

ON HILBERT MODULAR THREEFOLDS OF DISCRIMINANT 49

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To Don Zagier, on the occasion of his 60th birthday.

ABSTRACT. Let K be the totally real cubic field of discriminant 49, let \mathcal{O} be its ring of integers, and let $p \subset \mathcal{O}$ be the prime over 7. Let $\Gamma(p) \subset \Gamma = SL_2(\mathcal{O})$ be the principal congruence subgroup of level p . This paper investigates the geometry of the Hilbert modular threefold attached to $\Gamma(p)$ and some related varieties. In particular, we discover an octic in \mathbb{P}^3 with 84 isolated singular points of type A_2 .

1. INTRODUCTION

Let K be the totally real cubic field of discriminant 49, let \mathcal{O} be its ring of integers, and let $p \subset \mathcal{O}$ be the prime over 7. Let $\Gamma(p) \subset \Gamma = SL_2(\mathcal{O})$ be the principal congruence subgroup of level p . This paper investigates the geometry of the Hilbert modular threefold $X^\circ = \Gamma(p) \backslash \mathfrak{H}^3$ and some related varieties:

- (1) Let X be the minimal compactification of X° , and let X_{ch} be the singular toroidal compactification built using the fans determined by taking the cones on the faces of the convex hulls of the totally positive lattice points in the cusp data. Then X_{ch} is the canonical model of X (Theorem 8.4).
- (2) We construct parallel weight 1 Eisenstein series F_0, F_1, F_2, F_4 and a parallel weight 2 Eisenstein series E_2 that generate the ring of symmetric Hilbert modular forms of level p and parallel weight (i.e., the subring of the parallel weight Hilbert modular forms invariant under the action of the Galois group $\text{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/3\mathbb{Z}$) (Theorem 5.3).
- (3) There is a weighted homogeneous polynomial P of degree 8 with 42 terms such that $P(F_0, F_1, F_2, F_4, E_2) = 0$ (Proposition 6.7). This polynomial generates the ideal of relations on the F_i and E_2 , and the symmetric Hilbert modular threefold $X_{\text{Gal}} = X/\text{Gal}(K/\mathbb{Q})$ is the hypersurface cut out by $P = 0$ in the weighted projective space $\mathbb{P}(1, 1, 1, 1, 2)$ (Theorem 7.6).

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- (4) Let Q be the polynomial obtained from P by setting the weight 2 variable to zero. Then Q has 24 terms and defines a degree 8 hypersurface in \mathbb{P}^3 with singular locus being 84 quotient singularities of type A_2 (Proposition 9.1).

These results can be considered part of the venerable tradition of writing explicit equations for modular varieties, a tradition including (i) the Klein quartic, which presents the modular curve $X(7)$ as an explicit quartic in \mathbb{P}^2 , (ii) the Igusa quartic, which is the minimal compactification of the Siegel modular threefold of level 2 [17, 18], and (iii) the Burkhardt quartic, which is a three-dimensional ball quotient [17, 18]. Our results are in the spirit of results of van der Geer, Hirzebruch, van de Ven, and Zagier for Hilbert modular surfaces [14–16, 30–33], and use many of the same techniques: toroidal compactifications, the Shimizu trace formula [24], the action of the finite group $\Gamma/\Gamma(p) \simeq SL_2(\mathbb{F}_p)$, and explicit construction of modular forms.

We now give an overview of the paper. Section 2 collects basic facts about our field K , and Section 3 describes the toroidal resolutions X_{sm} and X_{ch} we consider. In Section 4 we compute intersection numbers of exceptional divisors on X_{sm} and X_{ch} . The modular forms we need are constructed in Section 5, and in Section 6 we compute the polynomial relation they satisfy. Section 7 treats the symmetric Hilbert modular threefold, and Section 8 contains the proof that X_{ch} is the canonical model. Finally, Section 9 describes the octic with 84 A_2 -singularities.

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2. THE CUBIC FIELD OF DISCRIMINANT 49

In this section we summarize standard facts about the totally real cubic field K of discriminant 49 and its ring of integers \mathcal{O} .

The field K is the maximal totally real subfield of the cyclotomic field $\mathbb{Q}(\zeta_7)$, where $\zeta_7 = \exp(2\pi i/7)$. Thus K is obtained by adjoining to \mathbb{Q} the number $w = \zeta_7 + \zeta_7^{-1}$. The extension K/\mathbb{Q} is Galois, with $\text{Gal}(K/\mathbb{Q}) \simeq \mathbb{Z}/3\mathbb{Z}$, and a generator of the Galois group maps w to $w^2 - 2$. The norm and trace of an element $r = a + bw + cw^2$ are given respectively by

$$N(r) = a^3 - a^2b + 5a^2c - 2ab^2 - abc + 6ac^2 + b^3 - b^2c - 2bc^2 + c^3$$

and

$$\text{Tr}(r) = 3a - b + 5c.$$

We define an isomorphism

$$(2.1) \quad K \otimes \mathbb{R} \simeq \mathbb{R}^3,$$

by sending $a \in K$ to $(a_1, a_2, a_3) \in \mathbb{R}^3$, where $a \mapsto a_k$ are the three embeddings $K \hookrightarrow \mathbb{R}$, numbered by $w_k = 2 \cos(2^k \pi/7) = 2 \cos(2k\pi/7)$. Thus $w_1 = w$, $w_2 = w^2 - 2$, and $w_3 = 1 - w - w^2$.

The elements $\{1, w, w^2\}$ form a \mathbb{Z} -basis of \mathcal{O} . Thus \mathcal{O} is isomorphic to $\mathbb{Z}[w]/\langle w^3 + w^2 - 2w - 1 \rangle$. It has class number 1, in other words it is a PID. The extension \mathcal{O}/\mathbb{Z} is ramified only over the prime 7. The ideal $\langle 7 \rangle \subset \mathbb{Z}$ factors as p^3 , where the prime $p \subset \mathcal{O}$ is generated by the totally positive element $2 - w$ of norm 7 and trace 7. The full unit group \mathcal{O}^\times is generated by the elements $-1, w$, and $w + 1$. The subgroup of units of norm 1 is generated by the pair $w, -w - 1$, and the subgroup of totally positive units is generated by $w^2, w^2 + 2w + 1$.

In what follows we will need the list of all totally positive elements of \mathcal{O} of traces 7 and 14, which are shown in Table 1. The columns are ordered by the Galois action. The first row of the trace 14 elements are twice those of trace 7. Note that the elements of trace 7 give a \mathbb{Z} -basis of p .

Trace 7	$2 - w$	$4 - w^2$	$1 + w + w^2$
Trace 14	$4 - 2w$	$8 - 2w^2$	$2 + 2w + 2w^2$
	$5 + w$	$3 + w^2$	$6 - w - w^2$
	$w + 3w^2$	$7 - 3w - 2w^2$	$7 + 2w - w^2$
	$1 - w + 2w^2$	$9 - 2w - 3w^2$	$4 + 3w + w^2$
	$3 + 5w + 2w^2$	$-1 - 2w + 3w^2$	$12 - 3w - 5w^2$

TABLE 1. Totally positive elements of \mathcal{O} of trace 7 and 14

3. RESOLUTION OF THE CUSPS

Let \mathfrak{H} be the upper half plane. We denote elements of the product \mathfrak{H}^3 using multi-index notation, i.e. given $z \in \mathfrak{H}^3$ we write $z = (z_1, z_2, z_3)$. Let $\Gamma = \Gamma(1)$ be the Hilbert modular group $\mathrm{SL}(2, \mathcal{O})$, and let $\Gamma(p) \subset \Gamma$ be the principal congruence subgroup of matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with $b, c \in p$ and $a, d = 1 \pmod{p}$. Let X° be the Hilbert modular threefold $\Gamma(p) \backslash \mathfrak{H}^3$, where as usual $\Gamma(p)$ acts on \mathfrak{H}^3 via the three real embeddings of K in \mathbb{R} corresponding to (2.1). Thus we have

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \left(\frac{a_1 z_1 + b_1}{c_1 z_1 + d_1}, \frac{a_2 z_2 + b_2}{c_2 z_2 + d_2}, \frac{a_3 z_3 + b_3}{c_3 z_3 + d_3} \right), \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(p), z \in \mathfrak{H}^3.$$

It is easy to see that the group $\Gamma(p)$ has no elliptic elements (elements of finite order), so that the quotient X° is a smooth complex threefold. The finite group $G = \Gamma(1)/\Gamma(p)$ acts on X , with the center $Z(G)$ acting trivially. The residue field \mathcal{O}/p is isomorphic to \mathbb{F}_7 , hence $G \simeq \mathrm{SL}(2, \mathbb{F}_7)$. The action of G on \mathfrak{H}^3 factors through the quotient $G/Z(G) \simeq \mathrm{PSL}(2, \mathbb{F}_7)$, the simple group of order 168.

Let X be the minimal (Baily–Borel–Satake) compactification of X° obtained by adjoining cusps. Recall that X is the quotient of the partial compactification $\overline{\mathfrak{H}}^3$ obtained by adjoining the set of cusps $\mathbb{P}^1(K)$ to \mathfrak{H}^3 and endowing the result with the Satake topology. The variety X is singular; resolutions of the cusp singularities of Hilbert modular varieties were described for quadratic fields by Hirzebruch [13], and in general by Ehlers [6] (see also [8, Appendix of III.7]). The goal of this section is to explicitly describe the resolution of singularities of the cusps of X .

Lemma 3.1. *The Hilbert modular threefold X has 8 cusps. The group G acts transitively on the cusps of X .*

Proof. A generalization of the standard argument for the principal congruence subgroup of $\mathrm{SL}(2, \mathbb{Z})$ (cf. [25, Lemma 1.42]) shows that the cusps are in bijection with the set of nonzero points in $(\mathcal{O}/p)^2$ modulo the induced action of the units. It is not hard to compute that the resulting set is isomorphic to $\mathbb{P}^1(\mathbb{F}_7)$. This proves the first statement.

