MODULAR SYMBOLS FOR $\mathbb{Q}$-RANK ONE GROUPS AND VORONOĬ REDUCTION

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ABSTRACT. Let G be a reductive algebraic group of $\mathbb{Q}$-rank one associated to a self-adjoint homogeneous cone defined over $\mathbb{Q}$, and let $\Gamma \subset G$ be a torsion-free arithmetic subgroup. Let d be the cohomological dimension of Γ. We present an algorithm to compute the action of the Hecke operators on $H^d(\Gamma; \mathbb{Z})$. This generalizes the classical modular symbol algorithm, when $\Gamma \subset SL_2(\mathbb{Z})$, to a setting including Bianchi groups and Hilbert modular groups. In addition, we generalize some results of Voronoĭ for real positive-definite quadratic forms to self-adjoint homogeneous cones of arbitrary $\mathbb{Q}$-rank.

1. INTRODUCTION

1.1. Let $\mathfrak{H}_2 = SL_2(\mathbb{R})/SO(2)$ be the upper-half plane, and let $\Gamma(N) \subset SL_2(\mathbb{Z})$ be the principal congruence subgroup of level $N > 2$, acting on $\mathfrak{H}_2$ from the left by linear fractional transformations. Then the cohomology group $H^1(\Gamma(N)\backslash\mathfrak{H}_2; \mathbb{C})$ is closely related to the space of all weight-two modular forms of level N. The *modular symbols* provide a concrete approach to the group $H^1(\Gamma(N)\backslash\mathfrak{H}_2; \mathbb{C})$ (§2.1) that has allowed the testing of many conjectures in number theory and has led to explicit formulas for L-functions and their derivatives [9][11][18]. Important to applications is the modular symbol algorithm developed by Manin [18]. An algebra of Hecke operators acts on $H^1(\Gamma(N)\backslash\mathfrak{H}_2; \mathbb{C})$, and using the algorithm one may compute their eigenvalues. Essentially this algorithm is the euclidean algorithm applied to pairs of integers (§2.2).

Now consider the case where Γ is a torsion-free Bianchi subgroup, that is, Γ is of finite index in $SL_2(\mathscr{O}_K)$, where $\mathscr{O}_K$ is the ring of integers in an imaginary quadratic extension $K/\mathbb{Q}$. The group Γ acts on hyperbolic three-space $\mathfrak{H}_3 = SL_2(\mathbb{C})/SU(2)$, and we consider the cohomology group $H^2(\Gamma\backslash\mathfrak{H}_3; \mathbb{C})$. As before, an algebra of Hecke operators acts on the cohomology, and one is interested in this Hecke-module for many reasons. For example, results of Grunewald and Schwermer [13] imply that, for all but a finite set of K, the rational cohomology of $SL_2(\mathscr{O}_K)$ contains cuspidal cohomology, which is important in the theory of automorphic forms. Also,

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the "Langlands philosophy" predicts a "Shimura-Taniyama-Weil" correspondence between Hecke eigenclasses and certain algebraic varieties defined over K. More precisely, let $\Gamma \subset SL_2(\mathscr{O}_K)$ be a congruence subgroup and let $\xi \in H^2(\Gamma\backslash\mathfrak{H}_3;\mathbb{C})$ be a cuspidal Hecke eigenclass. Then one hopes to associate to ξ an algebraic variety V/K—specifically an elliptic curve or an abelian variety of dimension two—so that the zeta function of V is assembled from the eigenvalues of ξ in a precise way. Results of Cremona [7] and Cremona and Whitley [8] when K has class number one support this. Hence one wishes to compute Hecke eigenvalues for general K. But in general the ring $\mathscr{O}_K$ is not a euclidean domain, and one cannot directly apply the modular symbol algorithm as described in [18] (however, see §1.2).

In this paper we present an analog of the modular symbol algorithm for $H^d(\Gamma;\mathbb{Z})$, where Γ is a torsion-free arithmetic group associated to certain self-adjoint homogeneous cones, and d is the cohomological dimension of Γ. This includes finite index subgroups of $SL_2(R)$, where R is

- $\mathbb{Z}$,
- the ring of integers in a CM field, or
- the ring of integers in a totally real field.

We replace the continued fractions of [18] with a study of the geometry of self-adjoint homogeneous cones. Thus our algorithm does not require that R be a euclidean domain.

Here is the organization of this paper. Section 2 contains a review of the classical modular symbol algorithm from [18] and presents our algorithm in that case. Section 3 contains a review of the reduction theory of self-adjoint homogeneous cones, and in Section 4 we generalize some results of [22] to this setting (Theorems 2 and 3). These results are valid for cones of any $\mathbb{Q}$-rank ≥ 1. Finally, Section 5 contains a description of our algorithm, in Theorem 4. Throughout the paper we comment on implementation issues related to the algorithm.

1.2. **Related work.** Let $\Gamma \subset SL_2(\mathscr{O}_K)$, where $\mathscr{O}_K$ is the ring of integers in an imaginary quadratic extension $K = \mathbb{Q}(\sqrt{-m})$. For K a non-euclidean ring with class number one ($m = 19, 43, 67, 163$), Whitley developed a "pseudo-euclidean algorithm" that allowed implementation of the modular symbol algorithm [23].

Also, I learned upon completion of this work that Jeremy Bygott has independently studied the modular symbol algorithm for the non-PID imaginary quadratic case in his forthcoming Ph.D. thesis [6].

1.3. **Acknowledgments.** The results in this paper depend heavily on results from the work of Avner Ash. I thank him for graciously and patiently explaining his work to me. I also thank Mark McConnell for a careful reading of an early version of this paper and many helpful comments. I also thank the referee for many helpful suggestions.

Finally, the results in this paper are an extension of some of the results in my Ph.D. thesis [14]. I thank heartily my advisor, Robert MacPherson, for the encouragement and inspiration he has given me.

2. A motivating example

In this section we illustrate our algorithm for $\Gamma \subset SL_2(\mathbb{Z})$. As in the introduction let $\mathfrak{H}_2 = SL_2(\mathbb{R})/SO_2$, and let $\mathfrak{H}_2^* = \mathfrak{H}_2 \cup \mathbb{Q} \cup \{\infty\}$ be the usual partial compactification of $\mathfrak{H}_2$ by adding cusps. We assume that $\mathfrak{H}_2^*$ is given the Satake topology, and we extend the action of $SL_2(\mathbb{R})$ to the cusps. We denote the quotient $\Gamma\backslash\mathfrak{H}_2^*$ by X_Γ. We assume that Γ is torsion-free, so that X_Γ is smooth.

