CRYSTAL GRAPHS, TOKUYAMA'S THEOREM, AND THE GINDIKIN–KARPELEVIČ FORMULA FOR G_2

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Abstract. We conjecture a deformation of the Weyl character formula for type G_2 in the spirit of Tokuyama's formula for type A. Using our conjecture we prove a combinatorial version of the Gindikin–Karpelevič formula for G_2, in the spirit of Bump–Nakasuji's formula for type A.

1. Introduction

Let $\mathfrak{g}$ be a simple complex Lie algebra, let Λ_W be its weight lattice, and let $\mathbb{C}[\Lambda_W]$ be the associated ring of Laurent polynomials. Let W be the Weyl group of $\mathfrak{g}$ and for any $w \in W$ let $\operatorname{sgn} w \in \{\pm 1\}$ be its sign. Given a dominant weight $\theta \in \Lambda_W$, let V_θ be the irreducible representation of highest weight θ. The Weyl character formula expresses the character $\chi_\theta \in \mathbb{C}[\Lambda_W]$ as a ratio of two polynomials:

$$\chi_\theta(\mathbf{x}) = \frac{\sum_{w\in W} (\operatorname{sgn} w)\mathbf{x}^{w(\theta+\rho)-\rho}}{\prod_{\alpha>0}(1-\mathbf{x}^{-\alpha})}. \tag{1}$$

Here the product is taken over the positive roots α, the Weyl vector ρ is $\frac{1}{2}\sum_{\alpha>0}\alpha$, and for any weight β we denote by $\mathbf{x}^\beta$ the corresponding monomial in $\mathbb{C}[\Lambda_W]$.

We can define a deformation of (1) by inserting a parameter into the denominator. Let q be a variable and put

$$D(\mathbf{x}) = \prod_{\alpha>0}(1-q^{-1}\mathbf{x}^{-\alpha}).$$

Then the product

$$N_\theta(\mathbf{x}) = \chi_\theta(\mathbf{x})D(\mathbf{x})$$

is a polynomial supported in the convex hull of the weights of the representation $V_{\theta+\rho}$. When $\mathfrak{g}$ has type A, Tokuyama [15] showed how to compute $N_\theta(\mathbf{x})$

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explicitly as a sum over the Gel′fand–Cetlin basis of $V_{\theta+\rho}$. His formula has recently played an imporant role in the study of Weyl group multiple Dirichlet series. These are series in several complex variables built from data attached to root systems; each has a group of functional equations isomorphic to the Weyl group of the root system that intermixes all the variables. Such series are related to p-adic Whittaker functions and in fact are conjectured to be Fourier–Whittaker coefficients of certain Eisenstein series on metaplectic groups (finite central covers of reductive groups). We refer to [3] for more information about this connection.

Tokuyama's formula has been generalized to other root systems with various combinatorial tools. For instance Hamel–King [9] gave a generalization to $\mathfrak{g}$ of type C, in which the Gel′fand–Cetlin basis was replaced by symplectic shifted tableaux. Conjectural generalizations to $\mathfrak{g}$ of types B, D were given in [1,5,8]; recently the case of type B was proved by Friedberg–Zhang [6]. For arbitrary Φ, the most general result is due to McNamara [14], who showed how p-adic Whittaker functions can be computed as sums over crystal graphs.([1]) When $\mathfrak{g}$ is type A, the sums can be taken over Gel′fand–Cetlin patterns and computed explicitly, and McNamara recovers Tokuyama's theorem. However, apart from this case, McNamara's formulas have not been explicitly computed for any other type.

In this paper, we present a conjectural analogue of Tokuyama's theorem when $\mathfrak{g}$ has type G_2 (Conjecture 4.3). We describe how to compute the polynomial $N_\theta(\mathbf{x})$ as a sum over certain weight vectors in $V_{\theta+\rho}$. As a combinatorial model for this representation, we use patterns due to Littelmann [13]; when $\mathfrak{g}$ has type A, these are equivalent to Gel′fand–Cetlin patterns. Although we are unable to prove our conjecture, we are able to treat the limiting case that the highest weight becomes infinite. In this case our formula (Theorem 5.2) becomes a combinatorial version of the Gindikin–Karpelevič formula [12], in the spirit of that proved by Bump–Nakasuji [4].

2. Background and the Tokuyama numerator

In this section we state Tokuyama's formula for characters of representations of GL_{r+1} and explain the connection to crystal graphs. We begin by describing what a formula of "Tokuyama-type" looks like. We will use slightly different normalizations from §1: in particular we will shift our characters so that they are supported on the root lattice, and will index representations by lowest weights. These conventions are somewhat unusual from the point of view

[1]Another approach also valid for an arbitrary Cartan–Killing type has been presented by Kim–Lee [11], who compute $N_\theta(\mathbf{x})$ as a sum over weights of $V_\theta \otimes V_\rho$.

of combinatorial representation theory, but they are more natural when one connects these constructions to p-adic Whittaker functions.