For the second, one observes that action of G on the cusps is the same as the induced action of G on $\mathbb{P}^1(\mathbb{F}_7)$ under the above bijection. \square

Lemma 3.1 implies that to resolve the cusps it suffices to look at the cusp corresponding to the image of $(i\infty, i\infty, i\infty)$ in $\overline{\mathfrak{H}}^3$. Before investigating this cusp, we first describe the cusp singularity and a toroidal resolution of the unique cusp of $Y = \Gamma(1) \backslash \overline{\mathfrak{H}}^3$, the minimal compactification of the Hilbert modular threefold attached to the full group $\Gamma(1)$.

The inverse different of the ring \mathcal{O} is the fractional ideal p^{-2} . Let $C \subset p^{-2} \cong \mathbb{Z}^3$ be the totally nonnegative cone. Namely C consists of the elements of p^{-2} whose images under (2.1) are nonnegative. Clearly, these are just 0 and the totally positive elements in p^{-2} . The group $U \cong \mathbb{Z}^2$ of totally positive units acts on C . The local (analytic) ring of the cusp for the group $\Gamma(1)$ is given by the U -invariant Fourier series with exponents in C that converge in some neighborhood of the cusp (cf. [8, Theorem 4.1]).

Partial toroidal resolutions of this singularity are described by rational polyhedral fans in the dual cone C^* in the dual of p^{-2} under the trace pairing, i.e. by fans in the totally nonnegative cone in the lattice \mathcal{O} . One needs the fan to be invariant under the natural action of U on C^* .

The resulting resolution will be nonsingular if and only if the fan is simplicial with each maximal cone generated by a \mathbb{Z} -basis of \mathcal{O} . One such resolution for our singularity was constructed explicitly by Grundman [10]. However, the goal there was to compute special values of partial zeta functions using Shintani’s formula [26], not to construct a geometrically natural resolution. Hence the resolution found in [10] is not the most useful for our purposes.

To describe a somewhat better resolution from our perspective, observe that there is always a natural *partial* resolution of a cusp singularity: one begins with the convex hull Π of the nonzero points in $C^* \cap \mathcal{O}$, and builds the fan whose cones of maximal dimension are the cones on the facets of Π . For quadratic fields, such fans lead to canonical resolutions of the cusp singularities of Hilbert modular surfaces. For fields of higher degree, the resulting varieties are usually singular; indeed, the fans one obtains need not be simplicial.

For the cusp of Y , we can describe this canonical fan as follows. We start with the fundamental parallelogram of the action of U as in [10, Figure 1], but then subdivide it by the *opposite* diagonal (Figure 1). Note that vertices in Figure 1 are labeled by units in \mathcal{O} .

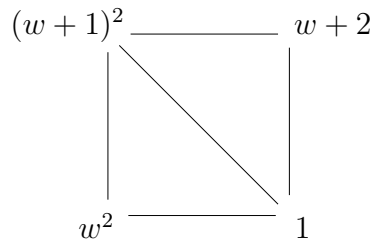


FIGURE 1. The canonical partial resolution Y_{ch} .

Proposition 3.2. *The translates of the two triangles of Figure 1 by U are precisely the facets of the convex hull of the set of totally positive integers.*

Proof. It is clear that the cones spanned by these triangles cover the entire cone C^* . Consequently, it is only necessary to show that each of these triangles is indeed a face of the boundary. This means that the linear functions that equal 1 on this face are bigger than 1 on any other totally positive element of \mathcal{O} .

Consider first the triangle with vertices 1 , w^2 , and $(w+1)^2$. The supporting inequality determined by this triangle on any integer $r = a+bw+cw^2$ is $2a-b+2c \geq 2$. Let ω be the totally positive integer $9 - 2w - 3w^2$. If r is a totally positive element in \mathcal{O} with $2a - b + 2c \leq 2$, then $\text{Tr}(r\omega) \leq 14$. Given that ω lies in p , we see that $\text{Tr}(r\omega)$ is in p , so it has to be 7 or 14.

According to Table 1, there are three totally positive integers of trace 7. One easily sees that none are divisible by ω in \mathcal{O} . The table also shows that there are 15 elements of trace 14. Exactly three of them are divisible by ω , and the ratios are 1, w^2 , and $(w+1)^2$. Thus this triangle is a facet of the convex hull.

The argument for the triangle with vertices 1 , $w+2$, and $(w+1)^2$ is similar. The supporting inequality on $r = a + bw + cw^2$ is $a - b + 2c \geq 1$. If $a - b + 2c \leq 1$, then $\text{Tr}(r\eta) \leq 7$, where $\eta = 2 - w$. Again $\eta \in p$, which means $\text{Tr}(r\eta) = 7$ and $r\eta \in \{4 - w^2, 2 - w, 1 + w + w^2\}$. Consequently, 1 , $w+2$, and $(w+1)^2$ are vertices

of a facet of the convex hull of totally positive elements of \mathcal{O} , which finishes the proof. \square

Using the partial resolution we can construct a resolution of the cusp singularity of Y . The points $1, w^2, 1 + 2w + w^2$ generate a sublattice of index 2 in \mathcal{O} , and thus the corresponding cone must be subdivided. We subdivide all translates of this cone into three cones. Geometrically this amounts to blowing up the unique singular point on the partial resolution. On the other hand the points $1, 2 + w, 1 + 2w + w^2$ form a \mathbb{Z} -basis of \mathcal{O} , and thus no blowups are needed on its translates. The resulting smooth triangulation is shown in Figure 2.

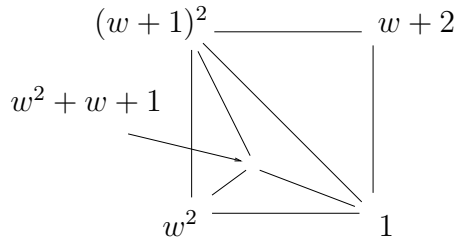


FIGURE 2. The smooth resolution Y_{sm} .

Now we want to resolve the singularities of the cusps of X . The discussion is essentially the same as above, although the lattice and unit groups change. The lattice \mathcal{O} needs to be replaced by the lattice p . Define the subgroup $U_1 \subset U$ by

$$(3.1) \quad U_1 = \{u \in U \mid u \equiv 1 \pmod{p}\}.$$

This is a subgroup of index 3 in U . Then we have

$$\begin{pmatrix} u & * \\ 0 & u^{-1} \end{pmatrix} \in \Gamma(p) \quad \text{if and only if} \quad u \in U_1.$$

Moreover, recall that the ideal p is generated by the totally positive element $2 - w$. Thus the resolution of the cusp of X is basically the same as that for Y . All one needs to do is to multiply the vertices of Figures 1 and 2 by $2 - w$ to dilate the lattice \mathcal{O} to p (cf. Remark 3.3), and then to take 3 copies of the domains in Figures 1 and 2 to reflect the smaller group of units. The resulting resolutions X_{ch} and X_{sm} are depicted in Figure 3.

- The partial resolution X_{ch} , which corresponds to taking the convex hull, is obtained by erasing the E_i and the lines emanating from them. The variety X_{ch} is not smooth: it has 24 isolated $\mathbb{Z}/2\mathbb{Z}$ quotient singularities. The exceptional divisors of $\pi_1: X_{ch} \rightarrow X$ over each cusp are D_1, D_2, D_3 . Since there are 8 cusps, there are altogether 24 exceptional divisors.
- The full resolution $\pi_2: X_{sm} \rightarrow X$ is given by the entire Figure 3. It is the blowup of X_{ch} at its 24 $\mathbb{Z}/2\mathbb{Z}$ quotient singularities. Thus the map π_2 has 48

exceptional divisors. Note that there is a natural map $X_{sm} \rightarrow X_{ch}$; the exceptional divisors are E_1, E_2, E_3 over each cusp, and so there are 24 components overall.

Remark 3.3. The points in Figure 3 correspond (left-to-right and then top to bottom) to elements $1 + 2w + w^2, 2 + w; 1 + w + w^2, 4 - w^2; w^2, 1, 5 - w - 2w^2; 2 - w; -w + w^2, 3 - w - w^2$ of the totally positive cone C^* in \mathcal{O} . The corresponding points in the cone of p are given by multiplication of these points by $2 - w$.

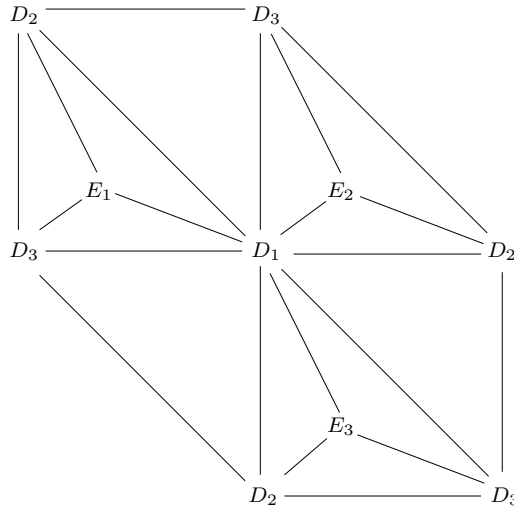


FIGURE 3. Divisors on X_{sm} .

We will introduce the following notations. Let D be the sum of the exceptional divisors of type D_i , and let E be the sum of the exceptional divisors of type E_i on X_{sm} . We will also abuse notation and use the same notation for the divisors D_i on X_{ch} and for their sum. We will summarize some of the easy facts that follow from the toric geometry.