2.1. We begin by paraphrasing aspects of Manin's work [18]. By Poincaré duality, $H^1(X_\Gamma;\mathbb{C})$ may be identified with $H_1(X_\Gamma;\mathbb{C})$, and so we may study the space of weight-two modular forms for Γ by studying the latter. Let q_1 and q_2 be cusps equivalent modulo Γ. Then any smooth path γ from q_1 to q_2 descends to a closed path on X_Γ representing a class in $H_1(X_\Gamma;\mathbb{Z})$. Furthermore, this class is independent of γ, and in fact depends only on the ordered pair (q_1, q_2).

More generally, suppose q_1 is not necessarily equivalent to q_2 modulo Γ. Then integration of one-forms $\omega \in H^1(X_\Gamma;\mathbb{R})$ along γ yields a functional $\int\colon H^1(X_\Gamma;\mathbb{R}) \to \mathbb{R}$, and this allows us to associate to the pair (q_1, q_2) a class in $H_1(X_\Gamma;\mathbb{R})$. By the theorem of Manin-Drinfeld ([16], p. 61), this class actually lies in $H_1(X_\Gamma;\mathbb{Q})$. We define a *modular symbol* to be the rational homology class constructed from an ordered pair of cusps in this way, and denote this class by $[q_1, q_2]$. This class agrees with the class in the previous paragraph when q_1 and q_2 are equivalent modulo Γ.

Proposition 1. [18] *The modular symbols satisfy the following:*

1. $[q_1, q_2] = -[q_2, q_1]$.
2. $[q_1, q_2] = [q_1, q_3] + [q_3, q_2]$.

Furthermore, $H_1(X_\Gamma;\mathbb{Z})$ *is spanned by modular symbols modulo* Γ.

2.2. The Hecke operators act on $H^1(X_\Gamma;\mathbb{C})$, and hence by duality on $H_1(X_\Gamma;\mathbb{C})$. On a modular symbol, an operator acts by

$$[q_1, q_2] \longmapsto \sum_{\alpha\in A} [\alpha q_1, \alpha q_2],$$

where A is a finite set of 2×2 integral matrices that depends on the operator. For example, let $\Gamma = \Gamma(N)$, and let p be a prime not dividing N. For the classical operator T_p we may take

$$\alpha \in \left\{ \sigma_p \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & p \end{pmatrix}, \dots, \begin{pmatrix} 1 & p-1 \\ 0 & p \end{pmatrix} \right\}.$$

Here $\sigma_p \in SL_2(\mathbb{Z})$ is a fixed matrix satisfying

$$\sigma_p \equiv \begin{pmatrix} p^{-1} & 0 \\ 0 & p \end{pmatrix} \mod N$$

([20], Prop. 3.36). Note that $\det \alpha \neq \pm 1$.

A finite basis of $H_1(X_\Gamma; \mathbb{C})$ is provided by the set of *unimodular symbols*. Write a cusp q in lowest terms as m/n, where the cusp ∞ is written formally as $1/0$. Then the unimodular symbols are the symbols $[q_1, q_2]$ satisfying

$$\det \begin{pmatrix} m_1 & m_2 \\ n_1 & n_2 \end{pmatrix} = \pm 1.$$

The Hecke operators do not preserve unimodularity, and it is necessary for eigenvalue computations to construct an explicit homology between a non-unimodular symbol and a cycle of unimodular symbols. This is done by the *modular symbol algorithm*. Assume that a non-unimodular symbol has the form $[0, q]$, where q is a positive rational number. Let $[\![a_1, \dots, a_k]\!]$ be the simple continued fraction expansion of q, i.e.

$$q = a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots + \cfrac{1}{a_k}}}$$

Let q_i be the i^{th} convergent $[\![a_1, \dots, a_i]\!]$. Then by applying (2) from Proposition 1, we have

$$[0, q] = [0, \infty] + [\infty, q_1] + \cdots + [q_{k-1}, q].$$

Furthermore, the basic properties of simple continued fractions imply that the modular symbols on the right are unimodular. Figure 1 illustrates the result for the modular symbol $[0, 12/5]$.

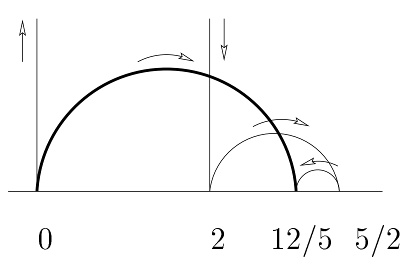

FIGURE 1. $12/5 = [\![2, 2, 2]\!]$ implies $[0, 12/5] = [0, \infty] + [\infty, 2] + [2, 5/2] + [5/2, 12/5]$

To complete the discussion, we note that $SL_2(\mathbb{Z})$ acts transitively on the cusps. Since the above algorithm is $SL_2(\mathbb{Z})$-equivariant, any modular symbol can be written as a sum of unimodular symbols.

2.3. Now we present our technique for writing a modular symbol as an equivalent sum of unimodular symbols. No use will be made of continued fractions; instead, we look at the relationship between a geodesic representing a modular symbol and a certain tessellation of $\mathfrak{H}_2$. In this simple case our algorithm will appear needlessly complicated, but it is formulated in a way that will generalize to other settings. It is also quite practical for machine computations.

To begin, we tile $\mathfrak{H}_2$ with the $SL_2(\mathbb{Z})$-translates of the ideal geodesic triangle with vertices 0, 1, and ∞ (see Figure 2). This tessellation descends to a finite triangulation of X_Γ. The edges of this tessellation are geodesics inducing the unimodular symbols, and every unimodular symbol arises in this way.

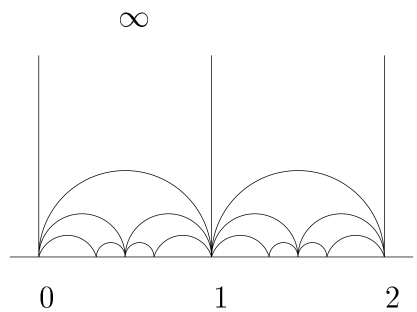

FIGURE 2. Ideal triangles in $\mathfrak{H}_2$

Given any point $x \in \mathfrak{H}_2$, let $R(x)$ be the set of vertices of the triangle (or edge) of the tessellation meeting x.

Let $[0, q]$ be a modular symbol as before, and let γ be the ideal geodesic in $\mathfrak{H}_2$ from 0 to q. Because γ is a geodesic between two rational cusps, one can show that γ will only meet a finite number of triangles in the tessellation. Hence we may choose a finite subset $x_1, \ldots, x_r \in \gamma$ (as in Figure 3) so that

1. $0 \in R(x_1)$,
2. $q \in R(x_r)$, and
3. $R(x_i) \cap R(x_{i+1}) \neq \emptyset$.