As before let $\mathfrak{g}$ be a simple complex Lie algebra of rank r. Let Φ be the root system of $\mathfrak{g}$ and $\Phi^+ \cup \Phi^-$ the partition into positive and negative roots, and $\Delta = \{\alpha_1, \dots, \alpha_r\}$ the simple roots. Let $\varpi_1, \dots, \varpi_r$ be the fundamental weights and $\rho = \frac{1}{2}\sum_{\alpha>0}\alpha = \sum \varpi_i$. Let W be the Weyl group of Φ with simple reflections $s_1, \dots, s_r$.

We let Λ be the lattice generated by the roots and $\mathbb{C}[\Lambda]$ the ring of Laurent polynomials determined by Λ. Given $\lambda \in \Lambda$, let $\mathbf{x}^\lambda \in \mathbb{C}[\Lambda]$ be the corresponding monomial. We may identify $\mathbb{C}[\Lambda]$ with $\mathbb{C}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$ via $\mathbf{x}^{\alpha_i} \mapsto x_i$. Let $\Lambda^+ \subset \Lambda$ be the cone generated by the positive roots (the codominant cone).

Let q be a parameter. We define the Weyl denominator by

$$\Delta(\mathbf{x}) = \prod_{\alpha>0}(1 - \mathbf{x}^\alpha)$$

(note the use of $\mathbf{x}^\alpha$, not $\mathbf{x}^{-\alpha}$) and a deformation $D(\mathbf{x})$ of $\Delta(\mathbf{x})$ by

$$D(\mathbf{x}) = \prod_{\alpha>0}(1 - q^{-1}\mathbf{x}^\alpha). \tag{2}$$

Let θ be a dominant weight and let V_θ be the irreducible representation of $\mathfrak{g}$ with *lowest weight* $-\theta$.([2]) Let χ_θ be the character of V_θ. As in §1, the character χ_θ is most properly thought of as an element of the group ring of the weight lattice, but we modify χ_θ to be an element of $\mathbb{C}[\Lambda]$ by shifting so that the term for the lowest weight is supported on the monomial $\mathbf{x}^0 \in \mathbb{C}[\Lambda]$; by abuse of notation we denote the resulting polynomial in $\mathbb{C}[\Lambda]$ also by χ_θ. With this convention, the support of χ_θ is contained in the codominant cone Λ^+, and χ_θ is actually a polynomial under the identification $\mathbb{C}[\Lambda] \simeq \mathbb{C}[x_1^{\pm 1}, \dots, x_r^{\pm 1}]$. For example, if $\Phi = A_2$ and $\theta = \varpi_2$, then V_θ is the standard representation. If we write $x = \mathbf{x}^{\alpha_1}, y = \mathbf{x}^{\alpha_2}$, then $\chi_\theta = 1 + y + xy$. Similarly if $\theta = \rho$, then V_θ is the adjoint representation, and $\chi_\theta = 1 + x + y + 2xy + x^2y + y^2x + x^2y^2$.

Definition 2.1. Let V_θ be an irreducible representation and let $\chi_\theta(\mathbf{x})$ be its character as above. Then the *Tokuyama numerator* $N_\theta(\mathbf{x}) \in \mathbb{C}[q^{-1}][\Lambda]$ is the polynomial $N_\theta(\mathbf{x}) = \chi_\theta(\mathbf{x})D(\mathbf{x})$.

Note that if $q = 1$, then $D(\mathbf{x}) = \Delta(\mathbf{x})$, and then by (1) $N_\theta(\mathbf{x})$ is a sum of signed monomials indexed by the Weyl group W. In general $N_\theta(\mathbf{x})$ is a polynomial supported on monomials $\mathbf{x}^\beta$ with β a weight of $V_{\theta+\rho}$. When $\Phi = A_r$, Tokuyama showed how to write $N_\theta(\mathbf{x})$ as a sum over certain weights in the

[2]For many root systems, including G_2, the representation V_θ as defined coincides with the representation with highest weight θ. For some, such as type A, they differ. This choice means that certain changes have to be made when comparing results we cite below with the original sources.

representation $V_{\theta+\rho}$, and thus gave an explicit expression for the numerator $N_\theta(\mathbf{x})$ (cf. Theorem 3.2). The goal of this paper is to give an explicit conjectural formula for the numerator when $\Phi = G_2$.

3. Crystal graphs and Littelmann Patterns

Recall that $\mathfrak{g}$ is a simple complex Lie algebra with root system Φ, θ is a dominant weight, and V_θ is the irreducible representation of lowest weight $-\theta$. Littelmann patterns [13] provide a combinatorial way to index a basis of V_θ. For instance when $\Phi = A_r$, Littelmann patterns are essentially the famous Gel′fand–Cetlin patterns that encode branching rules for SL_n [7]. In this section we recall how to construct Littelmann patterns, with an emphasis on G_2.

