# TORIC MODULAR FORMS OF HIGHER WEIGHT

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ABSTRACT. In the papers [1, 2] we used the geometry of complete polyhedral fans to construct a subring $\mathscr{T}(l)$ of the modular forms on $\Gamma_1(l)$, and showed that for weight two the cuspidal part of $\mathscr{T}(l)$ coincides with the space of cusp forms of analytic rank zero. In this paper we show that in weights greater than two, the cuspidal part of $\mathscr{T}(l)$ coincides with the space of all cusp forms.

## 1. Introduction

1.1. In [1, 2] we used the geometry of complete polyhedral fans to construct a subring $\mathscr{T}_*(l)$ of the modular forms on $\Gamma_1(l)$. If $l \geq 5$, we showed that $\mathscr{T}_*(l)$ is generated in weight one by certain Eisenstein series, and in [1, Theorem 4.11] we showed that for weight two the cuspidal part of $\mathscr{T}_*(l)$ coincides with the space of cusp forms of analytic rank zero. The main result of this paper, Theorem 5.10, is that in weights greater than two, the cuspidal part of $\mathscr{T}_*(l)$ coincides with the space of all cusp forms. In fact, we prove a stronger statement: we define certain weight $k$ toric modular forms $\widetilde{s}^{(k)}_{a/l}$, and show that any cusp form can be written as a $\mathbb{C}$-linear combination of the forms $\widetilde{s}^{(k)}_{a/l}$ and pairwise products of the form $\widetilde{s}^{(m)}_{a/l}\widetilde{s}^{(n)}_{b/l}$, where $m+n=k$ and $m,n>0$.

The proof of Theorem 5.10 is formally very similar to the proof of [1, Theorem 4.11]. Let $\mathscr{S}(l)$ be the space of weight $k$ holomorphic cusp forms on $\Gamma_1(l)$. We define a map $\rho\colon \mathscr{S}(l) \to \mathscr{S}(l)$, and show that its image contains all newforms for $k \geq 3$. We describe the map $\rho$ in terms of Manin symbols, which allows us to write $\rho(f)$ in terms of products of certain explicit toric Eisenstein series. A key role is played by certain weight $k$ Manin symbols $\{R_{(m,n)} | m,n \in \mathbb{Z}\}$ that satisfy relations similar to weight two Manin symbols.

1.2. Here is an outline of the paper. In Section 2 we review results about toric modular forms, and in Section 3 review results about Manin symbols and introduce the symbols $R_{(m,n)}$. In Section 4 we describe $(\bmod\, l)$-*polynomials*, a technical tool we use later to manipulate $q$-expansions. We prove the main result along with some corollaries in Section 5.

The remaining sections contain complements to the main result and results proved in [1]. In Section 6 we use products of Eisenstein series of higher weight to define a

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map $\mu$ from weight $k$ Manin symbols to a certain quotient of the space of weight $k$ modular forms. This map is analogous to the map $\mu$ in [1, Definition 3.11], but some complications do occur in the higher case. Finally, in Section 7 we show that the map from symbols to forms is compatible with the action of the Hecke operators.

Throughout the paper we keep our arguments as elementary as possible. In particular, we avoid using results of [2] that are based on the Hirzebruch-Riemann-Roch theorem for toric varieties. While this complicates the proofs a bit, it makes the paper accessible to readers with no knowledge of toric varieties. We outline an alternative approach to the results using toric geometry in Remarks 6.4 and 7.14.

## 2. Toric modular forms

2.1. We briefly review the definition of toric forms; more details can be found in [1, 2]. Let $k$ and $l$ be positive integers, and suppose $l \geq 5$. As usual let $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$, where $\tau$ is in the upper halfplane $\mathfrak{H}$. A holomorphic modular form of weight $k$ on the group $\Gamma_1(l)$ is called *toric* if it can be expressed as a homogeneous polynomial of degree $k$ in the functions $\widetilde{s}_{a/l}(q)$ given by

$$\widetilde{s}_{a/l}(q) := (\frac{1}{2} - \frac{a}{l}) + \sum_{n>0} q^n \sum_{d|n} (\delta_d^{a \bmod l} - \delta_d^{-a \bmod l}), \quad a = 1, \dots, l-1.$$

Here $\delta_d^{a \bmod l}$ is 1 if $a = d \bmod l$ and is 0 otherwise. The space of toric forms $\mathscr{T}_*(l)$ of all weights is thus generated as a graded ring by certain weight one Eisenstein series. By results of [1, 2], $\mathscr{T}_*(l)$ is known to be stable under the action of the Hecke operators, and under Atkin-Lehner lifting.

**Proposition 2.2.** *The ring $\mathscr{T}_*(l)$ contains the modular forms*

$$\widetilde{s}_{a/l}^{(k)} := C + \sum_{n>0} q^n \sum_{d|n} d^{k-1} (\delta_d^{a \bmod l} + (-1)^k \delta_d^{-a \bmod l}), \tag{1}$$

*where $k \geq 2$ and $a = 0, \dots, l$, except for $(a, k) = (0, 2)$. Here $C$ is a constant determined uniquely by the modularity of $\widetilde{s}_{a/l}^{(k)}$.*

*Proof.* Let $\vartheta(z, \tau)$ be Jacobi's theta function [4]. Then it is easy to construct a given $\widetilde{s}_{a/l}^{(k)}$ as a linear combination of the modular forms

$$\begin{aligned} s_{a/l}^{(k)}(q) &:= (2\pi\mathrm{i})^{-k} \left( \frac{\partial^k}{\partial z^k} \right)_{z=0} \log \left( \frac{z\vartheta(z + a/l, \tau)\vartheta'(0, \tau)}{\vartheta(z, \tau)\vartheta(a/l, \tau)} \right) \\ &= C + \sum_{n>0} q^n \sum_{d|n} d^{k-1} (\mathrm{e}^{2\pi\mathrm{i}da/l} + (-1)^k \mathrm{e}^{-2\pi\mathrm{i}da/l} - 2\delta_k^{0 \bmod 2}) \end{aligned}$$

of [2, Section 4.4], and the standard level one Eisenstein series

$$E_k := C + \sum_{n>0} q^n \sum_{d|n} d^{k-1} \tag{2}$$

for even $k \geq 4$. The forms $s_{a/l}^{(k)}$ and $E_k$ for $k > 2$ are toric, see [2, Theorem 4.11 and Remark 4.13]. We use here and in what follows the convention denoting constant terms whose exact value is irrelevant by $C$. $\square$

2.3. We must exclude $(a, k) = (0, 2)$ from the statement since $E_2$ is not modular. However, it will be convenient to allow $\widetilde{s}_{0/l}^{(2)}$ in later arguments, which merely amounts to working in the larger ring $\mathscr{T}_*(l)[E_2]$. In fact, since we will never multiply more than two of the $\widetilde{s}$'s together, we will be working in $\mathscr{T}_*(l) + E_2\mathscr{T}_*(l)$. We call elements of this ring *toric quasimodular forms* (cf. [7]).

*Remark* 2.4. The statement of Proposition 2.2 is true for all $l = 2, 3, 4$, but the definition of toric modular forms there is a bit more complicated. However, it turns out that for these levels all modular forms are toric, as defined in [2]. From now on we will call any polynomial in $\widetilde{s}_{a/l}^{(k)}$ a toric quasimodular form of level $l$. Then the rest of the paper works for an arbitrary level $l \geq 1$.

**Definition 2.5.** By a slight abuse of notations we say that a weight $k$ quasimodular form $f$ can be written as a *linear combination of pairs* if $f$ can be written as a $\mathbb{C}$-linear combination of the forms $\widetilde{s}_{a/l}^{(k)}$ and $\widetilde{s}_{a/l}^{(m)}\widetilde{s}_{b/l}^{(n)}$ where $m, n > 0$, $m + n = k$, and $a, b = 0, \ldots, l-1$.

**Proposition 2.6.** *The space of toric quasimodular forms contains the derivatives $\partial_\tau \widetilde{s}_{a/l}^{(k)}$. Moreover, each $\partial_\tau \widetilde{s}_{a/l}^{(k)}$ can be written as a linear combination of pairs.*

*Proof.* The span of $\widetilde{s}^{(k)}$ is the same as the span of $s^{(k)}$ and $E_k$, so we will instead consider their derivatives. The $q$-expansion of $\partial_\tau s_{a/l}^{(k)}$ is

$$2\pi\mathrm{i} \sum_{n>0} q^n \sum_{d|n} nd^{k-1}(\mathrm{e}^{2\pi\mathrm{i}da/l} + (-1)^k \mathrm{e}^{-2\pi\mathrm{i}da/l} - 2\delta_k^{0 \bmod 2}). \tag{3}$$

Now take (cf. [1, proof of Prop. 3.8])

$$s_\alpha^{(2)} + s_\alpha^2 = \frac{1}{6} - 2\sum_{n>0} q^n \sum_{d|n} \frac{n}{d}(\mathrm{e}^{2\pi\mathrm{i}\alpha d} + \mathrm{e}^{-2\pi\mathrm{i}\alpha d}), \tag{4}$$

and differentiate it $k$ times with respect to $\alpha$. Let $F_\alpha(q)$ be the resulting right hand side. It is easy to express (3) as a linear combination of $\{F_{a/l}(q)\}$ and the derivatives $\partial_\tau E_k$, so it suffices to show that these forms can be expressed as a linear combination of pairs.