We call such a collection a *sufficiently fine partition* of γ. From each $R(x_i) \cap R(x_{i+1})$ choose a cusp q_i. Then we claim that we have a homology

$$[0, q] = [0, q_1] + [q_1, q_2] + \cdots + [q_r, q], \tag{1}$$

and that each term on the right is a unimodular symbol.

First, we may see we have a homology either by repeatedly applying (2) of Proposition 1, or by continuously deforming γ into geodesics inducing the classes on the right-hand side (see Figure 4).

Finally, each nontrivial term on the right of (1) corresponds to an edge in the tessellation because q_i and q_{i+1} are both vertices of a triangle containing x_{i+1}. In

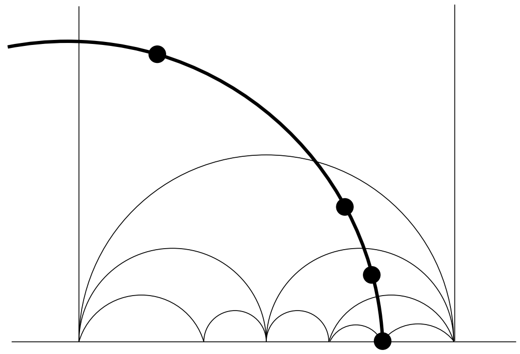

FIGURE 3. A partition of γ

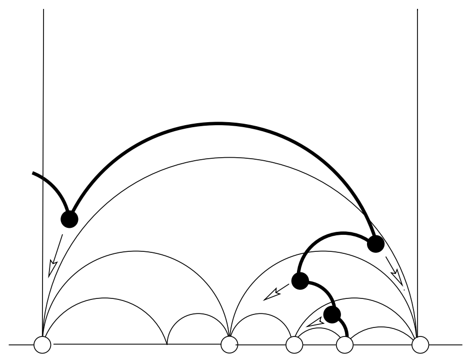

FIGURE 4. A deformation of γ

fact, as shown in [14], if γ is oriented then one may choose the q_i canonically, and our algorithm in this case is equivalent to the modular symbol algorithm.

3. Self-adjoint homogeneous cones

In this section we present the geometric context of our algorithm. This specifies the arithmetic groups to which our algorithm applies and describes the constructions replacing $\mathfrak{H}_2$ and its tessellation from §2.

The results in §3.1–3.4 and §4.1 are due to A. Ash and originally appeared in [1] and [3]. Our exposition closely follows the former.

3.1. Let V be a real vector space defined over $\mathbb{Q}$, and let $C \subset V$ be an open cone. That is, C contains no straight line, and C is closed under homotheties: if $x \in C$ and $\lambda \in \mathbb{R}^{>0}$, then $\lambda x \in C$. The cone C is called *self-adjoint* if there exists a scalar product $\langle\, ,\, \rangle$ on C such that

$$C = \left\{\, x \in V \mid \langle y, x\rangle > 0 \quad \text{for all } y \in \bar{C} - \{0\} \,\right\}.$$

Let G denote the connected component of the identity of the linear automorphism group of C, i.e. $G = \{\, g \in GL(V) \mid gC = C \,\}^0$. The cone C is called *homogeneous* if G acts transitively on C. If K denotes the isotropy group of a fixed point in C, then we may identify C with G/K. The fact that C is self-adjoint implies that G is reductive and C modulo homotheties is a Riemannian symmetric space.

We also assume that all these notions are compatible with the $\mathbb{Q}$-structure on V. That is, as a subgroup of $GL(V)$, G is defined by rational equations, and the scalar product $\langle\, ,\, \rangle$ is defined over $\mathbb{Q}$. This is stronger than saying that G is defined over $\mathbb{Q}$. In particular, the group of real points $G(\mathbb{R})$ must be isomorphic to a product of the following groups ([10], p. 97):

1. $GL_n(\mathbb{R})$
2. $GL_n(\mathbb{C})$
3. $GL_n(\mathbb{H})$
4. $O(1, n-1) \times \mathbb{R}^\times$
5. The noncompact Lie group with Lie algebra $\mathfrak{e}_{6(-26)} \oplus \mathbb{R}$

In each case V is a set of hermitian symmetric matrices. In other words, V is the set of $n \times n$ matrices over an appropriate $\mathbb{R}$-algebra with involution τ, in which $A \in V$ if and only if $A^t = A^\tau$. The cone C is then the subset of "positive-definite" matrices in an appropriate sense. For details we refer to ([10], Ch. V).

3.2. Let H be a hyperplane in V. We say that H is a *supporting hyperplane* of C if H is rational and $H \cap C = \emptyset$ but $H \cap \bar{C} \neq \emptyset$.

Given a supporting hyperplane H of C, let $C' = \mathrm{Int}(H \cap \bar{C})$. (Here $\mathrm{Int}(A)$ is the interior of A in its linear span.) Then C' is called a *rational boundary component*, and is a self-adjoint homogeneous cone of smaller dimension than C.

Definition 1. The *cusps* of C are the one-dimensional rational boundary components of C. The set of cusps is denoted $\Xi(C)$.

3.3. Let $L \subset V(\mathbb{Q})$ be a lattice, i.e. a discrete subgroup of $V(\mathbb{Q})$ such that $L \otimes \mathbb{Q} = V(\mathbb{Q})$. Let Γ_L denote the subgroup of $G(\mathbb{Q})$ preserving L. An *arithmetic subgroup* of G is a discrete subgroup commensurable with Γ_L for some L. Any torsion-free subgroup $\Gamma \subset \Gamma_L$ of finite index will act properly discontinuously and freely on C. Thus the quotient $\Gamma\backslash C$ is an Eilenberg-Mac Lane space for Γ, and the group cohomology $H^*(\Gamma)$ is $H^*(\Gamma\backslash C)$. In fact, since homotheties commute with the action of Γ, we may pass to $X := \mathbb{R}^{>0}\backslash C$, and compute $H^*(\Gamma\backslash X)$ instead.

3.4. Let $A \subset V(\mathbb{Q})$ be a finite set of nonzero points. The closed convex hull σ of the rays $\left\{\mathbb{R}^{\geq 0}x \;\middle|\; x \in A\right\}$ is called a *rational polyhedral cone*. The rays through the vertices of the convex hull of A are called the *spanning rays* of σ. We denote the set of spanning rays by $R(\sigma)$. The group $G(\mathbb{Q})$ acts naturally on the set of rational polyhedral cones, and we denote the action by a dot: $\sigma \mapsto g \cdot \sigma$.

Now we would like to partition C into convex subsets using a collection of rational polyhedral cones, in a manner compatible with the Γ_L-action. This requires some care, as C is open and any σ as above is closed.