Littelmann patterns encode weight vectors of V_θ by extracting data from the crystal graph $\mathscr{B}(\theta)$, so we begin by discussing the latter. We will not need much about crystal graphs and refer to [10] for a survey of their properties. For our purposes, we only need to know that $\mathscr{B}(\theta)$ is a finite directed graph with edges colored by the simple roots Δ. The vertices of $\mathscr{B}(\theta)$ are in bijection with certain weight vectors in V_θ; for $v \in \mathscr{B}(\theta)$ we write $v \mapsto \bar{v}$. Under this bijection, if there is an edge $v \to v'$ labelled by $\alpha \in \Delta$, then the weight of $\bar{v}$ is that of $\bar{v}'$ plus α. Thus the edges correspond to the lowering operators acting on V_θ. If we let $\theta \to \infty$, we obtain an infinite graph $\mathscr{B}(\infty)$. All the graphs $\mathscr{B}(\theta)$ appear as subgraphs of $\mathscr{B}(\infty)$.

Now choose a reduced expression for the longest Weyl word w_0. Littelmann proved that one can find a rational polyhedral cone $C_\infty \subset \mathbb{R}^N$, where $N = |\Phi^+|$, such that the lattice points $C_\infty \cap \mathbb{Z}^N$ are in bijection with the vertices of $\mathscr{B}(\infty)$. The inequalities defining the cone C_∞ depend only on w_0. Furthermore, after choosing a dominant weight θ, one can find a second set of rational inequalities depending on θ and w_0, such that if $C_\theta \subset C_\infty$ denotes the corresponding cone, then the lattice points $C_\theta \cap \mathbb{Z}^N$ are in bijection with the vertices of $\mathscr{B}(\theta)$. Finally he showed how to index these lattice points using tables of nonnegative integers that record the structure of certain paths in the crystal graph $\mathscr{B}(\theta)$. These tables are the *Littelmann patterns*; rather than giving their definition in full generality, we explain how they work for G_2 below and refer to [13] for more details. Given a Littelmann pattern π, we abuse notation and write $\pi \in \mathscr{B}(\theta)$ to indicate that π encodes a lattice point in C_θ indexing a vertex of $\mathscr{B}(\theta)$.

We now specialize to $\Phi = G_2$. The root system is shown in Figure 1 in §5; we have $|\Phi^+| = 6$, and the simple roots are α_1, α_2. The Weyl group has order 12 and the longest word w_0 has length 6. If we denote the simple reflection corresponding to the simple root α_i by s_i, then there are two reduced expressions for the longest word: $s_1s_2s_1s_2s_1s_2$ and $s_2s_1s_2s_1s_2s_1$. We will use the second expression. A Littelmann pattern for G_2 then has the form

$$\begin{bmatrix} a & b & c & d & e \\ & & f & & \end{bmatrix} \tag{3}$$

where $a, \dots, f$ are integers, called the *entries* of the pattern. To simplify notation, we usually write

$$[a, b, c, d, e][f] \tag{4}$$

for (3).

As described above, the entries are subject to certain inequalities determined by our choice of reduced expression for w_0 and by the highest weight θ. The first set of inequalities, which defines the infinite cone C_∞, gives lower bounds on the entries of a pattern: we have

$$2a \geq 2b \geq c \geq 2d \geq 2e \geq 0, \quad f \geq 0. \tag{5}$$

We call these the *circling*([3]) inequalities; if any of these is not strict, then we circle the entry in (4) that appears on the left side of the corresponding inequality. Thus e and f are circled if they vanish, d is circled if it equals e, and so on. We indicate circling of an entry u by a circle superscript: u°.

The second set of inequalities, which together with (5) defines C_θ, depends on the weight θ and provides upper bounds on pattern entries. Write $\theta = \ell_1\varpi_1 + \ell_2\varpi_2$. Then the entries must satisfy

$$\begin{aligned} &e \leq \ell_1, d \leq \ell_2 + e, c \leq \ell_1 + 3d - 2e, b \leq \ell_2 + c - 2d + e, \\ &\qquad a \leq \ell_1 + 3b - 2c + 3d - 2e, f \leq \ell_2 + a - 2b + c - 2d + e. \end{aligned} \tag{6}$$

We say that an entry u is boxed, denoted $\underline{u}$, if it reaches its upper bound in (6). Thus we write $\underline{e}$ if $e = \ell_1$, $\underline{d}$ if $d = \ell_2 + e$, and so on. To ease notation we sometimes give the boxing for a pattern in the form of a pattern itself with entries restricted to 0 and 1 and prefixed by $\mathtt{bx}$. In such a pattern a 1 indicates that the corresponding entry in the Littelmann pattern should be boxed, and 0 indicates it should be unboxed. For instance, the notation $\mathtt{bx}[0, 1, 0, 1, 0][1]$ means a Littelmann pattern of the form $[a, \underline{b}, c, \underline{d}, e][\underline{f}]$.