We consider first the derivatives of $s_\alpha^{(r)}$ with respect to $\alpha$. Putting $E_{\mathrm{odd}} = 0$, we have

$$(2\pi\mathrm{i})^{-1}\frac{\partial}{\partial\alpha}s_\alpha^{(r)} = s_\alpha^{(r+1)} - (2\pi\mathrm{i})^{-r-1}\left(\frac{\partial^{r+1}}{\partial z^{r+1}}\right)_{z=0} \log\left(\frac{z\vartheta'(0,\tau)}{\vartheta(z,\tau)}\right) = s_\alpha^{(r+1)} - 2E_{r+1},$$

and the statement follows from the fact that $E_r$ and $s^{(r)}$ can be written as linear combinations of $\widetilde{s}^{(r)}$. For $\partial_\tau E_k$ we argue as follows. Expand both sides of the equation (4) in a Laurent series in $\alpha$ around $\alpha = 0$. The coefficient at $\alpha^k$ on the right hand side of (4) is equal to $\partial_\tau E_k$, up to a multiplicative and an additive constant. To expand the left hand side notice that, up to the terms constant in $q$, the Laurent coefficient of $s_\alpha$ at $\alpha^k$ is a multiple of $E_{k+1}$, which follows from expanding $\mathrm{e}^{2\pi i d\alpha}$ in the definition of $s_\alpha$. It is easy to see that $s_\alpha$ has a simple pole at $\alpha = 0$ with a constant residue, so the coefficient of the Laurent expansion of $s_\alpha^2$ at $\alpha^k$ is a linear combination of $E_r E_{k+2-r}$, $E_{k+2}$ and some $E_r$ for $r < k+2$. Then the modular transformation properties of $\partial_\tau E_k$ finish the argument. □

Next we describe the action of $\Gamma_0(l)/\Gamma_1(l)$ on $\widetilde{s}$.

**Proposition 2.7.** *Let $\gamma \in \Gamma_0(l)$ have diagonal entries $p^{-1}$ and $p$ $\bmod l$ respectively. Then*

$$\gamma \widetilde{s}^{(k)}_{a/l} = \widetilde{s}^{(k)}_{p^{-1}a/l}.$$

*Proof.* The transformation properties of $\vartheta$ (cf. [2, Prop. 4.3]) imply

$$\gamma s^{(k)}_{a/l} = s^{(k)}_{pa/l}.$$

One can then use linear combinations of the forms in the proof of Proposition 2.2 to determine the action of $\gamma$ on $\widetilde{s}$. We leave the details to the reader. □

## 3. Manin symbols

3.1. This section closely follows [6], to which the reader is referred for more details. Let $l > 1$ be an integer, and let $E_l \subset (\mathbb{Z}/l\mathbb{Z})^2$ be the subset of pairs $(u, v)$ such that $\mathbb{Z}u + \mathbb{Z}v = \mathbb{Z}/l\mathbb{Z}$. The space of *Manin symbols* of weight $k$ and level $l$ is the $\mathbb{C}$-vector space generated by the symbols $x^r y^s(u, v)$, where $r$ and $s$ are nonnegative integers summing to $k - 2$ and $(u, v) \in E_l$, modulo the following relations:

1. $x^r y^s(u, v) + (-1)^r x^s y^r(v, -u) = 0$.
2. $x^r y^s(u, v) + (-1)^r y^r(x - y)^s(v, -u - v) + (-1)^s(y - x)^r x^s(-u - v, u) = 0$.

We denote the space of Manin symbols by $M$ (we omit the level and weight from the notation since it will be clear from the context). Two subspaces of $M$ will play an important role in what follows. Let $\iota\colon M \to M$ be the involution $x^r y^s(u, v) \mapsto (-1)^r x^r y^s(-u, v)$.

**Definition 3.2.** The space of *plus symbols* $M_+ \subset M$ is the subspace consisting of symbols $w$ satisfying $\iota(w) = w$. Similarly, the space of *minus symbols* $M_- \subset M$ is the subspace consisting of symbols $w$ satisfying $\iota(w) = -w$.

We have *symmetrization maps* $(\ ,\ )_\pm\colon M \to M_\pm$ given by $x^r y^s(u, v)_\pm := (x^r y^s(u, v) \pm (-1)^r x^r y^s(-u, v))/2$.

3.3. Let $M^* = \mathrm{Hom}_{\mathbb{C}}(M, \mathbb{C})$ be the dual of the space of Manin symbols. For any $\varphi \in M^*$, we define $\varphi$ on "degenerate" symbols $x^r y^s(u,v)$ with $\mathbb{Z}u + \mathbb{Z}v \neq \mathbb{Z}/l\mathbb{Z}$ by setting $\varphi(x^r y^s(u,v)) = 0$. This convention is somewhat artificial but turns out to be quite useful.

3.4. There exists a natural pairing between the spaces of Manin symbols and the spaces of cusp forms, see [6]. Let $\mathscr{M}(l)$ be the $\mathbb{C}$-vector space of weight $k$ holomorphic modular forms on $\Gamma_1(l)$, and let $\mathscr{S}(l) \subset \mathscr{M}(l)$ be the subspace of cusp forms. For $x^r y^s(u,v) \in M$ and $f \in \mathscr{S}(l)$ let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ be an element of $\Gamma(1)$ with $(c,d) = (u,v) \bmod l$. Then the integral

$$\int_0^{i\infty} (c\tau + d)^{-k} f\left(\frac{a\tau + b}{c\tau + d}\right) \tau^r \, d\tau$$

does not depend on the choice of $g$. Moreover, it is compatible with the relations on modular symbols, and we obtain a pairing

$$\begin{aligned} M \times \mathscr{S}(l) &\longrightarrow \mathbb{C}, \\ (x^r y^s(u,v), f) &\longmapsto \langle f, x^r y^s(u,v) \rangle. \end{aligned}$$

In general this pairing is degenerate, but one can identify a subspace of cuspidal Manin symbols $S$ such that the pairing is non-degenerate on $S_\pm \times \mathscr{S}(l)$. We will not use this fact, but details can be found in [6].

3.5. Next we present Merel's description of the Hecke action on the Manin symbols. Let $n \geq 1$ be an integer, and let $T_n$ be the associated Hecke operator. We denote the action of $T_n$ on a modular form $f$ by $f \big| T_n$. For any positive integer $n$, we define a set $H(n) \subset \mathbb{Z}^4$ by

$$H(n) = \{(a,b,c,d) \mid a > b \geq 0, \quad d > c \geq 0, \quad ad - bc = n\}. \tag{5}$$

**Theorem 3.6.** [6, Theorem 2 and Proposition 10] *For any positive integer $n$ coprime to $l$, define an operator $T'_n \colon M \to M$ by*

$$T'_n \, x^r y^s(u,v) = \sum_{H(n)} (ax + by)^r (cx + dy)^s (au + cv, bu + dv). \tag{6}$$

*If $n$ is not coprime to $l$, then we define $T'_n$ by (6) but omit terms with g.c.d.$(l, au + cv, bu + dv) > 1$. Then $T'_n$ is the adjoint of $T_n$ with respect to the pairing $\langle\ ,\ \rangle$, that is*

$$\langle f \big| T_n, x^r y^s(u,v) \rangle = \langle f, T'_n \, x^r y^s(u,v) \rangle.$$

We will abuse notation in what follows and write $T_n$ for $T'_n$. It is also proved in [6] that this Hecke action is compatible with the symmetrization maps:

**Proposition 3.7.** *We have*

$$T_n(x^r y^s(u,v)_\pm) = (T_n \, x^r y^s(u,v))_\pm.$$

3.8. To conclude this section we associate to every pair of integers $(m,n)$ a certain Manin symbol $R_{(m,n)}$. These symbols satisfy relations analogous to those satisfied by the weight two symbols.

**Definition 3.9.** Let $m,n\in\mathbb{Z}$. If $g.c.d.(m,n,l)=1$ then we let $R_{(m,n)}=(mx+ny)^{k-2}(m,n)$. If $g.c.d.(m,n,l)>1$ then we put $R_{(m,n)}=0$.

We remark that even though $R_{(m,n)}$ is built out of the Manin symbol $(m,n)$, its value depends on more than just the residues of $m$ and $n$ modulo $l$. It is straightforward to see that the symbols $R_{(m,n)}$ obey the following relations:

1. $R_{(m,n)}+R_{(-n,m)}=0$.
2. $R_{(m,n)}+R_{(-m-n,m)}+R_{(n,-m-n)}=0$.
3. $\big(R_{(m,n)}\big)_{\pm}=(R_{(m,n)}\pm R_{(-m,n)})/2$.