Definition 2. ([3], p. 117) Let $\Gamma \subset \Gamma_L$ be an arithmetic subgroup of G. A set of closed polyhedral cones $\{\sigma_\alpha\}$ is called a Γ-*admissible decomposition* of C when the following hold:

1. Each σ_α is the span of a finite number of rational rays.
2. For each α, the cone $\sigma_\alpha \subset \bar{C}$.
3. Every face of a σ_α is some σ_β in the decomposition.
4. $\sigma_\alpha \cap \sigma_\beta$ is a common face of σ_α and σ_β.
5. For any σ_α and any $\gamma \in \Gamma$, $\gamma\sigma_\alpha$ is some σ_β in the decomposition.
6. Modulo Γ, there are only a finite number of σ_α's.
7. $C = \bigcup_\alpha (\sigma_\alpha \cap C)$.

Note that a Γ-admissible decomposition descends to a decomposition of X into open cells.

We now describe a technique to construct Γ-admissible decompositions. The technique originates with Voronoĭ, and was generalized by Ash to all self-adjoint homogeneous cones. Let L' be $L - \{0\}$.

Definition 3. The *Voronoĭ polyhedron* Π is the closed convex hull of $L' \cap \Xi(C)$.

Theorem 1. *([3], p. 143) The cones over faces of Π form a Γ-admissible decomposition of C.*

We call the cones in this Γ-admissible decomposition of C the *Voronoĭ cones*, the decomposition of X associated to Π the *Voronoĭ decomposition*, and the open cells in X the *Voronoĭ cells*.[1] Here are two examples of this construction.

Example 1. The original example investigated by Voronoĭ [22] is the following. Let V be the vector space of symmetric $n \times n$ matrices, and let $C \subset V$ be the cone of positive-definite matrices. Then G is $GL_n(\mathbb{R})^0$, which acts on V by $A \mapsto gAg^t$, and K is SO_n. The scalar product is given by $\langle A, B\rangle = \mathrm{Tr}(AB)$.

Let L be the lattice of integral symmetric matrices. Then $\Gamma_L = SL_n(\mathbb{Z})$. The set of cusps $\Xi(C)$ is obtained as follows. Any nonzero integral (column) n-vector v determines a rank one quadratic form by $v \mapsto vv^t$. The cusps are the rays generated by all points in $\bar{C}$ of this form. Suppose that $z \in L' \cap \Xi(C)$ is a cusp arising in this manner from the n-vector v. Then if $y \in C$, the scalar product $\langle z, y\rangle$ is equal to the quadratic form y evaluated on v.

For $n = 2$ we have that $X = \mathfrak{H}_2$, and the Voronoĭ decomposition is the tessellation described in §2.3. For $n \geq 4$ there is more than one type of top-dimensional Voronoĭ cone modulo Γ_L, and not all the top-dimensional cones are simplicial. Complete information about these decompositions for $n = 3$ and 4 can be found in [17] and [19].

Example 2. Let $K/\mathbb{Q}$ be an imaginary quadratic extension, and let $\mathscr{O}_K$ be the ring of integers of K. Let V be the vector space of 2×2 hermitian symmetric matrices

[1] Our terminology is nonstandard. In [3], the polyhedron Π is referred to as a "kernel comparable to a Γ-polyhedral cocore" and is a specific example of a more general theory.

over $\mathbb{C}$, and let $C \subset V$ be the cone of positive-definite matrices. Then V is a four-dimensional vector space over $\mathbb{R}$, and X is three-dimensional hyperbolic space $\mathfrak{H}_3$. For L we may take the matrices in V with entries in $\mathscr{O}_K$, and then $\Gamma_L = SL_2(\mathscr{O}_K)$. In the classical picture of $\mathfrak{H}_3 \subset \mathbb{C} \times \mathbb{R}^{\geq 0}$, the rays generated by the vertices of Π become the points $K \cup \{\infty\}$, where K is pictured as a subset of $\mathbb{C} \times \{0\}$ and ∞ is pictured infinitely far above $\mathbb{C} \times \{0\}$ along $\mathbb{R}^{\geq 0}$.

The Voronoĭ decomposition becomes a tessellation of $\mathfrak{H}_3$ into ideal three-polytopes. In general these polytopes will not be simplices. For example, if $K = \mathbb{Q}(\sqrt{-1})$, then the unique polytope modulo Γ_L is an octahedron [7][12]. If $\mathscr{O}_K$ is euclidean there is only one type of top-dimensional polytope in the tessellation, but for general K there will be more than one type.

3.5. Here is the connection between the Voronoĭ decomposition of X and $H^*(\Gamma; \mathbb{Z})$. Let N be the dimension of X and d the cohomological dimension of Γ. Let C^k be the set of Voronoĭ cells of codimension k. The group Γ acts naturally on C^k by its action on rational polyhedral cones. We want to construct a Γ_L-equivariant "coboundary" map $\delta^k \colon \mathbb{Z}[C^k] \to \mathbb{Z}[C^{k+1}]$ so that the resulting chain complex modulo Γ computes $H^*(\Gamma; \mathbb{Z})$. We will call (C^*, δ^*) a *cocell complex*, and will say that the Voronoĭ decomposition gives X a *cocell structure*.

According to [1], there is a topological space $W \subset X$ such that the following hold:

1. W admits the structure of a Γ-equivariant regular cell complex with top-dimensional cells of dimension d.
2. W is a deformation retract of X, so that the homology of the chain complex associated to $\Gamma \backslash W$ is the homology of $\Gamma \backslash X$, and hence of Γ.
3. This cell structure is dual to the Voronoĭ decomposition in the following sense: every k-cell of W transversely intersects exactly one Voronoĭ cell of codimension k.

Let W_k denote the set of k-cells of W. Given $\tau \in W_k$, we denote its dual cell by $\hat{\tau} \in C^k$. Let $\mathbb{Z}[W_k]$ (respectively $\mathbb{Z}[C^k]$) denote the free abelian group on the elements of W_k (resp. C^k).

We may choose orientations compatibly between the cell and cocell structures in the following sense: for each pair $(\tau, \hat{\tau})$ we may fix orientations so that in the homeomorphism

$$\operatorname{Int} \tau \times \hat{\tau} \longrightarrow \mathbb{R}^N$$

the product of the orientations is carried to a fixed orientation of $\mathbb{R}^N$. This constructs a map $\mathbb{Z}[C^k] \to \operatorname{Hom}_{\mathbb{Z}}(\mathbb{Z}[W_k], \mathbb{Z})$.