Each pattern π determines a monomial $\mathbf{x}^\pi \in \mathbb{C}[\Lambda] \simeq \mathbb{C}[x^{\pm 1}, y^{\pm 1}]$: if $\pi = [a, b, c, d, e][f]$, then $\mathbf{x}^\pi = x^{a+c+e} y^{b+d+f}$ (the variable x corresponds to the short simple root). The pattern also determines a polynomial $H(\pi)$ in q^{-1}:

Definition 3.1. Let π be a boxed and circled Littelmann pattern. Then the *standard contribution* $H(\pi) \in \mathbb{Z}[q^{-1}]$ of π is defined to be $H(\pi) = \prod_{u \in \pi} h(u)$,

[3]The terminology circling and boxing comes from [2].

where the product is taken over the entries u of π, and

$$h(u) = \begin{cases} 0 & \text{if } u \text{ is both boxed and circled } (\underline{u}^\circ), \\ 1 & \text{if } u \text{ is not boxed and is circled } (u^\circ), \\ -1/q & \text{if } u \text{ is boxed and is not circled } (\underline{u}), \\ (1-1/q) & \text{if } u \text{ is neither boxed nor circled } (u). \end{cases}$$

We call the function $H(\pi)$ the standard contribution of a boxed and circled pattern π because that is what a pattern contributes in Tokuyama's original formula [15]. We state this formula here for the convenience of the reader, and thus for the moment let Φ be the root system A_r. Fix a dominant weight $\theta = \sum \ell_i \varpi_i$ and define χ_θ as above. The reduced expression $w_0 = s_1(s_2s_1)(s_3s_2s_1)\cdots$ determines a collection of circling and boxing inequalities; we refer to [13, Theorem 5.1, Corollary 1] for a complete description (cf. Example 3.3). A pattern π determines a monomial $\mathbf{x}^\pi$, and we have the following theorem:

Theorem 3.2. *For $\Phi = A_r$ and with the standard contributions in Definition 3.1, we have*

$$N_\theta(\mathbf{x}) = \chi_\theta(\mathbf{x})D(\mathbf{x}) = \sum_{\pi \in \mathscr{B}(\theta+\rho)} H(\pi)\mathbf{x}^\pi. \tag{7}$$

Example 3.3. If $\Phi = A_2$, then patterns have the form $\pi = [a, b][c]$; for such a π we have $\mathbf{x}^\pi = x^{b+c}y^a$. The circling inequalities are $a \geq b \geq 0, c \geq 0$, and the boxing inequalities are

$$b \leq \ell_1, a \leq \ell_2 + b, c \leq \ell_1 + a - 2b.$$

If we take $\theta = 0$, then $\chi_0 = 1$, thus (7) becomes a deformed version of the Weyl denominator formula. The sum is over the 8 patterns for $\mathscr{B}(\rho)$:

$$[0^\circ, 0^\circ][0^\circ], [0^\circ, 0^\circ][\underline{1}], [\underline{1}, 0^\circ][0^\circ], [\underline{1}, 0^\circ][1], [\underline{1}, 0^\circ][2], [1^\circ, \underline{1}][\underline{0}^\circ], [\underline{2}, \underline{1}][0^\circ], [\underline{2}, \underline{1}][\underline{1}].$$

The standard contributions are

$$1, -1/q, -1/q, -(1/q)(1-1/q), (-1/q)^2, 0, (-1/q)^2, (-1/q)^3,$$

and one can check that $N_\theta(\mathbf{x}) = 1 - q^{-1}x - q^{-1}y + (q^{-2} - q^{-1})xy + q^{-2}x^2y + q^{-2}xy^2 - q^{-3}x^2y^2 = D(\mathbf{x})$.

4. A Conjectural Tokuyama formula for G_2

We now present our conjectural generalization of Tokuyama's theorem for G_2. As a first approximation, define the polynomial

$$\sum_{\pi \in \mathscr{B}(\theta+\rho)} H(\pi)\mathbf{x}^\pi. \tag{8}$$

In other words, we simply take each pattern's contribution to be the standard contribution from Definition 3.1, where boxing and circling are computed as in

(5)–(6). One quickly sees that (8) is not correct: (8) does not equal $\chi_\theta(\mathbf{x})D(\mathbf{x})$. On the other hand, (8) is not that far from our goal: only certain coefficients in the sum are wrong, and the corresponding monomials all contain at least one pattern with a special form:

Definition 4.1. A G_2-Littelmann pattern $[a,b,c,d,e][f]$ is called *bad middle* if $c = b + d$ and $b = d + 1$.

Note that whether or not a pattern is bad middle depends only its top row, and is independent of the bottom row $[f]$. We are now ready to give the main definition needed for our conjecture.

Definition 4.2. Let $\pi = [a,b,c,d,e][f]$ be a boxed and circled G_2 Littelmann pattern. We define the contribution $\widehat{H}(\pi) \in \mathbb{Z}[q^{-1}]$ as follows.

First, if π is not bad middle, or if π is bad middle but the boxing is not specified below, or if π has an entry that is both boxed and circled, then put $\widehat{H}(\pi) = H(\pi)$, the standard contribution of π.

Otherwise, we put $\widehat{H}(\pi) = \widehat{T}(\pi')h(f)$, where π' denotes the top row of π, and $\widehat{T}$ is defined as follows:

(1) If π' has boxing $\mathtt{bx}[0,0,1,0,0]$, then we put $\widehat{T}(\pi') = 0$.