- Proposition 3.4.**
- (1) *The divisor $D + E$ on X_{sm} has simple normal crossings.*
 - (2) *Each component of the divisor E is isomorphic to \mathbb{P}^2 , with normal bundle $\mathcal{O}(-2)$.*
 - (3) *The component D_1 of the divisor D is isomorphic to a toric surface with the fan given in the left part of Figure 4, where the divisor names indicate the intersection with other components of $D + E$.*
 - (4) *The normal bundle of D_1 is given by the line bundle corresponding to the piecewise linear function on the fan whose values at the generators of one-dimensional cones of the fan are given in the right part of Figure 4.*

(5) *The discrepancies are given by*

$$K_{X_{ch}} = \pi_1^* K_X - D, \quad K_{X_{sm}} = \pi_2^*(K_{ch}) + \frac{1}{2}E = (\pi_2 \circ \pi_1)^* K_X - D - E.$$

Proof. The vertices of every triangle in Figure 3 correspond to three different components of $D+E$, which implies the simple normal crossing statement. The components of E are obtained by blowing up an isolated $\mathbb{Z}/2\mathbb{Z}$ -quotient singularity on X_{ch} ; this implies their description as a \mathbb{P}^2 with normal bundle $\mathcal{O}(-2)$.

To find the structure of the divisor D_1 from Figure 3, we project the surrounding vertices in Figure 3 to a dimension two lattice obtained by modding out the element that corresponds to D_1 . Remark 3.3 then leads to Figure 4. To calculate the normal bundle $D_1|_{D_1}$, observe that in a neighborhood of D_1 the divisor corresponding to the global linear function that calculates the first coordinate of the points in Remark 3.3 is trivial. By subtracting it from D_1 we arrive at the values in the second diagram of Figure 4.

The calculation of discrepancies is standard and is left to the reader. \square

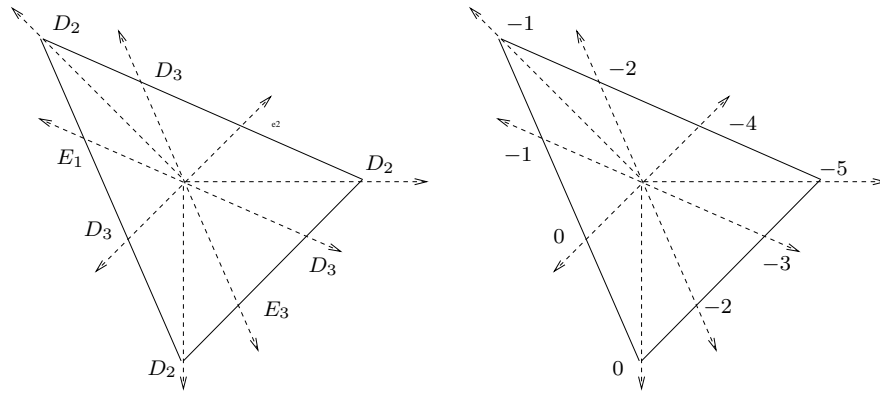


FIGURE 4. The component D_1 and its normal bundle.

4. INTERSECTION NUMBERS

The goal of this section is to calculate various intersection numbers on X_{sm} and X_{ch} . For background and basic definitions for Hilbert modular forms, we refer to [8].

Definition 4.1. We will denote by L the pullback from X of the line bundle of a (meromorphic) weight $(1, 1, 1)$ modular form. We will abuse notation and denote by L the pullback of this bundle to X_{sm} and to X_{ch} .

Note that the global sections $H^0(X_{sm}, kL)$ can be naturally identified with weight (k, k, k) modular forms with respect to $\Gamma(p)$. Similarly, global sections $H^0(X_{sm}, kL - D - E)$ are the cusp forms of weight (k, k, k) .

We will need the following proposition:

Proposition 4.2. [8, Lemma 4.6] *The canonical class K of X_{sm} is $2L - D - E$.*

Proposition 4.3. *We have*

$$\chi(X_{sm}, kL - D - E) = \chi(X_{sm}, kL) = 2(k-1)^3.$$

Proof. For even $k > 2$, we claim

$$(4.1) \quad \dim H^0(X_{sm}, kL - D - E) = 2(k-1)^3.$$

This implies both statements. Indeed, Proposition 4.2 and the Kawamata vanishing theorem [19] imply

$$H^0(X_{sm}, kL - D - E) = \chi(X_{sm}, kL - D - E).$$

This proves the statement about $\chi(X_{sm}, kL - D - E)$, since the Euler characteristic is polynomial in k . Then Serre duality implies $\chi(kL) = -\chi((2-k)L - D - E) = -2(1-k)^3$, which proves the statement about $\chi(X_{sm}, kL)$.

To prove (4.1), we apply the trace formula computations of Shimizu [24]. We have for even $k > 2$

$$(4.2) \quad \dim H^0(X_{sm}, kL - D - E) = \text{vol}(\Gamma(p) \backslash \mathfrak{H}^3)(k-1)^3 + e + c,$$

where the volume is computed with respect to a suitably normalized invariant measure and e (respectively c) is a contribution coming from the elliptic points of $\Gamma(p)$ (resp., the cusps of $\Gamma(p)$). Since $\Gamma(p)$ is torsion-free, we have $e = 0$. Furthermore c is essentially the sum of some special values of Hecke–Shimizu L -functions, each of which is attached to a pair (M, V) , where $M \subset K \otimes \mathbb{R} \simeq \mathbb{R}^3$ is a rank 3 \mathbb{Z} -module and V is a subgroup of the units fixing M . The action of -1 on M preserves this data and takes the special value into its negative (this happens for any totally real field of odd degree > 1). Thus $c = 0$, and the only contribution comes from the volume. By a theorem of Siegel [27, 28] we have

$$\text{vol}(\text{SL}(2, \mathcal{O}) \backslash \mathfrak{H}^3) = -\frac{\zeta_K(-1)}{4},$$

where $\zeta_K(s)$ is the Dedekind zeta function for K . This special value can be computed easily, e.g. using techniques in [11, 26], and we have

$$\text{vol}(\text{SL}(2, \mathcal{O}) \backslash \mathfrak{H}^3) = \frac{1}{84}.$$

Since the index of $\Gamma(p)$ in $\text{SL}(2, \mathcal{O})$ equals $\#\text{SL}(2, \mathbb{F}_7) = 336$ and the center of $\text{SL}(2, \mathcal{O})$ acts trivially, we find

$$\text{vol}(\Gamma(p) \backslash \mathfrak{H}^3) = 2,$$

which completes the proof of the claim. \square

Proposition 4.4. *We have the following intersection numbers on X_{sm} :*

$$L^3 = 12, \quad (K - \frac{1}{2}E)^3 = 36.$$

Proof. The Riemann-Roch formula for kL and Proposition 4.3 give $2(k-1)^3 = \frac{1}{6}k^3L^3 + \dots$, which implies $L^3 = 12$. To calculate $(K - \frac{1}{2}E)^3$ we will write it as $(K - \frac{1}{2}E)^2(2L - \frac{3}{2}E - D)$. Since L restricts trivially to D and E , we have $(K - \frac{1}{2}E)^2L = 4L^3 = 48$. Since $(K - \frac{1}{2}E)$ is the pullback of a \mathbb{Q} -Cartier divisor K_{ch} from X_{ch} , we have $(K - \frac{1}{2}E)^2E = 0$. So it remains to calculate $(K - \frac{1}{2}E)^2D = 24(2L - \frac{3}{2}E - D)^2D_1$. Since we know the normal bundle of D_1 from Proposition 3.4, we conclude that the restriction of $2L - \frac{3}{2}E - D$ to D_1 is given by the piecewise linear function on the left of Figure 5. The self-intersection is then an easy calculation in toric geometry. We get $0 - 2(\frac{1}{2})^2 + 2(-\frac{1}{2})(-1) - 2(-1)^2 + 2(-1)(-1) - (-1)^2 + 2(-1)(\frac{1}{2}) - 2(\frac{1}{2})^2 + 2(\frac{1}{2})2 - 2(2)^2 + 2(2)(4) - 4^2 + 2(\frac{5}{2})4 - 2(\frac{5}{2})^2 + 2\frac{5}{2}(1) - 2(1)^2$, which equals $\frac{5}{2}$. This gives $(K - \frac{1}{2}E)^3 = 96 - 24(\frac{5}{2}) = 36$.

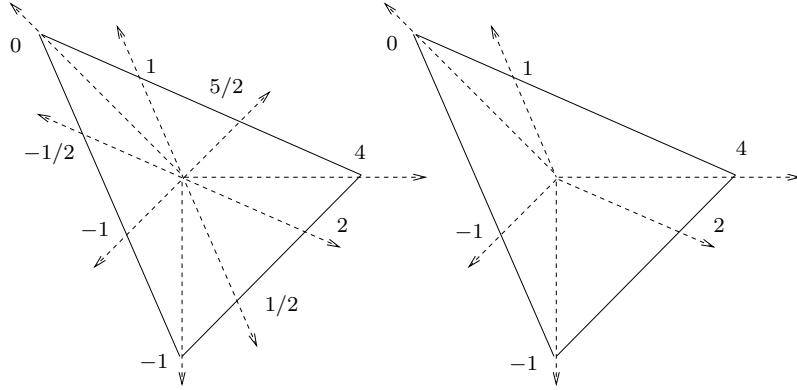


FIGURE 5. The restriction of $2L - \frac{3}{2}E - D$ to D_1 (proof of Proposition 4.4), and the toric surface D_1 on X_{ch} (proof of Lemma 8.3).

□

We will now calculate the dimensions of the spaces of modular forms of weight (k, k, k) for various k .

Proposition 4.5.

- (1) For $k \geq 3$ we have $\dim H^0(X_{sm}, kL - D - E) = 2(k-1)^3$ and $\dim H^0(X_{sm}, kL) = 2(k-1)^3 + 8$.
- (2) For $k = 2$ we have $\dim H^0(X_{sm}, 2L - D - E) = 3$ and $\dim H^0(X_{sm}, 2L) = 11$.
- (3) For $k = 1$ we have $\dim H^0(X_{sm}, L - D - E) = 0$. The dimension of $H^0(X_{sm}, L)$ is either 0 or 4.