Now to construct δ^*, we use the boundary map from W_*. Given two cells $\sigma, \tau \in W_*$, write $\sigma < \tau$ if σ appears in the closure of τ. Then $\partial_k \colon \mathbb{Z}[W_k] \to \mathbb{Z}[W_{k-1}]$ has the form

$$\tau \longmapsto \sum_{\sigma < \tau} [\sigma : \tau]\, \sigma,$$

where the $[\sigma : \tau] = \pm 1$ keeps track of the relative orientation between σ and τ. (Saying W is a regular cell complex makes $[\sigma : \tau]$ well defined.) We define $\delta^k : \mathbb{Z}[C^k] \to \mathbb{Z}[C^{k+1}]$ by

$$\hat{\tau} \longmapsto \sum_{\tau<\sigma} [\tau : \sigma]\, \hat{\sigma} \tag{2}$$

Proposition 2. *With the above coboundary map, $H^k(\Gamma; \mathbb{Z})$ is naturally isomorphic to the k^{th}-cohomology of the quotient modulo Γ of $(\mathbb{Z}[C^*], \delta^*)$.*

Proof. We must show that $\delta^2 \equiv 0$ and that δ is the adjoint of ∂ with respect to the pairing between $\mathbb{Z}[W_*]$ and $\mathbb{Z}[C^*]$. The former is purely formal using (2) and the fact that $\hat{\tau}$ is the map $\mathbb{Z}[W_k] \to \mathbb{Z}$ that takes τ to 1 and all others to 0. The latter is easily verified from the definitions and the choice of orientations. □

4. Voronoĭ reduction

In this section we address two questions:

1. How do we construct Π in practice?
2. Given a point $x \in C$, can we determine a top-dimensional Voronoĭ cone containing it? (Such a cone is unique for generic x, and for any given x there are at most a finite number of such cones containing it.)

In [22], Voronoĭ answers these in the setting of Example 1, where C is the cone of real positive-definite symmetric matrices. In this section we prove that Voronoĭ's results remain true in our more general context.

4.1. First we describe some geometric properties of Π which are proved in [1].

Let F be a facet of Π, that is, a codimension-one face of Π. Then there is a unique point $y_F \in C \cap V(\mathbb{Q})$ such that

1. $F = \{\, x \in \Pi \mid \langle x, y_F \rangle = 1\}$, and
2. for all $x \in \Pi - F$, we have $\langle x, y_F \rangle > 1$.

We say that y_F defines a *supporting hyperplane* of Π.

Given a facet F, let Z_F be the finite set of points $z \in L' \cap \Xi(C)$ such that $\langle z, y_F \rangle = 1$. Then F is the convex hull of Z_F, and as we range over all $w \in L' \cap \Xi(C)$ such that $w \notin Z_F$, the set of numbers $\langle w, y_F \rangle$ is bounded below away from 1. We call y_F the *perfect form* associated to F and Z_F the set of *minimal vectors* of y_F. In the case of Example 1, the y_F are perfect quadratic forms in the classical sense, with minimal vectors Z_F [22].

Let $\sigma \subset \bar{C}$ be a rational polyhedral cone. Then σ satisfies the "property of Siegel" with respect to the Voronoĭ decomposition. Specifically, the intersection $\sigma \cap \Pi$ is cut out from Π by a finite number of supporting hyperplanes ([1], p. 73). This implies that for any σ and for any $x \in C$, the orbit $\Gamma_L x$ meets σ in a finite set.

Given any $y \in C(\mathbb{Q})$, let $\pi(y) : V \to \mathbb{R}$ denote the linear map $x \mapsto \langle x, y \rangle$. We also need the following finiteness result.

Proposition 3. *Let $y \in C(\mathbb{Q})$. Then for any $\mu > 0$, the set*

$$\{z \in L' \cap \Xi(C) \mid 0 < \langle z, y\rangle \le \mu\}$$

is finite.

Proof. Given any $\lambda \in \mathbb{R}$, let H_λ be the affine hyperplane $\{x \in V \mid \langle x, y\rangle = \lambda\}$. Then $H_0 = \ker(\pi)$ is rational, since $y \in C(\mathbb{Q}) \subset V(\mathbb{Q})$. Hence the map $\pi(y)$ takes L onto a lattice in $\mathbb{R}$. Since some multiple of y lies in L', this lattice is nontrivial. Thus to prove the claim, it is enough to show that for any $\lambda > 0$, the set $H_\lambda \cap L' \cap \Xi(C)$ is finite.

To see this, consider the set $\bar{C} \cap H_\lambda$. Note that H_λ meets C, since $y' := \dfrac{\lambda y}{\langle y, y\rangle} \in H_\lambda$. Let $\ell \subset H_\lambda$ be any line through y', and let ∂C be $\bar{C} \setminus C$. Since C is a cone, ℓ must leave C, and hence $\ell \cap \partial C \neq \emptyset$. Let x be a point in $\ell \cap \partial C$. The self-adjointness of C implies that there is another point z on $\ell \cap \partial C$. (See Figure 5.)

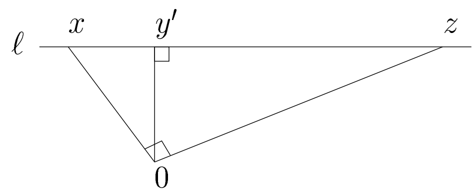

FIGURE 5.

Now let ℓ range over all lines in H_λ through y', and consider the set of lengths of the segments xy' and $y'z$. This set is bounded above, and hence $\bar{C} \cap H_\lambda$ is a closed and bounded subset of V. Since L is a lattice, the set $H_\lambda \cap L' \cap \Xi(C)$, if nonempty, is finite. □

4.2. Now we describe the construction of Π. Call two facets of Π *neighbors* if they meet along a codimension-two face of Π. We show how, given a facet F, one may systematically find all neighbors of F.

Lemma 1. *Let F and G be neighboring facets of Π with perfect forms y_F and y_G. Let $v \in V(\mathbb{Q})$ be orthogonal to the affine span of the origin and the polytope $F \cap G$, and such that $\langle x, v\rangle \ge 0$ for all $x \in F$. Then $y_G = y_F + \rho v$ for a unique $\rho \in \mathbb{R}^{>0}$.*

Proof. First note that the affine span of $F \cap G$ and the origin is a hyperplane, since $F \cap G$ is a codimension-two face of Π. Thus v is unique up to a scalar. Let $v' = y_G - y_F$. Then $\langle x, v'\rangle = 0$ for all $x \in F \cap G$, and $v' \neq 0$. Thus $\rho v = v'$ for some nonzero $\rho \in \mathbb{R}$, which shows that $y_G = \rho v + y_F$. We must show ρ is positive.

Let $x \in F - (F \cap G)$. Then $1 < \langle x, y_G\rangle = \langle x, y_F\rangle + \rho\langle x, v\rangle = 1 + \rho\langle x, v\rangle$, which means that $\rho\langle x, v\rangle > 0$. Since $\langle x, v\rangle \ge 0$ for all $x \in F$, the result follows.