We denote the images of $R_{(m,n)}$ under the symmetrization maps by $R^{\pm}_{(m,n)}$.

## 4. (Mod $l$)-polynomials

4.1. To simplify later manipulations with $q$-expansions, we now introduce certain functions. Fix a positive integer $l$.

**Definition 4.2.** A function $h\colon\mathbb{Z}\to\mathbb{C}$ is called a $(\operatorname{mod} l)$*-polynomial* if its restriction to each coset $l\mathbb{Z}+k$ is a polynomial.

One can think of a $(\operatorname{mod} l)$-polynomial as a set of $l$ ordinary polynomials, one for each residue modulo $l$. For example, the function that equals $m^2+m$ when $m$ is even and $m^3$ when $m$ is odd is a $(\operatorname{mod} 2)$-polynomial.

The set of all $(\operatorname{mod} l)$-polynomials forms a ring. One can analogously define (mod $l$)-polynomials $h(m,n)$ of two variables by requiring polynomiality on each pair of cosets $(l\mathbb{Z}+k_1,l\mathbb{Z}+k_2)$.

4.3. We say that a $(\operatorname{mod} l)$-polynomial $h$ is *odd* if $h(-m)=-h(m)$. Note that the individual polynomials constituting an odd $(\operatorname{mod} l)$-polynomial aren't independent, since the polynomials sitting over the residues $a \bmod l$ and $-a \bmod l$ are related. The space of odd $(\operatorname{mod} l)$-polynomials will be of particular importance to us, due to the following proposition.

**Proposition 4.4.** *Let $h$ be an odd $(\operatorname{mod} l)$-polynomial. Then up to a constant, the function*

$$f(q)=\sum_{D>0}q^D\sum_{d|D}h(d)$$

*is a linear combination of $\{\widetilde{s}^{(k)}_{a/l}\mid k\geq 1,\ a=1,\dots l-1\}$. Conversely, every linear combination of $\widetilde{s}^{(k)}_{a/l}$ has the above form, up to an additive constant.*

*Proof.* Let $r_{a,k}(m)$ be the $(\mathrm{mod}\, l)$-polynomial given by

$$r_{a,k}(m) = m^k \delta_m^{a \bmod l} - (-1)^k m^k \delta_m^{-a \bmod l}.$$

Then any odd $(\mathrm{mod}\, l)$-polynomial is a linear combination of the $r_{a,k}$'s. The result follows easily from the definition of $\widetilde{s}_{a/l}^{(k)}$ in (1). □

The following result allows one to construct odd one-variable $(\mathrm{mod}\, l)$-polynomials from even two-variable $(\mathrm{mod}\, l)$-polynomials.

**Proposition 4.5.** *Let $G\colon \mathbb{Z}^2 \to \mathbb{C}$ be a two-variable $(\mathrm{mod}\, l)$-polynomial such that $G(-n_1, -n_2) = G(n_1, n_2)$, and let $N$ be a positive integer. Then*

$$f(d) := \sum_{0<n<Nd} G(n,d) + \frac{1}{2}G(0,d) + \frac{1}{2}G(Nd,d)$$

*is an odd $(\mathrm{mod}\, l)$-polynomial.*

*Proof.* First we note that the space of all even two-variable $(\mathrm{mod}\, l)$-polynomials is spanned by the family of functions

$$G(n_1,n_2) = n_1^r n_2^s \mathrm{e}^{2\pi\mathrm{i}(k_1n_1+k_2n_2)/l} + (-n_1)^r(-n_2)^s \mathrm{e}^{-2\pi\mathrm{i}(k_1n_1+k_2n_2)/l}$$

for nonnegative integers $r$ and $s$ and integers $k_1$ and $k_2$. We use $\alpha_i = 2\pi\mathrm{i}k_i/l$ to write such a $G$ as

$$G(n_1,n_2) = n_1^r n_2^s (\mathrm{e}^{\alpha_1n_1+\alpha_2n_2} + (-1)^{r+s}\mathrm{e}^{-\alpha_1n_1-\alpha_2n_2}).$$

Now it suffices to treat the case $r = s = 0$, since all others can then be handled by partial differentiation with respect to $\alpha_1, \alpha_2$. For $r = s = 0$ and $\mathrm{e}^{\alpha_1} \neq 1$, an explicit calculation gives

$$\begin{aligned} f(d) &= \sum_{0<n<Nd} (\mathrm{e}^{\alpha_1n+\alpha_2d} + \mathrm{e}^{-\alpha_1n-\alpha_2d}) + \frac{1}{2}(\mathrm{e}^{\alpha_2d} + \mathrm{e}^{-\alpha_2d}) + \frac{1}{2}(\mathrm{e}^{(\alpha_1N+\alpha_2)d} + \mathrm{e}^{-(\alpha_1N+\alpha_2)d}) \\ &= \frac{(1+\mathrm{e}^{\alpha_1})}{2(1-\mathrm{e}^{\alpha_1})}\Big(\mathrm{e}^{\alpha_2d} - \mathrm{e}^{(\alpha_1N+\alpha_2)d} - \mathrm{e}^{-\alpha_2d} + \mathrm{e}^{-(\alpha_1N+\alpha_2)d}\Big). \end{aligned}$$

This is clearly an odd function in $d$, and after letting $\alpha_i = 2\pi\mathrm{i}k_i/l$ is obviously $(\mathrm{mod}\, l)$-polynomial in $d$. The case $\mathrm{e}^{\alpha_1} = 1$ follows by analytic continuation. □

4.6. The following technical statement will be needed for the proof of Lemma 5.8.

**Proposition 4.7.** *Fix a weight $k$ cusp form $f$ on $\Gamma_1(l)$, and define a function $h\colon \mathbb{Z}_{>0} \to \mathbb{C}$ by*

$$h(m) := \langle f, R^+_{(m,0)}\rangle + 2\langle f, \sum_{m>i>0} R^+_{(m,m-i)}\rangle.$$

*Then $h$ extends to an odd $(\mathrm{mod}\, l)$-polynomial.*

*Proof.* We use the symmetries of $R^+$ to rewrite $h(m)$ as

$$h(m) = \langle f, \sum_{-m<i<m} R^+_{(m,i)} \rangle = \sum_{0<i<2m} \langle f, R^+_{(m,i-m)} \rangle + \frac{1}{2}\langle f, R^+_{(m,-m)} \rangle + \frac{1}{2}\langle f, R^+_{(m,m)} \rangle.$$

Then Proposition 4.5 finishes the proof. □

## 5. Main theorem

5.1. Fix a weight $k \geq 3$ and a level $l$. In this section we define an endomorphism of the space $\mathscr{S}(l)$ of cusp forms of weight $k$ with respect to $\Gamma_1(l)$, and prove that its image contains all newforms. This definition is a generalization of [1, Definition 4.2] to $k > 2$.

**Definition 5.2.** Let $\rho\colon \mathscr{S}(l) \to \mathscr{S}(l)$ be the linear map

$$\rho(f) = \sum_{n=1}^{\infty} \left( \int_0^{\mathrm{i}\infty} (f \,|\, T_n)(s) ds \right) q^n.$$

**Proposition 5.3.** *The form $\rho(f)$ is a cusp form with nebentypus equal to that of $f$.*

*Proof.* The statement follows from [6, Theorem 6]; see also [1, Proposition 4.3]. □

The map $\rho$ was used in [1] because its image contains all newforms of weight two whose $L$-functions don't vanish at the center of the critical strip. The analogous statement for higher weights is the following:

**Proposition 5.4.** *The image of $\rho$ contains all newforms.*

*Proof.* One needs to show that for any newform $f$

$$\int_0^{\mathrm{i}\infty} f(\tau)\, d\tau \neq 0,$$

which is equivalent to $L(f, 1) \neq 0$. Without loss of generality we may assume that $f$ is a Hecke eigenform. If $k > 3$ then $L(f, 1)$ is a special value outside the critical strip, and so cannot vanish by absolute convergence of the Euler product. If $k = 3$ then $L(f, 1)$ is a special value on the boundary of the critical strip. By [5, Theorem 1.3] this special value cannot vanish. □

5.5. Fix a cusp form $f$. By Theorem 3.6 we can express $\rho(f)$ in terms of modular symbols as

$$\rho(f) = \sum_{n=1}^{\infty} q^n \langle f, T_n y^{k-2}(0,1) \rangle = \sum_{n=1}^{+\infty} q^n \langle f, T_n R^+_{(0,1)} \rangle = \sum_{n=1}^{\infty} q^n \langle f, \sum_{H(n)} R^+_{(c,d)} \rangle.$$