Proof. The statement for $k \geq 3$ about cusp forms follows from the proof of Proposition 4.3. Similarly, for $k = 2$, the dimension of $H^0(X_{sm}, 2L - D - E)$ can be computed using Riemann-Roch (cf. [8, Theorem II.4.8]); the computations are similar to those of the proof of Proposition 4.3. For even $k \geq 2$ these computations also give the dimension of $H^0(X_{sm}, kL)$, since by Lemma 3.1 the number of cusps of X_{sm} is 8.

For any $k \geq 3$ Kawamata vanishing implies there is a short exact sequence

$$0 \rightarrow H^0(X_{sm}, kL - D - E) \rightarrow H^0(X_{sm}, kL) \rightarrow H^0(X_{sm}, \mathcal{O}_{D \cup E}(kL)) \rightarrow 0.$$

We have $\mathcal{O}_{D \cup E}(kL) = \mathcal{O}_{D \cup E}$. Hence the dimension of $H^0(X_{sm}, \mathcal{O}_{D \cup E}(kL))$ is the number of connected components of $D \cup E$, which is 8. This completes the proof of (1) and (2).

We turn now to $k = 1$. Let us prove that $H^0(X_{sm}, L - D - E) = 0$. The space $H^0(X_{sm}, L - D - E) = 0$ is a representation of the group $G \simeq \mathrm{SL}(2, \mathbb{F}_7)$ such that the central involution acts by -1 . By investigating the character table of G [9, 23], we see that the smallest dimension of such a representation is 4. If there were a nonzero element in $H^0(X_{sm}, L - D - E) = 0$, a multiplication by it shows $\dim H^0(X_{sm}, L - D - E) \leq \dim H^0(2L - D - E) = 3$, which is a contradiction.

Finally, the space $H^0(X_{sm}, L)$ is also a representation of G with central involution acting by -1 . The dimensions of the irreducible representations of G with this property are 4, 6 and 8. There is an injection of G -representations $H^0(X_{sm}, L) \rightarrow H^0(X_{sm}, \mathcal{O}_{D \cup E}(L))$, where the latter has dimension 8. This shows that $\dim H^0(X_{sm}, L)$ cannot be 6. Suppose that $\dim H^0(X_{sm}, L) = 8$. Since $\dim H^0(X_{sm}, 2L) = 11$, the kernel of the multiplication map $H^0(X_{sm}, L)^{\otimes 2} \rightarrow H^0(X_{sm}, 2L)$ has codimension at most 11. Consequently, it contains a nonzero decomposable tensor, since the dimension of the image of the Segre embedding of $\mathbb{P}H^0(X_{sm}, L) \times \mathbb{P}H^0(X_{sm}, L)$ into $\mathbb{P}(H^0(X_{sm}, L)^{\otimes 2})$ is 14. However, this implies a product of two non-zero sections of $H^0(X_{sm}, L)$ is a non-zero section of $H^0(X_{sm}, 2L)$, contradiction. This leaves 0 and 4 as the only possibilities for $\dim H^0(X_{sm}, L)$. \square

Remark 4.6. We will later show in Corollary 5.6 that $\dim H^0(X_{sm}, L) = 4$ by explicitly exhibiting a basis of Eisenstein series.

5. EISENSTEIN SERIES

In this section we use Eisenstein series to construct explicitly some modular forms of weights $(1, 1, 1)$ and $(2, 2, 2)$ for the group $\Gamma(p)$. Recall that the class number of K is one, and thus every ideal of \mathcal{O} is principal.

Definition 5.1. For an ideal $(c) \subset \mathcal{O}$ we define $s(c) = 0$ if $p|c$. Otherwise, we define $s(c) = \mathrm{sgn}(N(c_1))$ where c_1 is a generator of (c) satisfying $c_1 = 1 \pmod{p}$.

We note that $s(c)$ is well-defined, since different generators c_1, c_2 of (c) that are equal to 1 mod p differ by a unit equal to 1 mod p , and all such units have positive norm. Alternatively, one could define $s(c)$ for $(c, p) = 1$ as the product of $\mathrm{sgn}(N(c))$

and the quadratic residue symbol $\left(\frac{c}{p}\right) \in \{\pm 1\}$ (the latter equals 1 if and only if the image of c is a square in $\mathcal{O}/p \simeq \mathbb{F}_7$).

Definition 5.2. For $z = (z_1, z_2, z_3) \in \mathfrak{H}^3$ and $a \in K$, we define $\text{Tr}(az) = a_1 z_1 + a_2 z_2 + a_3 z_3$. Then for $k \in \{0, 1, 2, 4\}$ we define

$$(5.1) \quad F_k(z) = c_k + \sum_{\substack{a \in \mathcal{O}, a \gg 0 \\ a \equiv k \pmod{p}}} \exp(2\pi i \text{Tr}(az)/7) \left(\sum_{(c)|(a)} s(c) \right)$$

where $c_0 = 1/14$ and $c_1 = c_2 = c_4 = 0$. We also define

$$(5.2) \quad E_2(z) = -\frac{1}{168} + \sum_{a \in p^{-2}, a \gg 0} \exp(2\pi i \text{Tr}(az)) \left(\sum_{(c)|(ap^2)} |N(c)| \right)$$

The use of the notation E_2 for the series (5.2) and an exceptional divisor in should cause no difficulty for the reader, since we will not be needing the latter in the sequel. It is easy to see that the series (5.1)–(5.2) converge. Our main result in this section is the following theorem:

Theorem 5.3. *For $k \in \{0, 1, 2, 4\}$ the series $F_k(z)$ is a weight $(1, 1, 1)$ modular form with respect to the group $\Gamma(p)$. The series $E_2(z)$ is a weight $(2, 2, 2)$ modular form with respect to the group $\Gamma(1)$.*

Before proving the theorem, we need to recall some work of Yang [34], who extended and corrected certain constructions of Hecke [12] for real quadratic fields to all totally real fields. First we need some notation.

Let L be a CM field with maximal real subfield K . Let \mathcal{O} be the ring of integers of K . Suppose $[K : \mathbb{Q}] = d$, and let χ be the quadratic Hecke character attached to the extension L/K . Let ∂_K be the different of K and let $d_{L/K}$ be the relative discriminant. Let $(\alpha_v) \in \prod_{v|d_{L/K}} F_v^\times$ be a tuple with $\text{ord}_v(\alpha_v) = \text{ord}_v(\partial_K)$. Let N be a square-free integral ideal of K coprime to $d_{L/K}$.

Theorem 5.4. [34, Theorem 1.2]

- (1) *There is a function $E(z, s; \Phi^{\alpha, N}) : \mathfrak{H}^d \times \mathbb{C} \rightarrow \mathbb{C}$ that as a function of s is meromorphic with possibly finitely many poles.*
- (2) *For s fixed and away from the poles, $E(z, s; \Phi^{\alpha, N})$ is a (non-holomorphic) Hilbert modular form of weight $(1, \dots, 1)$, level $d_{L/K}$, and character χ , where χ means*

$$\chi : (\mathcal{O}/d_{L/K}N)^\times \rightarrow (\mathcal{O}/d_{L/K})^\times \rightarrow \{\pm 1\}, \quad \chi(a) = \prod_{v|d_{L/K}} \chi_v(a).$$

(3) For $s = 0$, the function $E(z, 0; \Phi^{\alpha, N})$ is a holomorphic Hilbert modular form with Fourier expansion

$$(5.3) \quad E(z, 0; \Phi^{\alpha, N}) = (1 + \varepsilon(\alpha, N))L(0, \chi) + 2^d \varepsilon(\alpha, N) \sum_{\substack{t \in \partial_K^{-1} N \\ t \gg 0}} \delta(\alpha t) \rho_{L/K}(t \partial_K N^{-1}) \exp(2\pi i \operatorname{Tr}(tz)).$$

Here $L(s, \chi)$ is the Hecke L -function attached to the Hecke character χ and the quantity $\varepsilon(\alpha, N)$ is defined by

$$\varepsilon(\alpha, N) = (-1)^{o(N)} \prod_{v|d_{L/K}} \chi_v(\alpha_v) \prod_{v|\partial_K, v \nmid d_{L/K}} \chi_v(\partial_K)$$

where $o(N)$ is the number of prime factors of N . Furthermore

$$\delta(\alpha t) = \prod_{v|d_{L/K}} (1 + \chi_v(\alpha_v t))$$

and

$$\rho_{L/K}(\mathfrak{a}) = \#\{\mathfrak{A} \subset \mathcal{O}_L \mid N_{L/K} \mathfrak{A} = \mathfrak{a}\}$$

Now let $\Gamma_1(p)$ (resp. $\Gamma_0(p)$) be the subgroup of $\Gamma(1)$ given by the condition that its elements are upper-triangular unipotent (resp. upper-triangular) modulo p .

Lemma 5.5. *The series $F_0(z)$ converges to a modular form of weight $(1, 1, 1)$ for $\Gamma_1(p)$. Moreover, it has character χ for $\Gamma_0(p)$.*

Proof. Let $L = \mathbb{Q}(\zeta_7)$. Then L is a complex quadratic extension of K ; let χ be the corresponding Hecke character. The relative discriminant $d_{L/K}$ of the extension L/K is the prime ideal p , and the different ∂_K is p^2 .

Let $\alpha = (2 - w)^2 \in K \subset K_p$, and let $N = \mathcal{O}$. We claim that the series $E(z) = E(z, 0; \Phi^{\alpha, N})$ of Theorem 5.4 is nonzero and is a multiple of $F_0(z)$. Indeed, $\varepsilon(\alpha, N) = 1$ for this data since χ is quadratic and $o(N) = 0$. Using results of [11], we can compute that the constant term of $E(z)$ is $L(0, \chi) = 2/7$. Thus we must have

$$E(z, 0; \Phi^{\alpha, N}) \stackrel{?}{=} 16F_0(z);$$

we will check this by comparing q -expansions. By Theorem 5.4, this will complete the proof.