□

Suppose that we are given a facet F with corresponding perfect form y_F and minimal vectors Z_F. Choose a maximal proper face $E \subset F$, and let $Z_E \subset Z_F$ be the minimal vectors affinely spanning E. Let G be the facet neighboring F along E, and write $y_G = y_F + \bar{\rho} v$, where v is a vector satisfying the conditions of Lemma 1. Let Z_G be the set of minimal vectors of y_G. Define the function $\rho(x)$ by

$$\rho(x) := \frac{1 - \langle x, y_F \rangle}{\langle x, v \rangle}$$

and define S by

$$S := \{x \in L' \cap \Xi(C) \mid \langle x, v \rangle < 0 \quad \text{and} \quad y_F + \rho(x) v \in C\}.$$

Note that S is nonempty. This follows because whenever $z \in Z_G \setminus Z_E$, a computation shows $\rho(z) = \bar{\rho}$ and $y_F + \rho(z) v = y_G$, implying $z \in S$. This same computation shows conversely that if $z \in L' \cap \Xi(C)$ and $\rho(z) = \bar{\rho}$, then z is a minimal vector of y_G not in Z_E.

Lemma 2. *As x ranges over S, we have $\rho(x) \geq \bar{\rho}$.*

Proof. Assume there is some $x' \in S$ with $\rho(x') < \bar{\rho}$. Then

$$\begin{aligned}\langle x', y_G \rangle &= \langle x', y_F \rangle + \bar{\rho} \langle x', v \rangle \\ &< \langle x', y_F \rangle + \rho(x') \langle x', v \rangle \\ &= 1,\end{aligned}$$

which is a contradiction. $\square$

Now we show how to compute y_G. Choose any $\ell \in S$ and consider the point $y_\ell := y_F + \rho(\ell) v$. Let T be the set

$$T := \{x \in L' \cap \Xi(C) \mid \langle x, y_\ell \rangle \leq 1\}.$$

Lemma 3. *The set $T \cap (Z_G \setminus Z_E)$ is finite and nonempty.*

Proof. Since $v \in V(\mathbb{Q})$, we have that $y_\ell \in C(\mathbb{Q})$. Thus T is finite by Proposition 3.

We now show $T \cap (Z_G \setminus Z_E)$ is nonempty. Let $z \in (Z_G \setminus Z_E) \cap S$, a set we have seen is nonempty. Then

$$\begin{aligned}\langle z, y_\ell \rangle &= \langle z, y_F \rangle + \rho(\ell) \langle z, v \rangle \\ &\leq \langle z, y_F \rangle + \bar{\rho} \langle z, v \rangle \quad \text{[Lemma 2]} \\ &= \langle z, y_G \rangle \\ &= 1.\end{aligned}$$

Thus $z \in T$, and $T \cap (Z_G \setminus Z_E)$ is nonempty. $\square$

Theorem 2. *Given a facet F of the Voronoĭ polyhedron, all its neighbors may be determined in a finite number of steps.*

Proof. Let y_F be the perfect form corresponding to F, and let Z_F be the set of minimal vectors of y_F. Using standard techniques of convex geometry, such as Fourier-Motzkin elimination ([24], p. 37), we may determine all the maximal proper faces of F. Let E be such a face, and let $Z_E \subset Z_F$ be the minimal vectors affinely spanning E. Using Z_E, we may determine v as in Lemma 1. Let G be the facet of Π neighboring F along E, and let y_G be the corresponding perfect form. We need to compute y_G.

First we find an $\ell \in S$ by searching over L, and then using ℓ we construct the finite set T. The latter can done in a finite number of steps because it is equivalent to finding the set of vectors in a lattice on which a positive-definite quadratic form is less than a constant (cf. Example 1 and the final paragraph of §3.1). By Lemma 3, $T \cap (Z_G \setminus Z_E)$ is nonempty, where Z_G is the set of minimal vectors of y_G. Now let $Z \subset (T \setminus Z_E)$ be the set on which $\rho(x)$ attains its minimum. By Lemma 2 and the paragraph preceding it, $Z \subset Z_G \setminus Z_E$. Let H be the affine span of $Z \cup Z_E$. Then y_G is the unique point satisfying $\langle x, y_G\rangle = 1$ for all $x \in H$.

Repeating this procedure for each maximal face of F, we may determine all the neighbors of F. □

Remark 1. In practice, it may be the case that S *only* consists of $Z_G \setminus Z_E$, as a computation with $SL_2(\mathbb{Z})$ shows.

Hence one may find facets of Π provided one can construct an initial facet. In the setting of Example 1, Voronoĭ did this by showing that the quadratic form A_n, defined by

$$\sum_{i=1}^{n} x_i^2 + \sum_{1\leq i<j\leq n} (x_i - x_j)^2,$$

is perfect for all n ([22], §29).[2] In our more general setting, one cannot write down a perfect form that works for every case, even if one restricts to Bianchi groups (Example 2). However, in practice one may do the following.

First choose a large bounded set $U \subset V$ containing the origin, and let Σ be the convex hull of $L' \cap \Xi(C) \cap U$. Then Σ is a bounded polytope in V. Furthermore, since the facets of Π are bounded, if U is sufficiently large then most facets of Σ will be facets of Π.

To check if a facet $F \subset \Sigma$ is a facet of Π, one computes y_F and checks whether the $z \in L' \cap \Xi(C)$ such that $\langle z, y_F\rangle = 1$ are vertices of F.

Remark 2. In the Bianchi case, another possibility is to first construct the retract W from §3.5 using techniques in [21], and then use the duality between them to deduce the structure of Π.

4.3. To answer question (2), Voronoĭ describes a reduction algorithm. This algorithm is based on the following:

[2] Voronoĭ called this form the *principal perfect form*.

Proposition 4. *Fix an $x \in \Pi$ and a real number $\mu \geq 1$. Then there are only a finite number of perfect forms y_F satisfying*

$$\langle x, y_F \rangle \leq \mu.$$

Proof. Choose a facet F such that $\langle x, y_F \rangle \leq \mu$. Since $y_F \in C(\mathbb{Q})$, Proposition 3 implies the set $\{z \in L' \cap \Xi(C) \mid \langle z, y_F \rangle \leq \mu\}$ is finite. Hence the polyhedron

$$\Sigma = \{\, y \in \Pi \mid \langle y, y_F \rangle \leq \mu \,\}$$

is a bounded polytope in $\bar{C}$. If $\mu \in \pi(y_F)(L)$, where π is defined in §4.1, then the hyperplanes bounding Σ will be rational with respect to the $\mathbb{Q}$-structure on V, and thus Σ will have vertices in $V(\mathbb{Q})$. Hence, replacing μ by a slightly larger number if necessary, we may assume Σ has vertices in $V(\mathbb{Q})$.