Our goal is to show $\rho(f) \in \mathscr{T}_*(l)$ and is a linear combination of pairs. To this end, we consider the following linear combination of toric quasimodular forms:

$$\rho_1(f) = \sum_{r+s=k-2} \sum_{m,n=0}^{l-1} \frac{(r+s)!}{r!s!} \widetilde{s}_{m/l}^{(r+1)} \widetilde{s}_{n/l}^{(s+1)} \langle f, (x+y)^r y^s (m, m+n) - (x-y)^r y^s (m, m-n) \rangle$$

$$= C + \sum_{r=0}^{k-2} \sum_{m=0}^{l-1} c_{r,m} \widetilde{s}_{m/l}^{(r+1)} + \sum_{D>0} q^D \sum_{m,n,r,s} A \langle f, (x+y)^r y^s (m, m+n) - (x-y)^r y^s (m, m-n) \rangle.$$

Here $C, c_{r,m}$ are constants whose exact values will not be needed, and the constant $A = A(r, s, m, D)$ is defined by

$$A := \frac{(r+s)!}{r!s!} \sum_{I(D)} k_1^r k_2^s (\delta_{k_1}^{m \bmod l} + (-1)^{r-1} \delta_{k_1}^{-m \bmod l})(\delta_{k_1}^{m \bmod l} + (-1)^{r-1} \delta_{k_1}^{-m \bmod l}),$$

where $I(D) \subset \mathbb{Z}^4$ denotes the set

$$I(D) = \{(m_1, k_1, m_2, k_2) \mid m_1, k_1, m_2, k_2 > 0, \quad m_1 k_1 + m_2 k_2 = D\} \tag{7}$$

A linear combination similar to $\rho_1(f)$ appears in the proof of [1, Theorem 4.8] as the composition of several maps, one of which is induced by the intersection pairing on Manin symbols. Here, however, we just take this as a definition. After some simplification, the formula for $\rho_1(f)$ becomes

$$\rho_1(f) = C + \sum_{r=0}^{k-2} \sum_{m=0}^{l-1} c_{r,m} \widetilde{s}_{m/l}^{(r+1)} + 4 \sum_{D>0} q^D \langle f, \sum_{I(D)} (R^+_{(k_1, k_1+k_2)} - R^+_{(k_1, k_1-k_2)}) \rangle,$$

where $R^+_{(m,n)}$ is the Manin symbol from Section 3.8. The quasimodular form $\rho_1(f)$ and the modular form $\rho(f)$ are related as follows:

**Proposition 5.6.** *We have*

$$\rho_1(f) - 12\rho(f) = C + 4F_1 + 4F_2 + \sum_{r=0}^{k-2} \sum_{m=0}^{l-1} c_{r,m} \widetilde{s}_{m/l}^{(r+1)},$$

*where $C$ is a constant and*

$$F_1 = \sum_{n>0} q^n \sum_{d|n} \frac{2n}{d} \langle f, R^+_{(d,0)} \rangle,$$

$$F_2 = \sum_{n>0} q^n \sum_{d|n} \Big( \langle f, R^+_{(d,0)} \rangle + 2 \sum_{0<e<d} \langle f, R^+_{(d,d-e)} \rangle \Big).$$

*Proof.* This follows from the identity

$$\sum_{I(D)}(R^+_{(k_1,k_1-k_2)} - R^+_{(k_1,k_1+k_2)})$$
$$= -\sum_{d|n}(\frac{2n}{d}+1)R^+_{(d,0)} - 2\sum_{\substack{d|n\\ d>e>0}} R^+_{(d,d-e)} - 3\sum_{H(n)} R^+_{(c,d)}.$$

This identity with weight $k=2$ appears as an intermediate step of the proof of [1, Theorem 4.8]. However, its proof only uses relations among $R^+_{(m,n)}$ that are independent of the weight $k$. □

**Lemma 5.7.** *The quasimodular form $F_1$ is a linear combination of pairs.*

*Proof.* After some simplification, one can write

$$F_1 = 2\sum_{n>0} q^n \sum_{d|n} nd^{k-3}\langle f, x^{k-2}(d,0)_+\rangle.$$

Let $G$ be the $q$-series

$$G = \sum_{n>0} q^n \sum_{d|n} d^{k-3}\langle f, x^{k-2}(d,0)_+\rangle.$$

The complex number $\langle f, x^{k-2}(d,0)_+\rangle$ depends only on $d \bmod l$, and further satisfies

$$\langle f, x^{k-2}(-d,0)_+\rangle = (-1)^k\langle f, x^{k-2}(d,0)_+\rangle.$$

Hence $d^{k-3}\langle f, x^{k-2}(d,0)_+\rangle$ is an odd $(\bmod\, l)$-polynomial. By Proposition 4.4, $G$ is a linear combination of the $\widetilde{s}^{(k)}_{a/l}$ and a constant, and is hence toric quasimodular. Differentiating the linear combination for $G$ with respect to $\tau$ and applying Proposition 2.6 completes the proof. □

**Lemma 5.8.** *The quasimodular form $F_2 + C$ is a linear combination of pairs for a suitably chosen constant $C$.*

*Proof.* By Proposition 4.7, we know that the function

$$d \longmapsto \langle f, R^+_{(d,0)}\rangle + 2\sum_{0<e<d}\langle f, R^+_{(d,d-e)}\rangle$$

extends to a unique odd $(\bmod\, l)$-polynomial. The result then follows from Proposition 4.4, and weight considerations. □

5.9. We are now ready to prove our main theorem.

**Theorem 5.10.** *All cusp forms of weight three or more are toric. Moreover, any such cusp form can be written as a linear combination of pairs (Definition 2.5).*

*Proof.* One can easily see that lifts of the forms $\widetilde{s}^{(r)}_{a/l}$ can be written as linear combinations of $\widetilde{s}$ for the new level. Therefore, we may assume without loss of generality that $f$ is a newform. Hence by the proof of Proposition 5.4, $\rho(f)$ is a non-zero multiple of $f$. Proposition 5.6 and Lemmas 5.7 and 5.8 show that $\rho(f)$ can be written up to a constant as a linear combination of toric quasimodular forms $\widetilde{s}^{(m)}_{a/l}\widetilde{s}^{(n)}_{b/l}$ for $m+n=k$ and $\widetilde{s}^{(n)}_{a/l}$ for smaller $n \leq k$. The transformation properties under $\Gamma_1(l)$ insure that all lower weight forms come with zero coefficients, and that all the quasimodular forms used are actually modular, i.e. $E_2 s^{(k-2)}_{a/l}$ come with zero coefficients. ☐

**Corollary 5.11.** *If $l \geq 5$, then any cusp form of weight $k \geq 3$ can be written, up to a weight $k$ Eisenstein series, as a degree $k$ homogeneous polynomial in weight one Eisenstein series.*

*Proof.* This follows from Theorem 5.10, Proposition 2.2, and [2, Theorem 4.11]. ☐

**Corollary 5.12.** *The multiplication map*

$$\mathscr{M}_m(l) \otimes \mathscr{M}_n(l) \longrightarrow \mathscr{M}_{m+n}(l)$$

*is surjective for all $m \geq n \geq 1$, except for $m = n = 1$.*

*Proof.* Theorem 5.10 assures that the image of the above map contains all cusp forms, so it is enough to insure that the forms of the image take arbitrary values at the cusps. To obtain a form which vanishes at all but one cusp $p$ we multiply a form in $\mathscr{M}_m(l)$ that vanishes at all cusps except $p$ and perhaps one other cusp $q$ (relevant only if $m = 2$) by a form in $\mathscr{M}_n(l)$ that vanishes at $q$ but not at $p$. ☐

*Remark* 5.13. A slightly weaker statement can be proved directly by using the fact that the ring of modular forms is Cohen-Macaulay. However, we are not aware of any other proofs for $(m, n) = (2, 1)$ or $(2, 2)$.

*Remark* 5.14. One can also ask which Eisenstein series are toric. It is easy to see that for a prime level $p$ all Eisenstein series are toric. For composite levels, the situation is different. For example if $l = 25$ then weight two toric Eisenstein series form a subspace of codimension one in the space of all weight two Eisenstein series. We do not know any similar examples for higher weight.

**Theorem 5.15.** *For every level $l$ there exists an $N$ such that the ring of toric forms coincides with the ring of modular forms for weights $k \geq N$. When $l$ is prime, one can take $N = 3$.*

*Proof.* In view of Theorem 5.10, one needs to show that all Eisenstein series are eventually contained in the ring of toric forms. Because the ring of toric forms is Hecke stable [2, Theorem 5.3], it suffices to show that the values of toric forms at the cusps eventually span a $c$-dimensional space, where $c$ is the number of cusps. For this one needs to show that the values of $s_{a/l}$ for two different cusps are not proportional. This is accomplished by a direct calculation that we leave to the reader. ☐

*Remark* 5.16. Theorem 5.15 was used in [3] to analyze the embedding of the modular curve $X_1(p)$ given by the graded ring $\mathscr{T}_*(p)$.