(2) If π' has boxing $\mathtt{bx}[1,0,1,0,0]$, then we put

$$\widehat{T}(\pi') = \begin{cases} 0 & \text{if } d = 0, \\ T(\pi') & \text{if } d > 0. \end{cases}$$

Here and in what follows we write $T(\pi')$ for the product of $h(u)$ over the entries in the row $\pi' \subset \pi$ (in other words, this is what one would compute as the standard contribution of the top row π').

(3) If π' has boxing $\mathtt{bx}[1,0,0,0,0]$, then we put

$$\widehat{T}(\pi') = \begin{cases} (-q+1)/q^2 & \text{if } e = 0 \text{ and } d = 0, \\ (-q^3 + 2q^2 - 2q + 1)/q^4 & \text{if } e = 0 \text{ and } d > 0, \\ T(\pi') & \text{if } e > 0. \end{cases}$$

(4) If π' has boxing $\mathtt{bx}[0,1,0,1,0]$, then we put

$$\widehat{T}(\pi') = \begin{cases} T(\pi') & \text{if } a = b, \\ 0 & \text{if } b < a < c \text{ and } e = 0, \\ (-q^2 + 2q - 1)/q^5 & \text{if } b < a < c - e \text{ and } e > 0, \\ (q-1)/q^3 & \text{if } a = c \text{ and } e = 0, \\ (q^3 - 2q^2 + 2q - 1)/q^5 & \text{if } a = c - e \text{ and } e > 0, \\ 0 & \text{if } a > c \text{ and } e = 0, \\ (-q^2 + 2q - 1)/q^5 & \text{if } a > c - e \text{ and } e > 0. \end{cases}$$

If π' has boxing $\mathtt{bx}[0,0,0,0,0]$ and $e=0$, then we put

$$\widehat{T}(\pi') = \begin{cases} (q^2-2q+1)/q^2 & \text{if } a=b \text{ and } d>0,\\ (q^3-3q^2+3q-1)/q^3 & \text{if } b<a<c \text{ and } d>0,\\ (q^3-3q^2+4q-2)/q^3 & \text{if } a=c \text{ and } d>0,\\ (q^3-3q^2+3q-1)/q^3 & \text{if } a>c \text{ and } d>0,\\ (q-1)/q & \text{if } a=b \text{ and } d=0,\\ (q^2-2q+1)/q^2 & \text{if } a>b \text{ and } d=0. \end{cases}$$

Finally, if π' has boxing $\mathtt{bx}[0,0,0,0,0]$ and $e>0$, then we put

$$\widehat{T}(\pi') = \begin{cases} (q^4-3q^3+4q^2-3q+1)/q^4 & \text{if } a=b \text{ and } d>e,\\ (q^5-4q^4+7q^3-7q^2+4q-1)/q^5 & \text{if } a>b \text{ and } d>e,\\ (q^2-2q+1)/q^2 & \text{if } a=b \text{ and } d=e,\\ (q^4-3q^3+4q^2-3q+1)/q^4 & \text{if } a>b \text{ and } d=e. \end{cases}$$

We can now state our conjecture:

Conjecture 4.3. *Let $\Phi = G_2$ and put $N_\theta(\mathbf{x}) = \chi_\theta(\mathbf{x})D(\mathbf{x})$, where χ_θ is the character of the irreducible representation V_θ of G_2 of lowest weight $-\theta$, shifted to be an element of $\mathbb{C}[\Lambda]$ (as in the paragraph before Definition 2.1), and where $D(\mathbf{x}) = \prod_{\alpha>0}(1-q^{-1}\mathbf{x}^\alpha)$ is the deformed Weyl denominator* (2)*. Then we have*

$$N_\theta(\mathbf{x}) = \sum_{\pi\in\mathscr{B}(\theta+\rho)} \widehat{H}(\pi)\mathbf{x}^\pi. \tag{9}$$

Although we cannot currently prove Conjecture 4.3, we have checked it in many cases by computer:

Proposition 4.4. *Conjecture* (4.3) *is true for all weights $\theta = \ell_1\varpi_1 + \ell_2\varpi_2$ with $0 \le \ell_i \le 4$.*

Example 4.5. Let $\theta=0$. Then as in Example 3.3 the identity (9) becomes a deformed version of the Weyl denominator identity. The sum is taken over 64 patterns. On 24 of these $\widehat{H}$ vanishes since an entry is both boxed and circled. Of the remaining 40, there are 12 patterns that are bad middle, and 7 of these have their contributions altered by Conjecture 4.3:

(1) There are 2 patterns with top row $[1^\circ, 1, \underline{1}, 0^\circ, 0^\circ]$ and 3 with top row $[\underline{2}, 1, \underline{1}, 0^\circ, 0^\circ]$. All of these have $\widehat{H}=0$ (the first 2 by (1) and the second 3 by (2) in Definition 4.2).

(2) There are 2 patterns $[3, \underline{2}, 3, \underline{1}, 0^\circ][0^\circ]$ and $[3, \underline{2}, 3, \underline{1}, 0^\circ][\underline{1}]$. Using (4) in Definition 4.2, we compute that the first has $\widehat{H} = (q-1)/q^3$ and the second has $\widehat{H} = -(q-1)/q^4$.