Therefore the cone generated by the vertices of Σ is rational polyhedral, and only meets a finite subset of the orbit $\Gamma_L x$. By taking the adjoint action of Γ_L with respect to the scalar product, it follows that only finitely many facets F' that are Γ_L-equivalent to F satisfy $\langle x, y_{F'} \rangle \leq \mu$. Since there are only a finite number of facets modulo Γ_L, the result follows. □

The following lemma gives a local condition for when a given point of C lies in a cone over a facet of Π.

Lemma 4. *Let F be a facet of Π, and let $\mathscr{G}$ be the set of neighbors of F. Let $x \in C$. Suppose that $\langle x, y_F \rangle \leq \langle x, y_G \rangle$ for all $G \in \mathscr{G}$. Then x lies in the cone over F.*

Proof. Choose λ so that $x' := \lambda x$ satisfies $\langle x', y_F \rangle = 1$ and $\langle x', y_G \rangle \geq 1$ for $G \in \mathscr{G}$. For $\epsilon > 0$, let Σ_ϵ be the polyhedron defined by

$$\Sigma_\epsilon = \{x \mid 1 \leq \langle x, y_F \rangle \leq 1 + \epsilon \quad \text{and} \quad \langle x, y_G \rangle \geq 1 \quad \text{for } G \in \mathscr{G}.\}$$

If ϵ is sufficiently small, then $\Sigma_\epsilon \subset \Pi$. Hence $x' \in \Sigma_\epsilon$, and thus $x' \in \Pi$. Since x' also lies in the supporting affine hyperplane $\{x \mid \langle x, y_F \rangle = 1\}$, it must lie in F. □

Now we describe our Voronoĭ reduction algorithm.

Theorem 3. *Let $x \in \Pi$, and choose a facet F. Let $\mu = \langle x, y_F \rangle$. The following algorithm determines a cone in the Voronoĭ decomposition containing x:*

1. *For each neighbor G of F, compute $\langle x, y_G \rangle$.*
2. *If there exists a neighbor G with $\langle x, y_G \rangle < \mu$, replace F with G, μ with $\langle x, y_G \rangle$, and return to step one.*
3. *Otherwise, terminate the procedure: x lies in the cone generated by F.*

Proof. By Lemma 4, if the algorithm terminates then we have determined a cone containing x. We now prove that the algorithm terminates. Suppose that a neighbor G of F has $\langle x, y_G \rangle < \mu$. (Note that this quantity must be positive since C is self-adjoint and $x, y \in C$.) Then we return to step one, and we have decreased the scalar product. Since by Proposition 4 the set of facets satisfying $\langle x, y_{F'} \rangle \leq \mu$ is finite, the algorithm must terminate. □

Remark 3. The data needed to implement this algorithm is the same as that needed for the structure of Π modulo Γ_L, along with some additional information. In particular, one must determine:

1. A finite set $\mathscr{F}$ of representatives of the facets of Π modulo Γ_L.
2. For each $F \in \mathscr{F}$ and each neighbor G of F, a element $\gamma \in \Gamma_L$ such that $\gamma G \in \mathscr{F}$.

For an example of the implementation of this algorithm for $SL_2(\mathbb{Z})$, we refer to [15].

As far as we know, the computational complexity of this algorithm is unknown. However, in our experience it performs very well for $SL_3(\mathbb{Z})$ and $SL_4(\mathbb{Z})$.

5. The modular symbol algorithm

In this section we define modular symbols and describe our Hecke algorithm. Our definition of a modular symbol is closely related to the definitions appearing in [2] and [4]. As before, let N be the dimension of X and let d be the cohomological dimension of Γ. We assume from now on that G has $\mathbb{Q}$-rank one, so that $N = d + 1$.

5.1. Let $\rho(\Pi)$ be the set of rays in $\bar{C}$ generated by the vertices of the Voronoĭ polyhedron Π. Given an ordered pair $(u, v) \in \rho(\Pi) \times \rho(\Pi)$, we want to construct a class $[u, v] \in H^d(\Gamma; \mathbb{Z})$.

To this end we recall that X may be extended to a bordification $\bar{X}$ such that the quotient $\Gamma\backslash\bar{X}$, the *Borel-Serre compactification*, is a compact manifold with corners with interior $\Gamma\backslash X$ [5]. Let $\pi\colon \bar{X} \to \Gamma\backslash\bar{X}$ be the canonical projection. Given $u, v \in \rho(\Pi)$, we determine a path in $\Gamma\backslash\bar{X}$ as follows. First u and v determine a closed cone $\sigma \subset V$, and we let $\bar{\sigma}$ be the closure of $\sigma \cap V$ mod homotheties in $\bar{X}$. Then $\pi(\bar{\sigma})$ is a path with endpoints lying in $\partial(\Gamma\backslash\bar{X})$. Choosing an ordering (u, v) fixes an orientation of $\pi(\bar{\sigma})$ and thus determines a class in $H_1(\Gamma\backslash\bar{X}, \partial(\Gamma\backslash\bar{X}); \mathbb{Z})$. By Lefschetz duality,

$$H_1(\Gamma\backslash\bar{X}, \partial(\Gamma\backslash\bar{X}); \mathbb{Z}) = H^d(\Gamma\backslash\bar{X}; \mathbb{Z}),$$

and since $\Gamma\backslash\bar{X}$ is homotopy equivalent to $\Gamma\backslash X$, we have actually determined a class in $H^d(\Gamma; \mathbb{Z})$.

Definition 4. A *modular symbol* is a class in $H^d(\Gamma; \mathbb{Z})$ constructed as above from an ordered pair $(u, v) \in \rho(\Pi) \times \rho(\Pi)$. The class is denoted $[u, v]$.

Note that this definition is almost the same as that in §2.1 for $\Gamma \subset SL_2(\mathbb{Z})$, because using (u, v) to generate a cone is the same as choosing a specific path as in §2.1. However, as in the earlier definition, the class $[u, v]$ is independent of this path.

We have the following analogue of Proposition 1:

Proposition 5. *Let $u, v \in \rho(\Pi)$. The modular symbols satisfy the following:*

1. $[u, v] = -[v, u]$.
2. *If $w \in \rho(\Pi)$, then* $[u, v] = [u, w] + [w, v]$.
3. *The modular symbols span* $H^d(\Gamma; \mathbb{Z})$.

Proof. Only (2) and (3) require proof. To prove (2), let σ be the cone generated by u and v, and choose a ray $x \subset \sigma$ distinct from u and v. Let $\phi\colon [0,1] \to \bar{C}$ be a continuous family of rays such that $\phi(0) = x$ and $\phi(1) = w$. Let $\sigma_1(t)$ (respectively $\sigma_2(t)$) be the cone generated by u and $\phi(t)$ (resp. $\phi(t)$ and v). Then ϕ provides a continuous deformation of σ into $\sigma_1(1) \cup \sigma_2(1)$ that induces the homology $[u, v] = [u, w] + [w, v]$.