The sum in (5.1) is taken over all totally positive $a \in p$, whereas the sum in (5.3) is taken over all totally positive $t \in p^{-2}$. To compare these q -expansions, we set $a = 7t$. Thus we need to check that

$$2 \sum_{(c)|(a)} s(c) = \delta(\alpha a/7) \rho_{L/K}(a/(2 - w)), \quad a \in p, a \gg 0.$$

Note that $\sum_{(c)|(a)} s(c) = \sum_{(c)|(a/(2-w))} s(c)$, since the extra summands in the first sum vanish. The identities we will prove are

$$(5.4) \quad \sum_{(c)|(b)} s(c) = \rho_{L/K}(b), \quad b \in \mathcal{O},$$

$$(5.5) \quad \delta(\alpha a/7) = 2 \quad \text{if } a \in p, a \gg 0, \text{ and } \rho_{L/K}(a/(2-w)) \neq 0.$$

Since $d = 3$, this will complete the proof of the theorem.

Both sides of (5.4) are multiplicative functions, so it suffices to check them on powers of prime ideals $P^k \subset \mathcal{O}$. We have

$$\rho_{L/K}(P^k) = \begin{cases} k+1 & P \text{ split in } L, \\ 1 & P \text{ inert in } L, k \text{ even,} \\ 0 & P \text{ inert in } L, k \text{ odd,} \\ 1 & P \text{ ramified in } L. \end{cases}$$

On the other hand, from class field theory we know $s(P) = 1$ (respectively -1) if and only if P is split in L (respectively inert in L). Since the only prime that ramifies is p and $s(p) = 0$, this proves (5.4) and completes the proof of the lemma. \square

Proof of Theorem 5.3. We observe that since $p = (2-w)$, $F_0(z')$ is a modular form for $\Gamma(p)$, where

$$z' = (z_1/(2-w), z_2/(4-w^2), z_3/(1+w+w^2))$$

(the denominators are the Galois conjugates of $2-w$). Then we have

$$(5.6) \quad F_0(z') = \frac{1}{14} + \sum_{\substack{a \in \mathcal{O}, a \gg 0 \\ a=0 \pmod{p}}} \exp(2\pi i \operatorname{Tr}(az'/7)) \left(\sum_{(c)|(a)} s(c) \right)$$

$$(5.7) \quad = \frac{1}{14} + \sum_{a \in \mathcal{O}, a \gg 0} \exp(2\pi i \operatorname{Tr}(az/7)) \left(\sum_{(c)|(a)} s(c) \right).$$

Here we used $\sum_{(c)|(a(2-w))} s(c) = \sum_{(c)|(a)} s(c)$, which follows since the additional summands on the left vanish.

We now use the action of $z \mapsto z + (1, 1, 1)$ to separate (5.7) into eigenvectors of this action. This leads to F_0, F_1, F_2, F_4 , since it is rather straightforward to see that other possible values of $a \pmod{p}$ give series that are identically zero, since in these cases $s(c) = -s(a/c)$. It is also easy to observe that the F_i above are nonzero.

The statement for E_2 follows from the computations of Eisenstein series found in [8]. \square

Corollary 5.6. *The space of modular forms of weight $(1, 1, 1)$ for $\Gamma(p)$ has dimension 4 and basis $\{F_0, F_1, F_2, F_4\}$. As a consequence, the linear span of F_i is a four-dimensional irreducible representation of $\mathrm{SL}(2, \mathbb{F}_7)$.*

Proof. By Proposition 4.5 we know that the dimension of this space is either zero or four. It remains to check that the F_i are linearly independent, but this is obvious, since F_i is an eigenvector of the action $z \mapsto z + (1, 1, 1)$ with eigenvalue ζ_7^{3i} . \square

Remark 5.7. We can represent the modular forms for $\Gamma(p)$ as power series in variables q_1, q_2, q_3 by writing

$$\exp(2\pi i \mathrm{Tr}(az)/7) = q_1^{\mathrm{Tr}(a(2-w)/7)} q_2^{\mathrm{Tr}(a(4-w^2)/7)} q_3^{\mathrm{Tr}(a(1+w+w^2)/7)},$$

where $q_k = \exp(-2\pi i \mathrm{Tr}(w_k z)/7)$. In these coordinates the expansions are symmetric under cyclic permutation of the subscripts of the q_i , and so in the following we put

$$q(a_1, a_2, a_3) = \delta \sum_{\sigma \in \mathbb{Z}/3\mathbb{Z}} \prod q_{\sigma(i)}^{a_{\sigma(i)}},$$

where $\delta = 1/3$ if $a_1 = a_2 = a_3$, and is 1 otherwise. With these notations, the Eisenstein series become

$$\begin{aligned} F_0 &= 1/14 + q(2, 2, 3) + q(2, 5, 7) + q(3, 5, 6) \\ &\quad + q(3, 6, 5) + q(3, 7, 11) + 2q(4, 10, 14) + 2q(4, 4, 6) + \cdots, \end{aligned}$$

$$\begin{aligned} F_1 &= q(1, 1, 1) + 2q(2, 4, 4) + q(2, 3, 5) + q(2, 6, 9) + 2q(3, 3, 4) \\ &\quad + 2q(3, 6, 8) + 2q(4, 11, 16) + 2q(4, 5, 8) + 2q(4, 6, 7) \\ &\quad + 2q(4, 7, 6) + 2q(4, 9, 11) + 3q(4, 8, 12) + \cdots, \end{aligned}$$

$$\begin{aligned} F_2 &= q(1, 2, 3) + 2q(2, 2, 2) + q(2, 5, 6) + 2q(3, 4, 6) + 2q(3, 5, 5) \\ &\quad + 2q(3, 7, 10) + 2q(4, 10, 13) + 2q(4, 12, 18) + 2q(4, 4, 5) \\ &\quad + 2q(4, 6, 10) + 2q(4, 7, 9) + 2q(4, 9, 14) + 3q(4, 8, 8) + \cdots, \end{aligned}$$

$$\begin{aligned} F_4 &= q(1, 2, 2) + 2q(2, 4, 6) + 2q(3, 4, 5) + 2q(3, 5, 4) + 2q(3, 7, 9) \\ &\quad + q(3, 9, 14) + 2q(4, 10, 12) + 2q(4, 6, 9) + 2q(4, 7, 8) \\ &\quad + 2q(4, 8, 7) + 2q(4, 9, 13) + 3q(4, 4, 4) + \cdots, \end{aligned}$$

$$\begin{aligned} E_2 &= -1/168 + q(2, 2, 3) + q(2, 5, 7) + 8q(3, 5, 6) + q(3, 6, 5) \\ &\quad + q(3, 1, 11) + 9q(4, 4, 6) + 14q(4, 5, 5) + 14q(4, 7, 10) \\ &\quad + 14q(4, 8, 9) + 9q(4, 10, 14) + \cdots. \end{aligned}$$

6. RELATION AMONG F_0, F_1, F_2, F_4, E_2

In this section we will find a polynomial relation among the Eisenstein series F_i and E_2 . We begin by calculating the action of the group $G = \Gamma(1)/\Gamma(p) \cong \mathrm{SL}(2, \mathbb{F}_7)$ on the linear span of the F_i . Notice that this group is generated by the elements g_7 and g_4 given by

$$g_7: z \mapsto z + (1, 1, 1), \quad g_4: (z_1, z_2, z_3) \mapsto \left(-\frac{1}{z_1}, -\frac{1}{z_2}, -\frac{1}{z_3}\right);$$

the element g_k has order k .

First, the action of g_7 is given by $g_7(F_i) = \zeta_7^{3i} F_i$ for all i , which can be seen directly from Definition 5.2.

Next, to calculate the action of g_4 , we consider the restrictions $\bar{F}_i(\tau)$ of $F_i(z)$ to the diagonal $z = (\tau, \tau, \tau)$, $\tau \in \mathfrak{H}$. These are clearly modular forms of weight 3 for the principal congruence subgroup $\Gamma(7)$ of $\mathrm{SL}(2, \mathbb{Z})$. The action of g_4 can be recovered from the action of $\tau \mapsto -1/\tau$ on \bar{F}_i .

First we consider $\bar{F}_0(\tau)$. The calculation simplifies here, since $\bar{F}_0(\tau)$ is invariant under $\tau \mapsto \tau+1$ and is thus a modular form of weight 3 for the larger group $\Gamma_1(7)$. The ring of modular forms for $\Gamma_1(7)$ is easy to describe in terms of weight one Eisenstein series [2]. We will also need an explicit action of the Fricke involution on these weight one forms.

Proposition 6.1. [2, Theorem 4.11] *Let $q = \exp(2\pi i\tau)$. For $a = 1, 2, 3$ let s_a be the series*

$$s_a(\tau) = \frac{1}{2} - \frac{a}{7} + \sum_{n>0} q^n \sum_{d|n} (\delta_d^{a \bmod 7} - \delta_d^{-a \bmod 7}),$$

where δ_β^α takes the value 1 if α is the residue β , and is 0 otherwise. Then $\{s_1, s_2, s_3\}$ forms a basis of the space of modular forms of weight one for $\Gamma_1(7)$.

Remark 6.2. Explicitly,

$$\begin{aligned} s_1 &= 1/2 - 1/7 + q + q^2 + q^3 + q^4 + q^5 + q^7 + 2q^8 + q^9 + q^{10} + q^{11} + q^{14} + 2q^{15} + O(q^{16}), \\ s_2 &= 1/2 - 2/7 + q^2 + q^4 - q^5 + q^6 + q^8 + q^9 + q^{14} - q^{15} + O(q^{16}), \\ s_3 &= 1/2 - 3/7 + q^3 - q^4 + q^6 - q^8 + q^9 + q^{10} - q^{11} + q^{15} + O(q^{16}). \end{aligned}$$

In addition, these s_i satisfy the equation $7(s_1^2 + s_2^2 + s_3^2) = 5(s_1 + s_2 - s_3)^2$, which establishes an isomorphism of $X_1(7)$ with a conic in $\mathbb{P}^2(\mathbb{C})$. For more details see [3].