## 6. The map from symbols to forms in higher weight

6.1. A key step in the proof of [1, Theorem 4.11] was the analysis of a map $\mu$ from the minus space $M_-$ of weight 2 Manin symbols to a quotient of the space $\mathscr{M}_2(l)$ of weight 2 modular forms. Namely, we showed that the map

$$\mu\colon (m,n) \longmapsto \widetilde{s}_{m/l}\widetilde{s}_{n/l}$$

took $M_-$ into the quotient $\mathscr{M}_2(l)/\mathscr{E}_2(l)$, where $\mathscr{E}_2(l)$ is the space of weight 2 Eisenstein series ([1]). In this section we consider the analogous map in higher weight given by

$$\mu\colon x^r y^s (m,n) \longmapsto (-1)^s \widetilde{s}^{(s+1)}_{m/l}\widetilde{s}^{(r+1)}_{n/l} \tag{8}$$

and describe the relevant quotient containing the image.

**Theorem 6.2.** *Let $k>2$. The map $\mu$ in (8) applied to the space generated by the Manin symbols $x^s y^r(m,n)$ takes the relations*

$$x^r y^s(a,b) + (-1)^r y^r (x-y)^s (b,-a-b) + (-1)^s (y-x)^r x^s (-a-b,a) \tag{9}$$

*to the subspace generated by the modular forms $\widetilde{s}^{(k)}_{a/l}$ and the quasimodular forms $\partial_\tau \widetilde{s}^{(k-2)}_{a/l}$.*

*Proof.* The symbol (9) maps to

(10)

$$(-1)^s \widetilde{s}^{(s+1)}_{a/l}\widetilde{s}^{(r+1)}_{b/l} + \sum_{t=0}^{s} \frac{s!}{t!(s-t)!}\widetilde{s}^{(r+t+1)}_{b/l}\widetilde{s}^{(s-t+1)}_{-(a+b)/l} + \sum_{t=0}^{r} \frac{r!}{t!(r-t)!}\widetilde{s}^{(r-t+1)}_{-(a+b)/l}\widetilde{s}^{(s+t+1)}_{a/l}(-1)^{s+r}.$$

Up to quasimodular forms of lower weight and $\widetilde{s}^{(k)}_{a/l}$, the expression (10) can be simplified to

$$\sum_{D>0} q^D \sum_{I(D)} (A_{k_1,k_2} - A_{-k_1,k_2} + A_{k_2,-k_1-k_2} - A_{k_2,k_1-k_2} + A_{-k_1-k_2,k_1} - A_{k_1-k_2,-k_1}).$$

Here $I(D)$ is defined in (7) and

$$A_{k_1,k_2} = (-1)^s k_1^s k_2^r \bar{\delta}^{(a,b)}_{(k_1,k_2)},$$

where $\bar{\delta}^{(a,b)}_{(k_1,k_2)} = \delta^{a \bmod l}_{k_1}\delta^{b \bmod l}_{k_2} + (-1)^k \delta^{-a \bmod l}_{k_1}\delta^{-b \bmod l}_{k_2}$.

[1] This is slightly inaccurate: the map we're denoting by $\mu$ here is actually the composition of map called $\mu$ in [1] and the Fricke involution.

The set $I(D)$ can be partitioned into subsets corresponding to different "runs" of the Euclidean algorithm. Namely, there are partially defined maps $\Upsilon$ and $\Delta$ from $I(D)$ to itself given by

$$\Upsilon\colon (m_1,k_1,m_2,k_2) \longmapsto \begin{cases} (m_2, k_1+k_2, m_1-m_2, k_1), & \text{if } m_1 > m_2 \\ (m_2-m_1, k_2, m_1, k_1+k_2), & \text{if } m_1 < m_2 \\ \text{not defined}, & \text{if } m_1 = m_2 \end{cases}$$

$$\Delta\colon (m_1,k_1,m_2,k_2) \longmapsto \begin{cases} (m_1+m_2, k_2, m_1, k_1-k_2), & \text{if } k_1 > k_2 \\ (m_2, k_2-k_1, m_1+m_2, k_1), & \text{if } k_1 < k_2 \\ \text{not defined}, & \text{if } k_1 = k_2 \end{cases}$$

These maps are inverses of each other whenever their composition is defined. The whole set $I(D)$ can be pictured as a disjoint union of vertical threads, where each thread is obtained by starting at the top with a solution with $m_1 = m_2$ and applying $\Delta$ until arriving at a solution with $k_1 = k_2$([2]). The crucial observation is that for each thread $\Theta$, the sum

$$\sum_{\Theta} A_{k_1,k_2} + A_{k_2,-k_1-k_2} + A_{-k_1-k_2,k_1} - A_{-k_1,k_2} - A_{k_2,k_1-k_2} - A_{k_1-k_2,-k_1}$$

collapses. Indeed, the negative terms for elements $(m_1,k_1,m_2,k_2)$ cancel the positive terms for elements $\Delta(m_1,k_1,m_2,k_2)$. To see this, observe that if $k_1 > k_2$, then the positive terms of $\Delta(m_1,k_1,m_2,k_2)$ equal

$$A_{k_2,k_1-k_2} + A_{k_1-k_2,-k_1} + A_{-k_1,k_2}.$$

The $k_1 < k_2$ case is handled similarly, taking into account the symmetry $A_{-k_1,-k_2} = A_{k_1,k_2}$.

Hence, up to a linear combination of lower weight forms and the forms $\widetilde{s}^{(k)}_{a/l}$, the image of the relation (9) is equal to

$$\sum_{D>0} q^D \Big( \sum_{\{i\in I(D)|m_1=m_2\}} (A_{k_1,k_2} + A_{k_2,-k_1-k_2} + A_{-k_1-k_2,k_1}) \\ - \sum_{\{i\in I(D)|k_1=k_2\}} (A_{-k_1,k_2} + A_{k_2,k_1-k_2} + A_{k_1-k_2,-k_1}) \Big).$$

The coefficient of $q^D$ can be further simplified to

$$\sum_{d|D} \sum_{0<e<d} (A_{e,d-e} + A_{d-e,-d} + A_{-d,e}) - \sum_{d|D} (\frac{D}{d} - 1) \Big( d^{k-2} (\delta^{(a,b)}_{(-d,d)} + (-1)^k \delta^{(-a,-b)}_{(-d,d)}) \\ + (-1)^s d^s 0^r (\delta^{(a,b)}_{(d,0)} + (-1)^k \delta^{(-a,-b)}_{(d,0)}) + 0^s d^r (\delta^{(a,b)}_{(0,d)} + (-1)^k \delta^{(-a,-b)}_{(0,d)}) \Big),$$

where $\delta$ is now a Kronecker symbol for elements of $(\mathbb{Z}/l\mathbb{Z})^2$, and our convention is $0^s = 1$ if and only if $s = 0$.

[2] $\Upsilon$ and $\Delta$ stand for *up* and *down*.

To finish the proof we first observe that the contribution of the terms with $D/k$ is, up to an additive constant, a derivative with respect to $\tau$ of

$$-\delta^0_{a+b}\widetilde{s}^{(k-2)}_{b/l} + (-1)^{s+1}0^r\delta^0_b\widetilde{s}^{(k-2)}_{a/l} - 0^s\delta^0_a\widetilde{s}^{(k-2)}_{b/l},$$

where $\delta$ is the usual Kronecker function. To show that the remaining contributions give linear combinations of the forms $\widetilde{s}^{(\leq k)}_{a/l}$, it is enough to establish that for any $a, b, r, s$ the $(\bmod l)$-polynomial

$$h(d) := \sum_{0<e<d}(A_{e,d-e} + A_{d-e,-d} + A_{-d,e}) + A_{-d,d} + A_{d,0} + A_{0,d}$$

is odd. This follows easily from Proposition 4.5 and the symmetry of $A$. □

**Corollary 6.3.** *The map $\mu$ induces a map from the space of weight $k$ Manin symbols $M$ to the quotient $\mathscr{Q}$ of the space of weight $k$ quasimodular forms by subspace generated by the Eisenstein series $\widetilde{s}^{(k)}_{a/l}$ and the derivatives $\partial_\tau\widetilde{s}^{(k-2)}_{a/l}$.*

*Remark* 6.4. An alternative approach to Theorem 6.2 is to look at the identify

$$(s^{(1)}_\alpha + s^{(1)}_\beta + s^{(1)}_{-\alpha-\beta})^2 + \frac{1}{2}(s^{(2)}_\alpha + s^{(2)}_\beta + s^{(2)}_{-\alpha-\beta}) = 0$$

which comes from a calculation of certain toric form for the complex projective plane $\mathbb{P}^2$, see [2]. One can differentiate the above identity with respect to $\alpha$ and $\beta$ several times and plug in rational values of $\alpha$ and $\beta$. Then it remains to use the transformation that connects $\widetilde{s}^{(k)}_{a/l}$ and $s^{(k)}_{a/l}$. We leave the details to the reader.