Remark 4.6. We have checked (9) for larger weights than those in Proposition 4.4. The largest example we checked was $\theta = 6\varpi_1 + 6\varpi_2$. For this example the crystal graph $\mathscr{B}(\theta + \rho)$ has 262144 vertices.

Remark 4.7. The motivation to consider bad middle patterns comes from a similar investigation by the third-named author into an analogue of Tokuyama's theorem for the root system of type B [8]. Indeed the circling inequalities for the top row of the G_2-patterns are very similar to those for type B for a certain choice of reduced expression for w_0.

5. Gindikin–Karpelevič formula

Let F be a nonarchimedian local field with $\mathscr{O}$ its valuation ring. Let ϖ be a uniformizer and let q be the cardinality of the residue field $\mathscr{O}/\varpi\mathscr{O}$.

Let G be a simply-connected split Chevalley group over F; for us this will ultimately be of type G_2. Let $T \subset B \subset G$ be a maximal torus and a Borel subgroup. Let U^- be the opposite unipotent radical to B. Let $K \subset G$ be the maximal compact subgroup $G(\mathscr{O})$.

Let Φ be the root system of G determined by T and B, and let $\Delta \subset \Phi$ be the corresponding simple roots. As before let $\Phi = \Phi^+ \cup \Phi^-$ be the decomposition into positive and negative roots. For $\alpha \in \Phi$ let $e_\alpha \colon F \to G$ be the generator of the root subgroup corresponding to α, and let $h_\alpha \colon F \to G$ be the coroot corresponding to α. Thus T is the subgroup generated by $\{h_\alpha \mid \alpha \in \Delta\}$, B is generated by T and $\{e_\alpha \mid \alpha > 0\}$, and U^- is generated by $\{e_\alpha \mid \alpha < 0\}$.

Now we introduce the "spectral parameters." Let $\{z_\alpha\}$ be a set of nonzero complex numbers indexed by the simple roots. Given any root $\beta \in \Phi$, we define $\mathbf{z}^\beta \in \mathbb{C}$ by

$$\mathbf{z}^\beta = \prod_{\alpha \in \Delta} z_\alpha^{k_\alpha}, \quad \text{where } \beta = \sum_{\alpha \in \Delta} k_\alpha \alpha.$$

We can use the $\{z_\alpha\}$ to define a character $\chi \colon T \to \mathbb{C}$ by putting

$$\chi(\prod_{\alpha \in \Delta} h_\alpha(\varpi^{m_\alpha})) = \prod_{\alpha \in \Delta} z_\alpha^{m_\alpha}, \quad m_\alpha \in \mathbb{Z},$$

and then declaring that χ is trivial on $T \cap K$. We can extend χ to a character on B, and can then define the *principal series representation* V_χ by

$$V_\chi = \{f \colon G \to \mathbb{C} \mid f(bg) = \delta^{1/2}(b)\chi(b)f(g), b \in B\}.$$

Here δ is the modular quasi-character of B, and the action of G is given by right translations: $(g \cdot f)(g') := f(g'g)$. One can prove that the space of K-invariant vectors V_χ^K is one-dimensional. We choose a nonzero element $\varphi_K \in V_\chi^K$, called the *spherical vector*, such that

$$\varphi_K(bk) = \delta^{1/2}(b)\chi(b), \quad b \in B, k \in K.$$

We can now state the *Gindikin–Karpelevič formula*:

Theorem 5.1. *We have*

$$\int_{U^-(F)} \varphi_K(u)\,du = \prod_{\alpha>0} \frac{1-q^{-1}\mathbf{z}^\alpha}{1-\mathbf{z}^\alpha}. \tag{10}$$

We remark that Gindikin–Karpelevič proved their formula for F archimedian, in which case the right of (10) becomes a product of ratios of Gamma functions. The formula for F nonarchimedian was proved by Langlands [12].

Now let $\mathbb{C}[[\Lambda^+]] \simeq \mathbb{C}[[x_1,\dots,x_r]]$ be the formal power series ring on the codominant cone, and consider the generating function

$$\frac{D(\mathbf{x})}{\Delta(\mathbf{x})} = \prod_{\alpha>0} \frac{1-q^{-1}\mathbf{x}^\alpha}{1-\mathbf{x}^\alpha} \in \mathbb{C}[[\Lambda^+]]. \tag{11}$$

Up to a simple change of notation, (11) coincides with the right hand side of (10). Our goal is to express (11) as a sum over the infinite crystal $\mathscr{B}(\infty)$. This was done in type A by Bump–Nakasuji [4], and for all types by McNamara [14] and independently by Kim–Lee [11]. In type A these three results are equivalent, and take the form

$$\frac{D(\mathbf{x})}{\Delta(\mathbf{x})} = \sum_{\pi\in\mathscr{B}(\infty)} H(\pi)\mathbf{x}^\pi, \tag{12}$$

where $H(\pi)$ is the standard contribution of a Littelmann pattern. Our goal is now to prove the following theorem:

Theorem 5.2. *Let $\mathscr{B}(\infty)$ be the infinite crystal for $\Phi = G_2$. Then we have*

$$\frac{D(\mathbf{x})}{\Delta(\mathbf{x})} = \sum_{\pi\in\mathscr{B}(\infty)} \widehat{H}(\pi)\mathbf{x}^\pi, \tag{13}$$

where $\widehat{H}$ is defined in Definition 4.2.