Proposition 6.3. *The restriction $\bar{F}_0(\tau)$ of $F_0(z)$ is given by*

$$(6.1) \quad \bar{F}_0 = 7(s_1 + s_2 - s_3)^3 - 147s_1s_2s_3.$$

Proof. The line bundle of weight three forms on $X_1(7)$ has degree 6. One can compute that the difference of the left and right sides of (6.1) vanishes at $i\infty$ to order more than 6, which means the difference vanishes. \square

We now calculate the Fricke involution of $\bar{F}_0(\tau)$. We will only require a few terms in the resulting q -expansion.

Proposition 6.4. *We have*

$$(6.2) \quad \frac{1}{\tau^3} \bar{F}_0\left(-\frac{1}{7\tau}\right) = 49i\sqrt{7} \left(\frac{1}{14} + q^3 + 3q^5 + 5q^6 + 3q^7 + 15q^{10} + 21q^{12} + 21q^{13} + 15q^{14} + O(q^{15}) \right).$$

Proof. We use

$$\frac{1}{\tau} s_i\left(-\frac{1}{7\tau}\right) = C + \sum_{n>0} q^n \sum_{d|n} (\zeta_7^{di} - \zeta_7^{-di}),$$

where C is an irrelevant constant, which follows from the calculations of [1]. This shows

$$\begin{aligned} -\frac{1}{\tau} s_1\left(-\frac{1}{7\tau}\right) &= (\zeta_7 - \zeta_7^{-1})s_1 + (\zeta_7^2 - \zeta_7^{-2})s_2 + (\zeta_7^3 - \zeta_7^{-3})s_3 \\ -\frac{1}{\tau} s_2\left(-\frac{1}{7\tau}\right) &= (\zeta_7^2 - \zeta_7^{-2})s_1 + (\zeta_7^4 - \zeta_7^{-4})s_2 + (\zeta_7^6 - \zeta_7^{-6})s_3 \\ -\frac{1}{\tau} s_3\left(-\frac{1}{7\tau}\right) &= (\zeta_7^3 - \zeta_7^{-3})s_1 + (\zeta_7^6 - \zeta_7^{-6})s_2 + (\zeta_7^9 - \zeta_7^{-9})s_3 \end{aligned}$$

Together with Proposition 6.3 this yields (6.2). \square

Proposition 6.5. *The action of g_4 on F_0 is given by*

$$g_4(F_0) = -\frac{i}{\sqrt{7}}(F_0 + F_1 + F_2 + F_4).$$

Proof. By definition of g_4 ,

$$g_4 F_0(z_1, z_2, z_3) = \left(-\frac{1}{z_1 z_2 z_3}\right) F_0\left(-\frac{1}{z_1}, -\frac{1}{z_2}, -\frac{1}{z_3}\right).$$

We apply Proposition 6.4 with τ replaced by $\tau/7$ to get

$$g_4 F_0(\tau, \tau, \tau) = -\frac{1}{\tau^3} \bar{F}_0\left(-\frac{1}{\tau}\right) = -\frac{i}{\sqrt{7}} \left(\frac{1}{14} + q^{\frac{3}{7}} + 3q^{\frac{5}{7}} + 5q^{\frac{6}{7}} + 3q + O(q^{\frac{8}{7}}) \right).$$

Since $g_4 F_0(z_1, z_2, z_3)$ is a linear combination of F_0, F_1, F_2, F_4 , it remains to compare this expansion of it with those of the restrictions \bar{F}_i . \square

Proposition 6.6. *The action of g_4 and g_7 in the basis $\{F_0, F_1, F_2, F_4\}$ is given by the matrices*

$$\gamma_4 = -\frac{i}{\sqrt{7}} \begin{pmatrix} 1 & 2 & 2 & 2 \\ 1 & w_1 & w_3 & w_2 \\ 1 & w_3 & w_2 & w_1 \\ 1 & w_2 & w_1 & w_3 \end{pmatrix} \quad \gamma_7 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \zeta_7^3 & 0 & 0 \\ 0 & 0 & \zeta_7^6 & 0 \\ 0 & 0 & 0 & \zeta_7^5 \end{pmatrix}.$$

Proof. A computer computation shows that the above matrices γ_4 and γ_7 define a representation of G . It is well known that the group G has a unique (up to isomorphism) 4-dimensional complex representation such that g_7 has eigenvalues $1, \zeta_7^3, \zeta_7^5, \zeta_7^6$. Hence we can find a matrix $T \in \text{GL}(4, \mathbb{C})$ such that γ_4 and γ_7 are T -conjugates of the matrices of g_4 and g_7 in the basis F_0, F_1, F_2, F_4 .

Since the action of g_7 is known to be given by the matrix γ_7 , we get $T\gamma_7 = \gamma_7T$, which implies that T is diagonal. Write $T = \text{diag}(t_0, t_1, t_2, t_4)$. We have

$$g_4(F_0) = T\gamma_4T^{-1}(F_0) = -\frac{i}{\sqrt{7}}t_0^{-1}(t_0F_0 + t_1F_1 + t_2F_2 + t_4F_4).$$

Together with Proposition 6.5, this shows that T is a multiple of the identity, which finishes the proof. \square

We are now ready to state the polynomial relation satisfied by the Eisenstein series. We use a notation similar to that in Remark 5.7 and write

$$F(a_1, a_2, a_4) = \delta \sum_{\sigma \in \mathbb{Z}/3\mathbb{Z}} \prod F_{\sigma(i)}^{a_{\sigma(i)}},$$

where $\delta = 1/3$ if $a_1 = a_2 = a_4$, and is 1 otherwise ($\mathbb{Z}/3\mathbb{Z}$ acts by cyclically permuting 1, 2, 4). We also put $E_2^*(z) = \frac{6}{7}E_2(z)$.

Proposition 6.7. *The Eisenstein series E_2^*, F_0, F_1, F_2 and F_4 satisfy the polynomial relation $P_8(F_0, F_1, F_2, F_4, E_2^*) = 0$, where*

$$\begin{aligned} P_8 = & (E_2^*)^4 - 3(E_2^*)^2(2F_0^4 + 6F_0F(1, 1, 1) + F(0, 1, 3)) \\ & + E_2^*(-8F_0^6 + 20F_0^3F(1, 1, 1) + 10F_0^2F(0, 1, 3) \\ & + 10F_0F(0, 3, 2) + 15F(2, 2, 2) + F(0, 5, 1)) \\ & - 3F_0^8 + 38F_0^5F(1, 1, 1) + 13F_0^4F(0, 1, 3) \\ & - 46F_0^3(F(0, 3, 2) + F_0^2(5F(0, 5, 1) + 23F(2, 2, 2))) \\ & - 2F_0(F(0, 0, 7) + 4F(1, 2, 4)) + 2F(0, 2, 6) - 5F(1, 4, 3). \end{aligned}$$

Proof. The dimensions of the invariants of the k -th symmetric powers of the 4-dimensional representation of G for small k are given by the following table:

k	0	2	4	6	8
$\dim(\text{Sym}^k(V_4)^G)$	1	0	1	1	3

As a result, there is a 6-dimensional space of polynomials of total degree 8 in weight one variables F_i and weight two variable E_2 that are invariant under the action of G . Each such polynomial gives a weight $(8, 8, 8)$ modular form with respect to the group $\Gamma(1)$. We claim the space of $(8, 8, 8)$ modular forms for $\Gamma(1)$ has dimension 5.

To see this, we use the trace formula (4.2). This says that if $k \geq 2$ is even, the dimension of the space of cusp forms on $\Gamma(1)$ is

$$\text{vol}(\text{SL}(2, \mathcal{O}) \backslash \mathfrak{H}^3)(k-1)^3 + e + c,$$

where as before e (respectively c) represents the contribution of the elliptic points (resp., cusps). We have $c = 0$ as before, but now we have nontrivial elliptic points and must evaluate e . It is well-known that the elliptic elements have orders 2, 3, 7, and it is not hard to find representatives of them modulo $\text{SL}(2, \mathcal{O})$: there are four points of each order, giving 12 in all. (The elements of orders 2 and 3 come from conjugates of the usual elliptic elements of $\text{SL}(2, \mathbb{Z})$, and the elements of order 7 come from writing the ring of integers of $\mathbb{Q}(\zeta_7)$ as a free rank 2 \mathcal{O} -module and looking at the action of ζ_7 .) Each elliptic point P has a rational triple $\alpha = (\alpha_1, \alpha_2, \alpha_3)$ of “rotation numbers” attached to it that reflects the action of the stabilizer of P . The contribution of P to e is then

$$\frac{1}{N(P)} \sum_{j=1}^{N(P)} \prod_{\ell=1}^3 \frac{\exp(2\pi i k j \alpha_\ell)}{1 - \exp(2\pi i j \alpha_\ell)},$$

where $N(P)$ is the order of P . The rotation numbers for our 12 points are shown in Table 2. From this it is easy to compute that the dimension of the cuspidal subspace of parallel weight 8 is 4. There is one Eisenstein series of parallel weight 8 since the class number of \mathcal{O} is one, and hence the dimension of the full space of parallel weight 8 forms is 5.