## 7. Hecke equivariance of the symbols to forms map

7.1. It is not hard to see by explicit computation that the subspace spanned by the Eisenstein series and derivatives mentioned in Corollary 6.3 is invariant under the action of $\Gamma_0(l)/\Gamma_1(l)$, the Fricke involution, and the Hecke operators. Hence we can naturally extend their action to the quotient $\mathscr{Q}$. The goal of this section is to show that the map of Corollary 6.3 is compatible with the action of Hecke operators. For this, one needs to show that the map

$$x^r y^s(m,n) \mapsto (-1)^s\widetilde{s}^{(s+1)}_{m/l}\widetilde{s}^{(r+1)}_{n/l}$$

is compatible with the action of Hecke operators, up to linear combinations of $\widetilde{s}^{(k)}_{a/l}$ and $\partial_\tau\widetilde{s}^{(k-2)}_{a/l}$.

**Theorem 7.2.** *Let $p$ be a prime number coprime to $l$ and $T_p$ be the corresponding Hecke operator on $M_k$ and $\mathscr{M}_k$, where we abuse notations slightly. Let $\mu$ be the map defined in Theorem 6.2. Then for every $w \in M_k$, the image $\mu(T_p w)$ is equal to $T_p(\mu(\epsilon_{p^{-1}}w))$ modulo a linear combination of $s^{(k)}_{a/l}$ and $\partial_\tau s^{(k-2)}_{a/l}$. Here $\epsilon_{p^{-1}}$ is the action of the element of $\Gamma_0(l)/\Gamma_1(l)$ given by $x^r y^s(u,v) \mapsto x^r y^s(pu,pv)$, (cf. Proposition 2.7).*

Before we begin the proof of Theorem 7.2, we need a lemma giving a geometric interpretation of the set $H(p)$ involved in Merel's description of the $T_p$-action on Manin symbols (Theorem 3.6).

**Lemma 7.3.** [1, Theorem 3.16] *For each index $p$ sublattice $S \subset \mathbb{Z}^2$, consider the convex hull of all nonzero points of $S$ that lie in the first quadrant. Then the compact subset of the boundary of this convex hull is a union of segments. Moreover, the coordinates $(a, c)$, $(b, d)$ of the vertices of each segment (ordered from the $x$-axis) satisfy $ad - bc = p$ and $a > b \geq 0$, $d > c \geq 0$, and hence determine an element of $H(p)$. Conversely, all $(a, b, c, d) \in H(p)$ come from one such sublattice $S$ in this manner.*

Given an index $p$ sublattice $S \subset \mathbb{Z}^2$, we write $H(p, S)$ for the subset of those $(a, b, c, d) \in H(p)$ corresponding to $S$.

**Example 7.4.** Figure 1 shows the case $p = 2$. There are three sublattices of index 2, and altogether four distinct boundary segments. ¿From the segments we obtain the four elements of $H(2)$, namely $(1, 0, 0, 2)$, $(2, 1, 0, 1)$, $(1, 0, 1, 2)$ and $(2, 0, 0, 1)$.

FIGURE 1.

We will also need the following duality operation on the set of sublattices.

**Definition 7.5.** For an index $p$ sublattice $S$ we denote by $S^*$ the sublattice of all points $P$ in $\mathbb{Z}^2$ such that $P \cdot S \subseteq p\mathbb{Z}$. where $\cdot$ is the standard scalar product on $\mathbb{Z}^2$. It is clear that $S^{**} = S$. Moreover, $S^*$ can be obtained from $S$ by a $\pi/2$ rotation at the origin.

We are now ready to start the proof of Theorem 7.2.

*Proof.* It is enough to consider $w = x^r y^s(u, v)$. By Theorem 3.6 and the definition of $\mu$,

$$\mu(T_p x^r y^s(u, v)) \sim_l \sum_D q^D \sum_{h \in H(p)} \sum_{i \in I(D)} \Phi(h, i),$$

where

$$\Phi(h, i) = (ak_2 - bk_1)^r (ck_2 - dk_1)^s \bar{\delta}_{k_1, k_2}^{au+cv, bu+dv} - (ak_2 + bk_1)^r (ck_2 + dk_1)^s \bar{\delta}_{k_1, k_2}^{-au-cv, bu+dv}.$$

Here $\sim_l$ means that equality holds modulo linear combinations of $\widetilde{s}_{a/l}^{(<k)}$, and $I(D)$ is defined in (7). We can use $(p, l) = 1$ to rewrite the above as

$$\Phi(h, i) = A_{dk_1 - ck_2, ak_2 - bk_1} - A_{-dk_1 - ck_2, ak_2 + bk_1},$$

where $A_{\alpha,\beta} = \beta^r(-\alpha)^s\bar{\delta}^{pu,pv}_{\alpha,\beta}$. On the other hand,

$$T_p\mu(\epsilon_{p^{-1}}w) \sim_{pl} \sum_D q^D \sum_{I(pD)} (A_{k_1,k_2} - A_{-k_1,k_2}) \\ + p^{k-1}\sum_D q^D \sum_{I(D)} (-1)^s k_1^s k_2^r (\bar{\delta}^{u,v}_{k_1,k_2} - \bar{\delta}^{u,v}_{-k_1,k_2}). \tag{11}$$

For each $i \in I(pD)$ there exists a sublattice $S$ such that $(m_1, m_2) \in S$ and $(k_1, k_2) \in S^*$. Moreover, $S$ is unique unless $m_1, k_1, m_2, k_2 = 0 \bmod p$, in which case there are $(p+1)$ such sublattices $S$. To record this, we use the notation

$$I(pD, S) = \{i \in I(pD) \mid (m_1, m_2) \in S, \quad (k_1, k_2) \in S^*\}.$$

Let us further write, for any two subsets $U_1, U_2 \subset \mathbb{R}^2$,

$$I(pD, S; U_1, U_2) = \{i \in I(pD, S) \mid (m_1, m_2) \in U_1, (k_1, k_2) \in U_2\}.$$

Now we can rewrite (11) as

$$T_p\mu(\epsilon_{p^{-1}}w) \sim_{pl} \sum_D q^D \sum_S \Big( \sum_{I(pD,S;Q_I,Q_I)} A_{k_1,k_2} - \sum_{I(pD,S;Q_{II},Q_{II})} A_{k_1,k_2} \Big) \tag{12}$$

where $Q_I$ and $Q_{II}$ denote the open first and the second quadrants.

*Remark* 7.6. The reason we must write $\sim_{pl}$ here rather than $\sim_l$ is that the action of $T_p$ defined for weight $k$ on $\widetilde{s}^{(\leq k)}_{a/l}$ will be a linear combination $\widetilde{s}^{(\leq k)}_{a/pl}$.

Given any $h = (a, b, c, d) \in H(p)$, we also denote by $h$ the linear transformation $\mathbb{R}^2 \to \mathbb{R}^2$ given by the multiplying by matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ on the right. This allows us to write $\mu(T_p(w)) - T_p\mu(\epsilon_{p^{-1}}w)$ as

$$\mu(T_p(w)) - T_p\mu(\epsilon_{p^{-1}}w) \sim_{pl} \sum_D q^D \sum_S (\mathrm{Sum}_1 - \mathrm{Sum}_2 - \mathrm{Sum}_3 + \mathrm{Sum}_4), \tag{13}$$

where

$$\mathrm{Sum}_1 = \sum_{\substack{h\in H(p,S) \\ i\in I(pD,S;h^t(Q_I),h^{-1}(Q_I))}} A_{k_1,k_2}$$

$$\mathrm{Sum}_2 = \sum_{\substack{h\in H(p,S) \\ i\in I(pD,S;h^t(Q_{II}),h^{-1}(Q_{II}))}} A_{k_1,k_2}$$

$$\mathrm{Sum}_3 = \sum_{I(pD,S,Q_I)} A_{k_1,k_2}$$

$$\mathrm{Sum}_4 = \sum_{I(pD,S,Q_{II})} A_{k_1,k_2}$$

It is convenient to visualize $\mathrm{Sum}_1, \ldots, \mathrm{Sum}_4$ as indicated in Figure 2.

FIGURE 2.

7.7. Now we consider the right hand side of (13). It will turn out that most terms will cancel each other, but there will be some terms left over that will require careful consideration. To discuss these terms, we require some additional terminology.

For every sublattice $S$ of index $p$ we form the convex hulls of the nonzero points in *each* quadrant. The open 1-cones spanned by the points on the boundary of these hulls will be called *rays* of the lattice $S$, and will be denoted $\rho(S)$. The rays generated by the points $(\pm p, 0)$ and $(0, \pm p)$ will be called the *axis* rays; all others will be called *non-axis* rays. By abuse of notation, given a point $(x, y)$ we write $(x, y) \in \rho(S)$ to mean that $(x, y)$ lies on a ray of $S$. Finally, for any nonzero point $v \in S$, we define a rational cone $\operatorname{cone}(v)$ as follows:

- If $v$ lies on a ray $\rho$, then we put $\operatorname{cone}(v) = \rho$.
- Otherwise, we set $\operatorname{cone}(v)$ to be the interior of the unique 2-cone spanned by adjacent rays of $S$ and containing $v$.