Before we begin the proof, we need more notation. Recall that a *vector partition* on the positive roots Φ^+ is a function $\xi\colon \Phi^+ \to \mathbb{Z}_{\geq 0}$. Define the *index* $\iota(\xi)$ of a vector partition to be the number of $\alpha\in\Phi^+$ such that $\xi(\alpha)\neq 0$. Each vector partition determines a monomial $\mathbf{x}^\xi \in \mathbb{C}[\Lambda^+]$ by $\mathbf{x}^\xi = \mathbf{x}^\beta$, where

$$\beta = \beta(\xi) := \sum_{\alpha>0} \xi(\alpha)\alpha. \tag{14}$$

We sometimes abuse notation and write a vector partition as a sum as in (14).

Lemma 5.3. *We have*

$$\frac{D(\mathbf{x})}{\Delta(\mathbf{x})} = \sum_{\xi} (1-q^{-1})^{\iota(\xi)}\mathbf{x}^\xi,$$

where the sum is taken over all vector partitions on the positive roots.

Proof. This is a special case of [11, Theorem 1.6]. $\square$

Lemma 5.4. *There is a bijection between the G_2-Littelmann patterns satisfying the circling inequalities* (5) *and vector partitions on the positive roots for G_2 such that if π is taken to the the partition ξ, then $\mathbf{x}^\pi = \mathbf{x}^\xi$.*

Proof. Let $C = C_\infty \subset \mathbb{R}^6$ be the cone defined by (5). The simplicial cone C is generated by the points

$$v_1 = (0,0,0,0,0,1), v_2 = (1,0,0,0,0,0), v_3 = (1,1,0,0,0,0),\\ v_3' = (1,1,2,0,0,0), v_5 = (1,1,2,1,0,0), v_6 = (1,1,2,1,1,0), \tag{15}$$

and these are the primitive lattice points on the edges of C. One can check that C is not unimodular; that is, the sublattice of $\mathbb{Z}^6$ generated by the points (15) is not $\mathbb{Z}^6$, and is in fact a sublattice of index 2. One can decompose C as a union of unimodular cones $C_1 \cup C_2$ by including the point $v_4 = (1,1,1,0,0,0)$. In particular, we have

$$C_1 = \langle v_1, v_2, v_3, v_4, v_5, v_6 \rangle \quad \text{and} \quad C_2 = \langle v_1, v_2, v_3', v_4, v_5, v_6 \rangle.$$

Thus any lattice point in C can be uniquely written as a $\mathbb{Z}$-linear combination of the points $v_1, v_2, v_3, v_3', v_4, v_5, v_6$, where on C_1 (resp. C_2) we use all the v_i except v_3' (resp. v_3).

Number the roots as in Figure 1. We can use a lattice point $v \in C$ to determine a vector partition as follows:

- If $v \in C_1$, then $a_1v_1 + a_2v_2 + a_3v_3 + a_4v_4 + a_5v_5 + a_6v_6$ determines $a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 + a_4\alpha_4 + a_5\alpha_5 + a_6(\alpha_3 + \alpha_3')$.
- If $v \in C_2$, then $a_1v_1 + a_2v_2 + a_3'v_3' + a_4v_4 + a_5v_5 + a_6v_6$ determines $a_1\alpha_1 + a_2\alpha_2 + a_3'\alpha_3' + a_4\alpha_4 + a_5\alpha_5 + a_6(\alpha_3 + \alpha_3')$.

Hence lattice points in $C_1 \cap C_2$ correspond to partitions ξ such that $\xi(\alpha_3) = \xi(\alpha_3')$, whereas points in $C_1 \smallsetminus C_1 \cap C_2$ (resp. $C_2 \smallsetminus C_1 \cap C_2$) correspond to partitions such that $\xi(\alpha_3) > \xi(\alpha_3')$, (resp. $\xi(\alpha_3) < \xi(\alpha_3')$). It is easy to check that this correspondence is a bijection with the desired properties, which completes the proof of the lemma. $\square$

Proof of Theorem 5.2. The proof is by a direct computation with the contributions $\widehat{H}$ in Definition 4.2. Since the computation is easy but lengthy, we give the main points of the argument and leave the details to the reader.

The vertices of $\mathscr{B}(\infty)$ in the sum in (13) are parameterized by unboxed Littelmann patterns; the only requirement is that the entries of such a pattern satisfy the circling inequalities (5). By Lemmas 5.3 and 5.4, if for any such

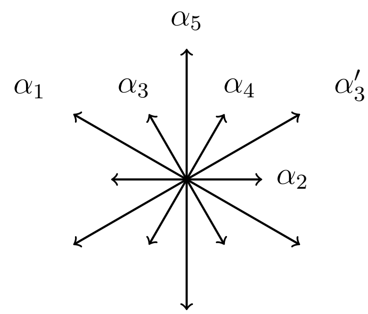

FIGURE 1.

pattern π we had $\widehat{H}(\pi) = (1-q^{-1})^{\iota(\xi)}$, where ξ is the vector partition attached to π in Lemma 5.4, then Theorem 5.2 would follow immediately.