Since the space of $(8, 8, 8)$ modular forms has dimension 5, the invariant polynomials in F_i, E_2 must satisfy at least one linear relation, and it then becomes a matter of finding it. We did this using Magma [4] to find a six-element spanning set of invariant degree 8 polynomials in F_i, E_2 . Looking at their Fourier expansions at the cusp $(i\infty)^3$, we observed that their span is of dimension at least five, and the only linear relation can be the multiple of the one in the statement. We leave the computational details to the reader. \square

Order	Multiplicity	Rotation numbers
2	4	$(1/2, 1/2, 1/2)$
3	1	$(1/3, 1/3, 1/3)$
3	3	$(1/3, 1/3, 2/3)$
7	1	$(1/7, 2/7, 4/7)$
7	3	$(1/7, 2/7, 3/7)$

TABLE 2. Rotation numbers of elliptic points

Remark 6.8. Looking at P_8 , one sees that if $F(a, b, c)$ occurs, then we have $a + 2b + 4c = 0 \pmod{7}$. This can be understood as follows. The hypersurface $P_8 = 0$ is preserved by the element γ_7 from Proposition 6.6, which means that the polynomial P_8 is preserved by γ_7 up to a constant. Since P_8 contains an $(E_2^*)^4$ term, this constant must be 1. By the way γ_7 acts on the F_i from Proposition 6.6, we have that any monomial $(E_2^*)^i F_0^j F_1^a F_2^b F_4^c$ must satisfy $(\zeta_7)^{3a+6b+5c} = 1$, or equivalently $a + 2b + 4c = 0 \pmod{7}$.

Remark 6.9. It is striking that the polynomial P_8 in Proposition 6.7 and the polynomials in the upcoming Proposition 8.2 have much simpler coefficients when expressed in terms of $E_2^* = \frac{6}{7}E_2$ and the F_i rather than E_2 and the F_i , even though E_2 itself has (except for the constant term) integral coefficients that are not divisible by 7, and has a constant term that already has a 7 in the denominator. This suggests that this ring of modular forms has a natural \mathbb{Z} -structure in which E_2 is $\frac{7}{6}$ times a primitive lattice element, although we do not know an explanation for this.

7. THE SYMMETRIC HILBERT MODULAR THREEFOLD

Recall that K/\mathbb{Q} is a Galois extension with Galois group $\text{Gal}(K/\mathbb{Q})$ isomorphic to $\mathbb{Z}/3\mathbb{Z}$. The action of $\text{Gal}(K/\mathbb{Q})$ on the real places of K induces an action of $\mathbb{Z}/3\mathbb{Z}$ on \mathfrak{H}^3 by cyclically permuting the coordinates. This extends to an action on the Hilbert modular threefold X .

Definition 7.1. The *symmetric Hilbert modular threefold* X_{Gal} is the quotient of X by the induced action of $\mathbb{Z}/3\mathbb{Z}$.

The goal of this section is to explicitly describe X_{Gal} .

Proposition 7.2. *The polynomial P_8 from Proposition 6.7 generates the ideal of relations on F_i, E_2 .*

Proof. We first observe that P_8 is irreducible. Indeed, any factors would be acted upon by the group G . Since some of the degrees of the factors in E_2 are four, and G has no nontrivial permutation representation on four or fewer elements, and since it has no non-trivial one-dimensional representations, each factor would have to be G -invariant. There are no invariant polynomials of degree two in F_i , so the only possibility is $P_8 = (E_2^2 + aP_4(F_i))(E_2^2 + bP_4(F_i))$ where P_4 is the generator of the one-dimensional invariant space $\text{Sym}^4(V_4)^G$. But this is easily seen to be impossible by looking at the coefficient by E_2 .

Then it remains to show that the Eisenstein series F_i are algebraically independent, as any other relation on (F_i, E_2) together with P_8 would lead to a relation among the F_i . In order to see this, we observe the transformation properties of the F_i imply that any polynomial relation among the F_i must be homogeneous. The existence of such a relation would in turn imply a polynomial relation among $T_1 = F_1/F_0, T_2 = F_2/F_0$

and $T_3 = F_4/F_0$. Consequently, the Jacobian $\det(\partial_i T_j)$ would have to be zero, where $\partial_i = \partial/\partial q_i$. But computing with Fourier expansions shows that up to a constant, this Jacobian equals

$$\begin{aligned} & -2744q_2^5q_1^4q_0^2 + (5488q_2^{13}q_1^9 - 2744q_2^8q_1^7 - 2744q_2^9q_1^6 \\ & \quad + 5488q_2^3q_1^5 - 2744q_2^4q_1^4 + 5488q_2^5q_1^3)q_0^3 + O(q_0^4), \end{aligned}$$

and is therefore nonzero for most points in the neighborhood of the cusp. \square

We will need some lemmas that describe the fixed loci of the Galois action. Let $\sigma : X_{sm} \rightarrow X_{sm}$ be the generator of the Galois group. It is the extension of the cyclic permutation of the coordinates of \mathfrak{H}^3 .

Lemma 7.3. *The fixed loci of σ on X_{sm} are of dimension at most one.*

Proof. We first consider the fixed points in the finite part. Suppose that an image of $(z_1, z_2, z_3) \in \mathfrak{H}^3$ is fixed by σ in X . This means that there exists $A \in \Gamma(p)$ such that $(A, A^\sigma, A^{\sigma^2})(z_1, z_2, z_3) = (z_3, z_1, z_2)$. This leads to $AA^\sigma A^{\sigma^2} z_3 = z_3$. The Galois conjugate of $B = AA^\sigma A^{\sigma^2}$ is $B^\sigma = A^\sigma A^{\sigma^2} A = A^{-1}BA$, so $\text{Tr}(B)$ is rational. On the other hand, $\text{Tr}(B)$ is in \mathcal{O} and is equal to 2 mod p . This shows that $\text{Tr}(B)$ is an integer which is equal to 2 mod 7. Since $z_3 \in \mathfrak{H}$, the only possibility is $B = \text{Id}$. For each A with this property we get a curve in X_{sm} .

As far as the boundary divisors go, we only need to show that σ does not fix E_i or D_i pointwise. Indeed, σ permutes D_i , and hence it permutes their intersections with E_i . Thus σ acts nontrivially on each E_i . \square

Remark 7.4. We believe that the only fixed points of σ in the finite part of X_{sm} are the image of the diagonal in \mathfrak{H}^3 , but we will not need this fact.

Consider the quotient map by the Galois group $X_{sm} \rightarrow Y$.

Lemma 7.5. *Consider the natural linearization of the line bundle L given by the pullbacks of holomorphic functions on \mathfrak{H}^3 . Then for every σ -fixed point x of X_{sm} , the corresponding character of the Galois group in the fiber L_x is trivial.*

Proof. The fixed locus of any finite order automorphism of a smooth variety is a disjoint union of smooth subvarieties, and the character of the restriction of a line bundle is locally constant.

Over cusps, the linearized line bundle L is isomorphic to \mathcal{O} , so the statement is clear. The proof of Lemma 7.3 shows that every σ -fixed curve in the finite part contains a point at infinity in its closure in X_{sm} . Hence, the character of σ on L_x is trivial on each of these components as well. \square

Theorem 7.6. *The symmetric Hilbert modular threefold X_{Gal} is isomorphic to the degree 8 hypersurface in $\mathbb{P}(1, 1, 1, 1, 2)$ given by $P_8 = 0$.*

Proof. We will prove that the ring $M = M^{\text{Gal}}$ of modular forms of parallel weight invariant under the Galois action is isomorphic to the subring M' generated by F_i and E_2 , which clearly suffices in view of Proposition 7.2.

First of all M' is a subring of M . Indeed, F_i and E_2 are invariant under the permutation of variables $(z_1, z_2, z_3) \rightarrow (z_2, z_3, z_1)$, since the Galois group acts trivially on \mathcal{O}/p . By Proposition 7.2 the Poincaré series of M' is $f(t) = (1+t^2+t^4+t^6)(1-t)^{-4}$. so it suffices to show that the Poincaré series of M is the same. For degree $k \leq 3$ all the modular forms of weight (k, k, k) (Galois invariant or not) lie in M' , since the dimension of the degree k part of M' is equal to $h^0(kL)$. For $k \geq 4$ the dimension of $M'_{\text{deg}=k}$ is

$$\frac{2}{3}k^3 - 2k^3 + \frac{34}{3}k - 10$$

so all one needs to do is to calculate the dimension of the Galois-invariant part of the space of (k, k, k) forms.

The Eisenstein series of weight (k, k, k) are Galois-invariant, because the Galois group fixes the cusps. Thus the dimension of the degree k part of M is 8 plus the dimension of the Galois-invariant part of $H^0(X_{sm}, kL - D - E)$. The line bundle $\mathcal{O}_{X_{sm}}(kL - D - E)$ for $k \geq 3$ satisfies Kawamata vanishing, so it has no higher cohomology and we can use the holomorphic Lefschetz formula for the trace of the action of the Galois group on the cohomology of $H^i(L)$. Since the Euler characteristics of $\mathcal{O}_{X_{sm}}(kL - D - E)$ is $2(k-1)^3$, the Euler characteristics of the Galois-invariant part is $\frac{2}{3}(k-1)^3 + h(k) = \frac{2}{3}(k-1)^3 + 2\text{Re}(\text{Trace}(\sigma))$ where σ is the operator on the cohomology of the linearized line bundle $\mathcal{O}_{X_{sm}}(kL - D - E)$ that corresponds to the generator of the Galois group.

We will now use the Lemmas 7.3 and 7.5 to show that $h(k)$ is a linear function of k . Indeed, the holomorphic Lefschetz formula gives a sum over the components Z_i of the fixed locus of σ of integrals of some fixed class times the equivariant Chern character of kL restricted to Z_i . This will give a mod 3 quasi-polynomial by Lemma 7.3. Moreover, it will in fact be polynomial, since there will be no factors of the form $\exp((2\pi i/3)k)$ by Lemma 7.5.

Thus it suffices to find $h(k)$ for two values of k in order to calculate the graded dimension of M . For $k = 2$ we are dealing with the canonical class $2L - D - E$ in its natural linearization. The only cohomology occurs at H^0 and H^3 . The cohomology for H^0 is Galois-invariant, since it can be written in terms of the elements of M' . The cohomology at H^3 is Serre dual to $H^0(X_{sm}, \mathcal{O})$, and the natural linearizations of K and \mathcal{O} are compatible with Serre duality. Thus we have $h(2) = (3-1) - \frac{2}{3}(2-1)^3 = \frac{4}{3}$. For $k = 3$ we have $\mathcal{O}(3L - D - E)$. Since all weight $(3, 3, 3)$ forms are Galois invariant, so are all cusp forms of this weight. There are no higher cohomology, and we have $\chi(3L - D - E) = 2(3-1)^3 = 16$. Hence we have

$$h(3) = 16 - \frac{2}{3}(3-1)^3 = \frac{32}{3}.$$

This leads to $h(k) = \frac{1}{3}(28k - 52)$, which then shows that for $k \geq 4$ $\dim M_k = \dim M'_k$. \square

8. THE CANONICAL MODEL

The goal of this section is to prove that the partial resolution X_{ch} is the canonical model of the field of modular functions for $\Gamma(p)$.

