608766764066834928980643050N^2 + 1608766742866282277626475400N$
25	$126410606437752N^3 + 25222836206508763559400612000N^2 + 25222836066273206697399198876N$
26	$495918532948104N^3 + 395766462495661160636578778460N^2 + 395766461567312931123662337504N$
27	$1946939425648112N^3 + 6214446125959394141657011559232N^2 + 6214446119809324376212824301816N$
28	$7648690600760440N^3 + 97647511242174540487608932875252N^2 + 97647511201404346543865732597200N$

Table 1. The polynomials $\left\langle \mathrm{Tr}\, X^{4m} \right\rangle_{2m}$ for various m.

m	$P_{2m,3}(N)$
1	$5N^4 + 10N^2$
2	$57N^4 + 715N^3 + 2991N^2 + 2012N$
3	$860N^4 + 53214N^3 + 1140454N^2 + 1664328N$
4	$15225N^4 + 3968087N^3 + 483128259N^2 + 1090473724N$
5	$299880N^4 + 319033030N^3 + 249112347045N^2 + 675734451935N$
6	$6358044N^4 + 27351953116N^3 + 144661285111944N^2 + 419433316997096N$
7	$141858288N^4 + 2456767685792N^3 + 89115824374493808N^2 + 264310496367750112N$
8	$3279398265N^4 + 228032530114903N^3 + 5663532405600489 7923N^2 + 16925535864252143 5964N$
9	$77730738800N^4 + 21665759515089126N^3 + 3668647198070964434 1697N^2 + 10991679426775361 9873127N$
10	$1875933348432N^4 + 2093828838718703120N^3 + 2408427007262037845 6338401N^2$ $+ 7222093024662821086 3587697N$
11	$45882224217648N^4 + 204952224882223813324N^3 + 1597395299645145316 2136090332N^2$ $+ 4791461595820439965 6971326696N$
12	$1133728265594652N^4 + 20260640463398745554900N^3 + 1068278728769010186 2034633426336N^2$ $+ 3204669390138106824 0448254609112N$
13	$28240495384558800N^4 + 2018693356691719867568864N^3 + 7193631461776444705 190769635498784N^2$ $+ 2158050595829087519 3026086420843808N$
14	$708064561500246000N^4 + 202433132515998915941341072N^3 + 4872505050842274403 228753037344910544N^2$ $+ 1461742385331744301 280882844201677 3264N$
15	$17849706012216423360N^4 + 20409382526463993156412539904N^3$ $+ 3316996638652912728 2527442068954053363 84N^2$ $+ 9950968289248550591 6319998363295961422 08N$
16	$452052794695103608185N^4 + 2067140897618143068480820110167N^3$ $+ 2268007122287693777 3395761362289660357 50019N^2$ $+ 6804016210546600711 6328072765864664746 77564N$
17	$11494037187436243492800N^4 + 210200930632671059865866117731558N^3$ $+ 1556754869151969792 5042464394388726325 99914801N^2$ $+ 4670263371286628536 2688482301562185567 84490591N$
18	$293268344389135999653000N^4 + 21449528858806475000358293855272820N^3$ $+ 1072214813013740985 7871915553902204867 96784074061N^2$ $+ 3216644141290694613 8667992337853394135 36734731829N$
19	$7505725776437016547650000N^4 + 2195485483980801935619242747107823320N^3$ $+ 7407459754396070422 5072881037590220750 7522345766640N^2$ $+ 2222237854311481677 2431677729124376790 81998585455640N$
20	$192624656075289400261899600N^4 + 225341438539793329136178845209819555124N^3$ $+ 5131536292064386873 1293693381220087992 865527128977 2773N^2$ $+ 1539460870144325438 3049209866231582238 0478935006184 9673N$

TABLE 2. The polynomials $\left\langle \mathrm{Tr}\, X^{6m} \right\rangle_{2m}$ for various m.

m	$P_{2m,4}(N)$
1	$14N^5 + 70N^3 + 21N$
2	$678N^5 + 19405N^4 + 228190N^3 + 1151300N^2 + 1228052N$
3	$57200N^5 + 9249672N^4 + 663296368N^3 + 23015663304N^2 + 72509379000N$
4	$7043750N^5 + 5133996301N^4 + 2194206265774N^3 + 649578371471156N^2 + 3496601133322644N$
5	$1112865264N^5 + 3292356247240N^4 + 8934623322380920N^3 + 2620737801759328952 0N^2$
	$+ 1698403870270139765 36N$
6	$203356067376N^5 + 2327989224345286N^4 + 4235853959599628566 0N^3 + 1252484603218971137182922N^2$
	$+ 857286059683584915198 2156N$
7	$40309820014464N^5 + 1760470794016891600N^4 + 2208756281685465167 20800N^3$
	$+ 6448748233621864789256 0499440N^2 + 4484021274269168999124 04473696N$
8	$8379037668637350N^5 + 1397674190499809027085N^4 + 1220475253273708488464567982N^3$
	$+ 345473434226909506166381984 1923124N^2 + 2413371312104473167388767 1026753684N$
9	$1795172959531094000N^5 + 1151182660249369448162472N^4 + 6995785268932128999554007655768N^3$
	$+ 1899209167489695720998095404166 41589392N^2 + 1328599222453987153177911406958 644513368N$
10	$392800279200915370928N^5 + 975656238155129500067949730N^4$
	$+ 411061905784452997872640238363 82940N^3 + 1064153911144033132935093384542340 9379274150N^2$
	$+ 744757950149460036397258259455 42355037009252N$
11	$87320368271147678319744N^5 + 845866517161545514341421235664N^4$
	$+ 245905372276288767953660446911 752974176N^3$
	$+ 605285088234616975494014612412 57793070921 3913264N^2$
	$+ 423672469347343731998947492683 95849040809 87437152N$
12	$19656886903997074769845808N^5 + 746867552705760727566904485163270N^4$
	$+ 149143475086965731940838721825 65737 24385180N^3$
	$+ 348541507134977305102124679289 33753390730 606921429450N^2$
	$+ 243974070260097400637009962309 70634246809 5564073801292N$
13	$4470981124924626 7888976 80000N^5 + 6693561533005877523300126926584 91792N^4$
	$+ 914601538912495928127062240959 9491598939420448N^3$
	$+ 202778037846411408653795929492 88519305725765287006094 56944N^2$
	$+ 141943697145779570294416267243 84345718936643410238390073440N$
14	$1025836407005311164005708400000N^5 + 607304198247408170031119018846109803356N^4$
	$+ 566014679278204067175653792772 1198607015153 0841304N^3$
	$+ 119013143695723056191287211622 5521398287139428828889 50522427956N^2$
	$+ 833090254865760852286285074073 343498022658025500053370799984984N$

Table 3. The polynomials $\left\langle \mathrm{Tr}\, X^{8m} \right\rangle_{2m}$ for various m.

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