The first step in investigating (13) is the following lemma, which is the heart of the proof.

**Lemma 7.8.** *Let $S \subset \mathbb{Z}^2$ have index $p$. Then for every $(m_1, k_1, m_2, k_2)$ such that $(m_1, m_2) \notin \rho(S)$ and $(k_1, k_2) \notin \rho(S^*)$, the total of the contributions of* $\mathrm{Sum}_1$*,* $\mathrm{Sum}_2$*,* $\mathrm{Sum}_3$ *and* $\mathrm{Sum}_4$ *at $(m_1, k_1, m_2, k_2)$ and $-(m_1, k_1, m_2, k_2)$ is zero.*

*Proof of Lemma 7.8.* Clearly, it is enough to check this if $(k_1, k_2)$ is in the first or second quadrant.

First assume $(k_1, k_2) \in Q_I$. Then the only nontrivial contributions come from $\mathrm{Sum}_1$ and $\mathrm{Sum}_3$ when $(m_1, m_2) \in Q_I$, and in this case we claim $\mathrm{Sum}_1$ contributes $A_{k_1,k_2}$ and $\mathrm{Sum}_3$ contributes $-A_{k_1,k_2}$. Indeed, if $(m_1, m_2) \in Q_I$, there is a contribution of exactly one $(a, b, c, d)$ in $\mathrm{Sum}_1$, which corresponds to $\operatorname{cone}(m_1, m_2)$. Hence the total contribution is zero.

Next assume $(k_1, k_2) \in Q_{II}$. In this case $\mathrm{Sum}_3$ doesn't contribute, and we split the contributions of the remaining sums into types:

- ($\mathrm{Sum}_1$, type 1) We assume $(m_1, m_2) \in Q_I$ lies in a cone above $\operatorname{cone}(k_2, -k_1)$ (see Figure 3, graph 1). Then there is a unique $(a, b, c, d)$ in $\mathrm{Sum}_1$, that corresponds to $\operatorname{cone}(m_1, m_2)$, and the contribution of $\mathrm{Sum}_1$ is $A_{k_1,k_2}$.
- ($\mathrm{Sum}_1$, type 2) We assume $(m_1, m_2) \in Q_I$ lies in a cone below the $\operatorname{cone}(k_2, -k_1)$ (see Figure 3, graph 2). Again, there is one $(a, b, c, d)$, and the contribution is $A_{-k_1,-k_2}$.
- ($\mathrm{Sum}_2$, type 1) We assume $(m_1, m_2) \in Q_I$ lies in a cone above $\operatorname{cone}(k_2, -k_1)$. Then there is a unique $(a, b, c, d)$ in $\mathrm{Sum}_2$ that corresponds to $\operatorname{cone}(k_2, -k_1)$, and the contribution of $\mathrm{Sum}_2$ is $-A_{k_1,k_2}$.

- ($\mathrm{Sum}_2$, type 2) We assume $(m_1, m_2) \in Q_I$ such that $(-m_1, -m_2)$ lies in a cone below $\mathrm{cone}(k_2, -k_1)$. Then there is a unique $(a, b, c, d)$ in $\mathrm{Sum}_2$ that corresponds to $\mathrm{cone}(k_2, -k_1)$, and the contribution of $\mathrm{Sum}_2$ is $-A_{k_1,k_2}$.
- ($\mathrm{Sum}_2$, type 3) If $(m_1, m_2) \in Q_{II}$, then there is a contribution of $-A_{k_1,k_2}$. Indeed, the unique $(a, b, c, d)$ corresponds to $\mathrm{cone}(k_2, -k_1)$.

Clearly, the type 1 contributions of $\mathrm{Sum}_1$ and $\mathrm{Sum}_2$ cancel; after we apply the symmetry $A_{-k_1,-k_2} = A_{k_1,k_2}$, the type 2 contributions of $\mathrm{Sum}_1$ and $\mathrm{Sum}_2$ cancel as well. Finally, the contribution of $\mathrm{Sum}_4$ cancels the type 3 contribution of $\mathrm{Sum}_2$, which completes the proof of Lemma 7.8. □

FIGURE 3.

*Remark* 7.9. In the terms that cancel, the matrices $(a, b, c, d)$ are different, which makes Lemma 7.3 crucial to the success of the proof.

We now return to the proof of Theorem 7.2. Having handled the bulk of the terms in $\mathrm{Sum}_1$–$\mathrm{Sum}_4$, we now examine the cases when $(m_1, m_2)$ or $(k_1, k_2)$ lie on a ray. For any point $(u, v)$, we let $(u, v)^\perp$ be the set of all $(x, y)$ with $ux + vy = 0$.

For each $(m_1, m_2) \in \rho(S)$ we define a subset $C(m_1, m_2) \subset S^*$ as follows. If $(m_1, m_2)$ is not on a coordinate axis, then we let $C(m_1, m_2)$ be the set of all points with positive scalar product with $(m_1, m_2)$ *except* those that lie in one of the closed cones adjacent to $(m_1, m_2)^\perp$ (Figure 4). We use the same notation to denote the similar set $C(k_1, k_2)$ constructed from a point $(k_1, k_2) \in \rho(S^*)$. If $(m_1, m_2)$ or $(k_1, k_2)$ lies on an axis, then we define $C(m_1, m_2)$ and $C(k_1, k_2)$ using the small diagrams in Figure 4.

FIGURE 4.

The rays of the boundary of $C(m_1, m_2)$ (excluding the origin) will be denoted by $\partial C(m_1, m_2)$ and similarly for $\partial C(k_1, k_2)$. For any cone $C$, we write ${\sum}'_C$ to indicate that the sum is taken over $C \cup \partial C$ with terms lying in $\partial C$ taken with weight $1/2$.

**Lemma 7.10.** *With the above notation,*

$$\begin{aligned}(14)\quad \mu(T_p(w)) - T_p\mu(\epsilon_{p^{-1}}w) \sim_{pl} \\ \sum_S \sum_{(k_1,k_2)\in\rho(S^*)\cap Q'_{II}} \sideset{}{'}\sum_{(m_1,m_2)\in C(k_1,k_2)} q^{(m_1k_1+m_2k_2)/p} A_{k_1,k_2} \\ - \sum_S \sum_{(m_1,m_2)\in\rho(S)\cap Q'_{I}} \sideset{}{'}\sum_{(k_1,k_2)\in C(m_1,m_2)} q^{(m_1k_1+m_2k_2)/p} A_{k_1,k_2},\end{aligned}$$

*where $Q'_I$ and $Q'_{II}$ are the closures of the first and second quadrants.*

*Proof of Lemma 7.10.* Because of Lemma 7.8, we need to examine the contribution of $\mathrm{Sum}_1$, $\mathrm{Sum}_2$, $\mathrm{Sum}_3$ and $\mathrm{Sum}_4$ to the quadruples $\pm(m_1,k_1,m_2,k_2)$ where at least one of $(m_1,m_2)$ and $(k_1,k_2)$ lie on the ray of the corresponding lattice. We will have to be especially careful when one of these vectors is located on a coordinate axis. In what follows we will fix lattices $S$ and $S^*$.

First, let us deal with the case when $(k_1,k_2)\in\rho(S^*)$ and is not on an axis, and $(m_1,m_2)\notin\rho(S)$. If $(k_1,k_2)$ is in the first or third quadrant, then $\mathrm{Sum}_2$ and $\mathrm{Sum}_4$ do not contribute, and the contributions of $\mathrm{Sum}_1$ and $\mathrm{Sum}_3$ cancel since they are respectively $A_{k_1,k_2}$ and $-A_{k_1,k_2}$. Therefore, it is enough to consider when $(k_1,k_2)$ lies in the second or fourth quadrants. Now the terms of $\mathrm{Sum}_2$ and $\mathrm{Sum}_3$ do not contribute, and the total contribution of $\mathrm{Sum}_1$ and $\mathrm{Sum}_4$ equals $A_{k_1,k_2}$ if and only if $(m_1,m_2)\in C(k_1,k_2)$, and is zero otherwise. This clearly corresponds to the terms we get on the right of (14).

Now suppose $(k_1,k_2)$ lies on a coordinate axis. We may assume that it lies in the positive portion. If $(m_1,m_2)\notin\rho(S)$, then only $\mathrm{Sum}_1$ contributes, and the contribution is $A_{k_1,k_2}$ if any only if $(m_1,m_2)\in C(k_1,k_2)$. Clearly this corresponds exactly to the contribution on the right of (14).

Analogously one can treat the case of $(m_1,m_2)\in\rho(S)$ with $(k_1,k_2)\notin\rho(S^*)$. We have therefore shown that Lemma 7.10 holds up to the contributions of $\pm(m_1,k_1,m_2,k_2)$ with both $(m_1,m_2)$ and $(k_1,k_2)$ on the rays of the corresponding lattices.