Now to prove (3), note that the results of §3.5 imply that any class in $H^d(\Gamma; \mathbb{Z})$ may be written as a cocycle for δ using the Voronoĭ cones of codimension-d modulo Γ. But the Voronoĭ cones of codimension d are cones generated by pairs of vertices of Π, and so any such cycle is in the span of the modular symbols. ☐

Remark 4. The third statement is also a consequence of the results in [2].

5.2. Now we describe our replacement for the notion of a unimodular symbol. Recall that $R(\sigma)$ is the set of spanning rays for a cone σ (§3.4).

Definition 5. Let $\sigma \subset \bar{C}$ be a rational polyhedral cone. Then σ is *Voronoĭ-reduced* if there is a top-dimensional Voronoĭ cone σ_α such that

$$R(\sigma) \subset R(\sigma_\alpha).$$

Note that a Voronoĭ-reduced cone is spanned by cusps. A Voronoĭ-reduced cone need not be a Voronoĭ cone, since the top-dimensional Voronoĭ cones are not simplicial in general (cf. Examples 1 and 2). However, since every Voronoĭ cone is generated by a finite set of cusps, and because of the finiteness properties of any Γ-admissible decomposition, we have

Proposition 6. *Modulo Γ, there are only finitely many Voronoĭ-reduced cones.*

We say that a modular symbol is Voronoĭ-reduced if it is induced by a Voronoĭ-reduced cone. Propositions 5 and 6 imply that the Voronoĭ-reduced modular symbols provide a finite spanning set for $H^d(\Gamma; \mathbb{Z})$.

Remark 5. Although the Voronoĭ-reduced modular symbols provide a finite spanning set for $H^d(\Gamma; \mathbb{Z})$, they are not a basis of $H^d(\Gamma; \mathbb{Z})$. In fact, they are not even a basis of $\mathbb{Z}[C^*]$ modulo Γ, because they are not necessarily supported on codimension-d Voronoĭ cones. However, in practice this does not affect their usefulness (cf. Remark 7).

5.3. Let $u, v \in \rho(\Pi)$, and let σ be the rational polyhedral cone generated by u and v. We are now ready to describe and prove our algorithm.

Given $x \in C$, let $\sigma(x)$ be the unique Voronoĭ cone containing x, and by abuse of notation let $R(x)$ be $R(\sigma(x))$.

Let $x_1, \dots, x_r$ be points in $\sigma \cap C$ such that the rays $\mathbb{R}^{>0}x_1, \dots, \mathbb{R}^{>0}x_r$ are distinct. These points subdivide σ into a collection of cones, namely those generated by the pairs $(u, x_1), \dots, (x_r, v)$.

Definition 6. The decomposition of σ by the x_i as above is called a *sufficiently fine partition* if

1. $u \in R(x_1)$,
2. $v \in R(x_r)$, and
3. for $i = 1, \ldots, r-1$, we have $R(x_i) \cap R(x_{i+1}) \neq \emptyset$.

Lemma 5. *Sufficiently fine partitions of γ exist.*

Proof. Since σ is rational polyhedral, the Siegel property implies that $\sigma \cap \Pi$ is cut out by finitely many supporting hyperplanes. Hence $\sigma \cap C$ meets only finitely many top-dimensional cones, and the intersection of these cones with σ subdivides the latter into finitely many 2-cones $\{\sigma_i\}$. We may take x_i to be any nonzero point in the interior of σ_i. Conditions (1) and (2) of Definition 6 are trivially satisfied. Also, $R(x_i) \cap R(x_{i+1}) \neq \emptyset$ because $\sigma_i \subset V_i$ and $\sigma_{i+1} \subset V_{i+1}$, where V_i and V_{i+1} are Voronoĭ cones that have a face in common.

□

Remark 6. For computational purposes, one may construct sufficiently fine partitions of γ as follows. Let $u, v \in \rho(\Pi)$ and let σ be the cone generated by u and v. Choose points $\bar{u} \in u$ and $\bar{v} \in v$. Let $\bar{x}$ be the midpoint of the segment between $\bar{u}$ and $\bar{v}$, and let x be the cone generated by $\bar{x}$. Now apply Theorem 3 to check whether $u \in R(\bar{x})$ and $v \in R(\bar{x})$. If these conditions are not satisfied, bisect the segments between $\bar{u}$, $\bar{x}$ and $\bar{x}$, $\bar{v}$, and check conditions (1), (2), and (3). Eventually, by the Siegel property, after a finite number of iterations one will have constructed a sufficiently fine partition of σ.

Now we present our algorithm.

Theorem 4. *Given a modular symbol $[u, v]$, the following constructs a chain of Voronoĭ-reduced modular symbols homologous to $[u, v]$:*

1. *Choose a set of points $\{x_i\}$ inducing a sufficiently fine partition of the cone generated by u and v.*
2. *For $i = 1, \ldots, r-1$, choose a ray $q_i \in R(x_i) \cap R(x_{i+1})$.*

Then $[u, v] = [u, q_1] + [q_1, q_2] + \cdots + [q_r, v]$.

Proof. First note that each modular symbol on the right hand side is Voronoĭ-reduced, since q_i and q_{i+1} are both rays from $R(x_{i+1})$. We must show there is a homology between the right side and the left. Notice that

$$[u, v] = [u, q_1] + [q_1, v]$$

by Proposition 5. Repeatedly applying this proposition, we see that

$$[q_i, v] = [q_i, q_{i+1}] + [q_{i+1}, v],$$

which completes the proof. □

Remark 7. For computational purposes, to determine the action of a Hecke operator we must write any modular symbol in terms of a basis of $H^d(\Gamma; \mathbb{Z})$, and by Remark 5 the technique in Theorem 4 is not sufficient to do this. However, in practice we may precompute explicit homologies between Voronoĭ-reduced modular symbols and modular symbols supported on Voronoĭ cones, as follows.

Let $\mathscr{F}$ be a set of representatives of the facets of Π modulo Γ_L. For each $F \in \mathscr{F}$, let u, v be any two vertices of F. Then $[u, v]$ is a Voronoĭ-reduced modular symbol. To write $[u, v]$ in terms of the basis of codimension-d Voronoĭ cones, choose any sequence of vertices $u = u_0, u_1, \dots, u_k = v$ of F such that u_i and u_{i+1} are joined by an edge of F. Then

$$[u, v] = [u, u_1] + \cdots + [u_{k-1}, v]$$

is the desired homology. Now repeat for all u, v and all $F \in \mathscr{F}$.

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