Unfortunately this is not the case: for many patterns π the contribution $\widehat{H}(\pi)$ is quite different. The simplest example is the unboxed pattern $\pi = [1,1,1,0,0][0]$. According to Lemma 5.4, this corresponds to the vector partition $1 \cdot \alpha_4$. On the other hand, we have $\widehat{H}(\pi) = (1-q^{-1})^2$. Another example is provided by the pattern $\pi' = [1,1,2,1,1][0]$. We have $\widehat{H}(\pi') = (1-q^{-1})$, yet the vector partition is $1 \cdot \alpha_3 + 1 \cdot \alpha_3'$.

However, in some sense these two patterns, which correspond to the primitive generators of the rays $\mathbb{R}v_4$ and $\mathbb{R}v_6$, are the main difficulty: all the patterns whose contributions under Definition 4.2 and Lemma 5.4 disagree live in the 4-dimensional intersection $C' = C_1 \cap C_2 = \langle v_1, v_2, v_4, v_5, v_6 \rangle$, and involve the rays generated by v_4 and v_6 in an essential way.

More precisely, let us indicate the relative interiors of subcones of the intersection C' by subsets of $\{1,2,4,5,6\}$. Thus for instance $\{2,4,6\}$ means the subset of C' of the form $\{av_2 + bv_4 + cv_6 \mid a,b,c \in \mathbb{R}_{>0}\}$; we abbreviate the notation further by eliminating braces and commas and write simply 246. Then investigation of Definition 4.2 shows that the only subcones where (i) there is a discrepancy between $\widehat{H}(\pi)$ and $(1-q^{-1})^{\iota(\xi)}$, or (ii) $\widehat{H}(\pi) \neq H(\pi)$ are those that appear in Table 1. In this table a mark in row “vp” (resp. row “corr”) indicates possibility (i) (resp. possibility (ii)).

To complete the proof of the theorem, one must systematically go through Table 1 and check that the corrections in Definition 4.2 exactly compensate for the difference between $\widehat{H}(\pi)$ and $(1-q^{-1})^{\iota(\xi)}$. We illustrate this with the cones 4 and 6, which typify the process.

Consider the lattice points av_4 and bv_6, where $a, b \geq 1$. The patterns (ignoring the bottom row, which plays no role) are $\pi_4(a) := [a,a,a,0,0]$ and

$\pi_6(b) = [b, b, 2b, b, b]$. Suppose $a = 2b$ is even. Then $\pi_4(a)$ and $\pi_6(b)$ contribute to the same monomial, and their total contribution is $(1 - q^{-1}) + (1 - q^{-1})^2$, which is what one expects from Lemma 5.3. Now suppose $a = 2b + 1$ is odd. If $a = 1$, then there is an explicit correction in Definition 4.2 that sets $\widehat{H}(\pi_4(1)) = (1 - q^{-1})$. If $a > 1$, then the patterns $av_4, v_4 + bv_6$ and $b(v_2 + v_5) + v_4 \in 245$ all contribute to the same monomial. According to Defintion 4.2 the pattern av_4 contributes $(1-q^{-1})^2$, the pattern $v_4 + bv_6$ contributes $(1-q^{-1})^2$ as well, and $b(v_2+v_5)+v_4$ contributes $1-3q^{-1}+4q^{-2}-2q^{-3}$. Adding up all these contributions one obtains $3 - 7q^{-1} + 6q^{-2} - 2q^{-3}$. This exactly equals the contributions of these patterns one wants from Lemma 5.4. Indeed, when one computes the vector partitions and their indices, one finds that these patterns *should* respectively contribute $1 - q^{-1}$, $(1 - q^{-1})^3$, $(1 - q^{-1})^3$. Since $1 - q^{-1} + (1 - q^{-1})^3 + (1 - q^{-1})^3 = 3 - 7q^{-1} + 6q^{-2} - 2q^{-3}$ we have perfect agreement. Note this computation has simultaneously accounted for (i) the "vp" and "corr" rows under 4, (ii) the "vp" row under 6, (iii) the "corr" row under 46, and (iv) the "corr" row under 245 for those patterns in 245 of the form $rv_2 + sv_4 + tv_5$ with $r = t$.

The remaining computations to complete Table 1 are entirely similar. The most complicated case to check is the cone 245. There the correction to the pattern corresponding to $av_2 + bv_4 + cv_5$ depends on whether $a < c$, $a = c$, or $a > c$; as we saw above we have already accounted for $a = c$.

	4	6	24	26	45	46	56	245	246	256	456	2456
vp	•	•	•	•	•		•	•		•		
corr	•		•		•	•		•	•		•	•

TABLE 1. Patterns where either $\widehat{H}$ doesn't agree with the contribution computed from the bijection in Lemma 5.4 (indicated by "vp") or where $H \neq \widehat{H}$ (indicated by "corr").

□

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