It is well-known that global holomorphic top forms on X_{sm} can be identified with cusp forms of parallel weight $(2, 2, 2)$. In fact, we can calculate the basis of these forms in our situation.

Proposition 8.1. *The dimension of the space $H^0(X_{sm}, K_{sm})$ is 3. Its basis is given by $2F_0F_1 - F_4^2, 2F_0F_2 - F_1^2, 2F_0F_4 - F_2^2$.*

Proof. The dimension statement follows from the trace formula, see [8]. To find the basis, observe that the space of modular forms of weight $(2, 2, 2)$ is a G -module of dimension 11. It is easy to show that E_2 and the degree two monomials in F_0, F_1, F_2, F_4 are linearly independent and hence give a basis of the weight $(2, 2, 2)$ forms. As a G -module, the space of weight $(2, 2, 2)$ forms decomposes into the direct sum of irreducibles of degrees 1, 3, 7. This implies that the cusp forms of parallel weight 2 correspond to the three-dimensional summand of the space of quadrics in F_0, \dots, F_4 . It is easy to check that the three quadrics in the statement generate this subspace. Alternatively, one can look at the eight translates of the cusp $(i\infty)^3$ given by $(F_0 : F_1 : F_2 : F_4) = (1 : 0 : 0 : 0)$ by the action of $\mathrm{SL}(2, \mathbb{F}_7)$ and look at the quadrics that vanish at these points. \square

Proposition 8.2. *Identify the space of sections of $2K$ on X_{ch} (or on any other model of its function field with canonical singularities) with the subspace of $H^0(X_{ch}, 2L)$ that vanishes twice at the exceptional divisor D . Under this identification, the following eight elements of $H^0(X_{ch}, 2L)$ give sections of $2K$:*

$$\begin{aligned} & F_0^4 - (E_2^*)^2 + 3F_0F_1F_2F_4 + \frac{1}{2}(F_2F_4^3 + F_4F_1^3 + F_1F_2^3), \\ & 4F_0^4 + 4E_2^*F_0^2 - 10F_4F_2F_1F_0 + F_4F_1^3 + F_2^3F_1 + F_4^3F_2, \\ & 4F_1F_0(F_0^2 + E_2^*) - 10F_4^2F_0^2 - 4F_4F_2^2F_0 - 4F_4F_2F_1^2 + F_2^4 + 2E_2^*F_4^2, \\ & 4F_2F_0(F_0^2 + E_2^*) - 10F_1^2F_0^2 - 4F_1F_4^2F_0 - 4F_1F_4F_2^2 + F_4^4 + 2E_2^*F_1^2, \\ & 4F_4F_0(F_0^2 + E_2^*) - 10F_2^2F_0^2 - 4F_2F_1^2F_0 - 4F_2F_1F_4^2 + F_1^4 + 2E_2^*F_2^2, \\ & 2F_2F_1F_0^2 + 2F_1^3F_0 - 2F_4^2F_2F_0 + F_4^2F_1^2 - 2E_2^*F_2F_1 + 2F_4F_2^3, \\ & 2F_1F_4F_0^2 + 2F_4^3F_0 - 2F_2^2F_1F_0 + F_2^2F_4^2 - 2E_2^*F_1F_4 + 2F_2F_1^3, \\ & 2F_4F_2F_0^2 + 2F_2^3F_0 - 2F_1^2F_4F_0 + F_1^2F_2^2 - 2E_2^*F_4F_2 + 2F_1F_4^3. \end{aligned}$$

Proof. It is straightforward to see that these forms vanish twice at the part of D that sits over the image of $(i\infty)^3$ in view of the explicit expansions in Remark 5.7. It suffices to check that the span of the above series is invariant under the action of G given by Proposition 6.6. We have done the latter calculation by GP-Pari. The

first element is invariant under G , while the latter seven span a seven-dimensional irreducible representation of G . \square

Lemma 8.3. *The \mathbb{Q} -Cartier divisor K_{ch} is ample on the divisor D_1 on X_{ch} .*

Proof. The fan of the toric surface D_1 on X_{ch} is given on the right of Figure 5, with $K_{ch}|_{D_1}$ given by the piecewise-linear function whose values on the generators of one-dimensional cones of the fan are given in the Figure. Since the cone of effective curves on a toric surface is a finite polyhedral cone spanned by the infinity divisors, it suffices to check that the intersection number $K_{ch}M$ is positive for each of the six infinity divisors M . A standard toric geometry calculation shows that for each infinity divisor M we have $K_{ch}M = 1/2$. If one would rather avoid the singularities and the fractional intersection numbers, the calculation can be done on $D_1 \subset X_{sm}$, in view of the projection formula. \square

Theorem 8.4. *The partial resolution X_{ch} is the canonical model of the field of modular functions of $\Gamma(p)$.*

Proof. Recall that every field of functions of general type has a unique canonical model which is characterized in its birational class by the properties of having canonical singularities and an ample canonical line bundle.

Clearly, X_{ch} has canonical singularities, so it suffices to show that $K_{X_{ch}}$ is ample. Note that $K_{X_{ch}}$ is only a Weil divisor, but $2K_{X_{ch}}$ is Cartier. We will use the Nakai-Moishezon criterion, see [20, Theorem 1.37]. By Proposition 4.4 we have $K_{X_{ch}}^3 > 0$, since the divisor $K - \frac{1}{2}E$ is the pullback of $K_{X_{ch}}$. Let C be an irreducible curve on X_{ch} . If C lies in an infinity divisor, then $K_{X_{ch}}C > 0$ by Lemma 8.3. So we can consider the case where the generic point of C lies in the finite part of X_{ch} .

Denote by $R_j, j = 1, \dots, 11$ the polynomials in F_i and E_2 from Propositions 8.1 and 8.2. If $K_{X_{ch}}C \leq 0$ then C maps to a point in the weighted projective space $\mathbb{P}(1, 1, 1, 2, 2, 2, 2, 2, 2, 2, 2)$ given by the R_j . Consider the 5×11 matrix of the partial derivatives of R_j with respect to F_i and E_2 . For every point in the image of the finite part of C in X_{Gal} the rank of this matrix is less than five. We have calculated in Magma [4] the Hilbert series of the scheme defined by the size five minors of this matrix, to show that this scheme is zero-dimensional. This shows that $K_{ch}C > 0$.

It remains to show that $K_{X_{ch}}^2 S > 0$ for any surface $S \subset X_{ch}$. It is easy to see that the quadrics from Proposition 8.1 have no common components, so $K_{X_{ch}}S$ is an effective curve except perhaps if S is a divisor at infinity. Then Lemma 8.3 finishes the argument. \square

9. AN OCTIC WITH 84 SINGULARITIES OF TYPE A_2

In this section we describe explicitly an octic $W \subseteq \mathbb{P}^3$ with 84 isolated singular points of type A_2 .

Consider the intersection of the Hilbert modular threefold X_{Gal} with the hypersurface $E_2 = 0$. The resulting surface W is given by the degree 8 equation $Q(F_0, F_1, F_2, F_4) = 0$, where

$$\begin{aligned} Q = & -3F_0^8 + 38F_0^5F(1, 1, 1) + 13F_0^4F(0, 1, 3) \\ & - 46F_0^3(F(0, 3, 2) + F_0^2(5F(0, 5, 1) + 23F(2, 2, 2))) \\ & - 2F_0(F(0, 0, 7) + 4F(1, 2, 4)) + 2F(0, 2, 6) - 5F(1, 4, 3). \end{aligned}$$

in the homogeneous coordinates F_0, F_1, F_2, F_4 (the notation $F(a, b, c)$ is defined before Proposition 6.7).

Proposition 9.1. *The octic W in \mathbb{P}^3 has 84 singular points of type A_2 .*

Proof. Note that X_{Gal} has singularities of type A_2 along the image of the diagonal in \mathfrak{H}^3 . The Hilbert modular form E_2 restricts to the $SL_2(\mathbb{Z})$ -Eisenstein series g_6 on the diagonal. In fact, since g_6 has simple zeroes, at each such zero the surface $E_2 = 0$ in \mathfrak{H}^3 is nonsingular. The gradient of E_2 is Galois-invariant, so the tangent space of $E_2 = 0$ has eigenvalues $(-\frac{1}{2} \pm \frac{\sqrt{3}}{2})$. This shows that $E_2 = 0$ is transversal to the diagonal. Thus, W is singular at the images of the zeroes of g_6 , with singularities of type A_2 , and perhaps at some other points or curves. There are 84 such zeroes of g_6 modulo the group action, which form the orbit of (i, i, i) .

A Magma calculation shows that the Hilbert polynomial of the radical of the Jacobian ideal of the equation of W is 84. Consequently, W has exactly 84 isolated singular points, which are the zeroes of g_6 as above. \square

Remark 9.2. We believe that the 84 singular points of type A_2 is the best lower bound for octics to date. The upper bound of 98 can be established by the work of Miyaoka [22]. See also [21] for detailed discussion about related problems. It would be interesting to see if our octic is in any way related to the Endraß octics [7] that have 168 singular points of type A_1 . We note that $PSL_2(\mathbb{F}_7)$ -invariant octic surfaces in \mathbb{P}^3 form a net that was described by Edge; see [5], pp. 202–3 and the table on p. 205. In his notation, our octic is given by $-3/4\Pi^8 + 10S^8 + 18E^8$.

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