If both $(m_1,m_2)$ and $(k_1,k_2)$ belong to non-axis rays of $S$ and $S^*$, the contributions of $\mathrm{Sum}_1$ and $\mathrm{Sum}_2$ are zero. Hence, the contribution of $-A_{k_1,k_2}$ occurs if both of them lie in the first or third quadrant and the contribution of $A_{k_1,k_2}$ occurs if both lie in the second or fourth quadrant. To show that this is consistent with the right hand side of the equation of the lemma, observe that if $(k_1,k_2)\in Q_{II}$ and $(m_1,m_2)\in Q_I$, the contributions of the two $\sum'$ cancel. Indeed, in this case $(k_1,k_2)\in C(m_1,m_2)$ and $(k_1,k_2)\in\partial C(m_1,m_2)$ is equivalent to $(m_1,m_2)\in C(k_1,k_2)$ and $(m_1,m_2)\in\partial C(k_1,k_2)$, respectively.

The remaining case of one or both of $(m_1,m_2)$ and $(k_1,k_2)$ on the axis with both of them on the rays is treated similarly and is left to the reader. ◻

Continuing now with the proof of Theorem 7.2, we investigate the sums on the right of (14). We divide the contributions to the sums over $\rho(S^*)$ into two types: those coming from non-axis rays, and those coming from axis rays.

**Lemma 7.11.** *In the sums over $S$ in (14), the contributions of the non-axis rays give a linear combination of $\widetilde{s}^{(\le k)}_{a/pl}$ and $\partial_\tau\widetilde{s}^{(k-2)}_{a/l}$.*

*Proof of Lemma 7.11.* First we calculate the contribution of a $(k_1,k_2)\in\rho(S^*)$ such that $(k_1,k_2)$ lies on the ray $\mathbb{R}_{>0}(-c,a)$, where $(a,c)$ is in the first quadrant.

Let $(b,d)$ (respectively $(b_1,d_1)$) be the generator of the ray of $S$ adjacent to the ray generated by $(a,c)$ in the counterclockwise (resp. clockwise) direction. Then the sets of vectors $\{(a,c),(b,d)\}$ and $\{(a,c),(b_1,d_1)\}$ form a $\mathbb{Z}$-basis of $S$, which implies $(b,d)+(b_1,d_1) = N(a,c)$ where $N$ is a positive integer. Then any $(m_1,m_2) \in C(k_1,k_2)$ can be written

$$(m_1,m_2) = -\alpha(a,c) + \beta(b,d),$$

where

$$\alpha,\beta \in \mathbb{Z}, \quad (m_1,m_2)\cdot(-d,b) > 0, \quad (m_1,m_2)\cdot(-d_1,b_1) > 0. \tag{15}$$

The conditions (15) translate into the inequality $0 < \alpha < N\beta$ on $\alpha$, which has $N\beta - 1$ solutions for a given $\beta$. Note that the terms in $\partial C(K_1,K_2)$ correspond to $\alpha = 0$ and $\alpha = N\beta$, which contributes an extra $A_{k_1,k_2}$ for each value of $\beta$.

Now if we write $(k_1,k_2) = t(-c,a)$ for some positive integer $\mathbb{Z}$, then $(m_1k_1 + m_2k_2)/p = t\beta$, so that the contribution of the complete ray $\mathbb{R}_{>0}(-c,a) \in \rho(S^*)$ to the first term of Lemma 7.10 is

$$\sum_{t>0}\sum_{\beta>0} q^{t\beta}(N\beta - 1)A_{-tc,ta} = \sum_{D>0} q^D \sum_{t|D} \frac{ND}{t} A_{-tc,ta}.$$

When one recalls the definition of $A$, this is easily seen to be a linear combination of $\partial_\tau \widetilde{s}^{(k-2)}_{a/l}$.

Next we calculate the contribution of an $(m_1,m_2)$ that lies on ray $\mathbb{R}_{>0}(a,c)$ of $S$. The computation is very similar to the above. As before we denote by $(b,d)$ and $(b_1,d_1)$ the generators of the rays of $S$ adjacent to $\mathbb{R}_{>0}(a,c)$. Then in the second summation of (14), the pairs $(k_1,k_2)$ are of the form

$$(k_1,k_2) = -\alpha(c,-a) + \beta(d,-b), \ \alpha,\beta \in \mathbb{Z},$$

where as before $0 < \alpha < N\beta$ for $(m_1,m_2) \in C(m_1,m_2)$ and $\alpha = 0$ or $\alpha = N\beta$ for $(m_1,m_2) \in \partial C(m_1,m_2)$. If we write $(m_1,m_2) = t(a,c)$ for $t$ a positive integer, then we obtain

$$-\sum_{t>0}\sum_{\beta>0} q^{t\beta}\Big(\sum_{0<\alpha<N\beta} A_{-\alpha c+\beta d, \alpha a-\beta b} + \frac{1}{2}A_{\beta d,-\beta b} + \frac{1}{2}A_{\beta(-Nd+c),\beta(Na-b)}\Big).$$

It remains to use Propositions 4.5 and 4.4 to see that the above is a linear combination of $s^{(\leq k)}_{a/l}$. This completes the proof of the lemma. □

**Lemma 7.12.** *In the sums over $S$ in (14), the contributions of the axis rays give a linear combination of $\widetilde{s}^{(\leq k)}_{a/pl}$ and $\partial_\tau \widetilde{s}^{(k-2)}_{a/l}$.*

*Proof of Lemma 7.12.* First, if $S$ or $S^*$ contain $(0,1)$ or $(1,0)$, then the contributions of the two sums in Lemma 7.10 cancel. Hence we may ignore lattices of this type.

If $(k_1,k_2)$ is on the positive half of the $x$-axis, then $k_1$ is a multiple of $p$. The top cone of $S$ in the first quadrant is the span of the positive half of $y$-axis and $(1,a)$, with $a$ taking all values from 0 to $p-1$, depending on $S$. One then observes that the contribution of $S_1$ and $S_2$ with $a_1 + a_2 = p$ can be thought of as the sum over

$(m_1, m_2)$ in the interior of the cone spanned by $(1, a_1)$ and $(1, -a_2)$, plus half the sum for $(m_1, m_2)$ on the boundary of the cone. It is then easily seen to give a linear combination of $\partial_\tau \widetilde{s}^{(k-2)}_{a/l}$. The case of $(k_1, k_2)$ on the positive half of the $y$-axis is treated similarly.

If $(m_1, m_2)$ is on one of the axes, then we observe that the sum of $A_{k_1,k_2}$ over $C(m_1, m_2)$ and its boundary can be thought of as the sum over all points of $\mathbb{Z}^2$ that lie in that cone of an even two-variable mod $pl$-polynomial $\hat{A}_{k_1,k_2}$, which we define to equal $A_{k_1,k_2}$ if $(k_1, k_2) \in S^*$ and zero otherwise. One then again invokes Propositions 4.5 and 4.4 to conclude that these terms contribute a linear combination of $\widetilde{s}^{(\le k)}_{a/pl}$. □

*Completion of the proof of Theorem 7.2.* By Lemmas 7.11 and 7.12, we have that

$$\mu(T_p(w)) - T_p\mu(\epsilon_{p^{-1}}w)$$

is a linear combination of $\partial_\tau \widetilde{s}^{(k-2)}_{a/l}$ and $\widetilde{s}^{(\le k)}_{a/pl}$. The modular transformation properties then imply that only $\widetilde{s}^{(k)}_{a/l}$ and $\partial_\tau \widetilde{s}^{(k-2)}_{a/l}$ appear, which finishes the proof of Theorem 7.2. □

*Remark* 7.13. Another way to state Theorem 7.2 is to say that the composition of $\mu$ and Fricke involution is Hecke-equivariant.

*Remark* 7.14. The discussion of this section simplifies a bit if one uses the geometry of toric varieties. More specifically, one has to consider toric modular forms $f_{\mathbb{Z}^2,\deg}$ defined in [2] and then differentiate them with respect to the components of the degree function deg. Then the Hecke action described in [2] can be interchanged with these partial differentiations, which gives the desired result. It worth mentioning that our proof is in some sense parallel to this calculation. For example, the number $N$ that appears in the treatment of the second sum of Lemma 7.10 is related to the self-intersection numbers of the boundary divisors on the toric surface given by the fans that correspond to the subgroups $S$.

*Remark* 7.15. It may be interesting to analyze products of more than two $\widetilde{s}$. Every such product may be associated to a symbol

$$x_1^{r_1} \cdots x_n^{r_n}(a_1, \cdots, a_n)$$

where $a_i \in \mathbb{Z}/l\mathbb{Z}$. Then one expects to be able to develop a generalization of the theory of Manin symbols, by introducing relations on these symbols that come from linear relations on the products. The action of Hecke operators will then come from toric geometry, and will be related to subgroups of index $p$ in $\mathbb{Z}^n$ as in [2].

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