A CHARACTERIZATION OF DYNKIN ELEMENTS

PAUL E. GUNNELLS AND ERIC SOMMERS

ABSTRACT. We give a characterization of the Dynkin elements of a simple Lie algebra. Namely, we prove that one-half of a Dynkin element is the unique point of minimal length in its N-region. In type A_n this translates into a statement about the regions determined by the canonical left Kazhdan-Lusztig cells.

1. INTRODUCTION

Let G be a connected simple algebraic group over the complex numbers and $\mathfrak{g}$ its Lie algebra. We begin by recalling the Dynkin-Kostant classification of nilpotent G-orbits in $\mathfrak{g}$ (see [4]).

Let $H = \left[\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right]$ and $E = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ be elements of $\mathfrak{sl}_2(\mathbf{C})$. Let $\mathcal{A}_{hom}$ denote the G-conjugacy classes of Lie algebra homomorphisms from $\mathfrak{sl}_2(\mathbf{C})$ to $\mathfrak{g}$. Then the map $\Omega : \mathcal{A}_{hom} \to \{\text{nilpotent } G\text{-orbits in } \mathfrak{g}\}$ given by $\Omega(\phi) = \mathrm{Ad}(G)\phi(E)$ is a bijection: surjectivity is just the Jacobson-Morozov theorem and injectivity follows from a theorem of Kostant. Let $\Upsilon : \mathcal{A}_{hom} \to \{\text{semi-simple } G\text{-orbits in } \mathfrak{g}\}$ be the map $\Omega(\phi) = \mathrm{Ad}(G)\phi(H)$. A theorem of Mal'cev shows that Υ is injective. Hence nilpotent orbits in $\mathfrak{g}$ are parametrized by the image of Υ. This finite set was completely determined by Dynkin [5].

If $\mathcal{O}$ is a nilpotent orbit, then an element in the conjugacy class $\Upsilon(\Omega^{-1}(\mathcal{O}))$ is called a *Dynkin element* for $\mathcal{O}$ (or for any element $e \in \mathcal{O}$). If we fix a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ and a set of simple roots $\Pi \subset \mathfrak{h}^*$, then there is a unique Dynkin element h for $\mathcal{O}$ which lies in $\mathfrak{h}$ and such that $\alpha(h) \geq 0$ for all $\alpha \in \Pi$. We often refer to this element as **the** Dynkin element for $\mathcal{O}$; the numbers $\alpha(h)$ are assigned to the Dynkin diagram of G and yield the weighted Dynkin diagram of $\mathcal{O}$. To summarize, this diagram completely determines the Dynkin elements for $\mathcal{O}$ and hence $\mathcal{O}$ itself.

The aim of this paper is to give a new characterization of the Dynkin elements. We begin by partitioning the dominant Weyl chamber into regions indexed by nilpotent orbits (which we call N-regions). These regions arise by generalizing in the most straightforward fashion the notion of S-cells (also called geometric cells) from the finite Weyl group to the affine Weyl group; they are closely related to Lusztig's $\tilde{S}$-cells [12]. Our main theorem states that the point of minimal length in the closure of the N-region corresponding to an orbit $\mathcal{O}$ is one-half the Dynkin element for $\mathcal{O}$. In type A_n, the closure of an N-region coincides with the region determined by a canonical left Kazhdan-Lusztig cell, so our theorem becomes a statement about Kazhdan-Lusztig cells.

The Dynkin elements for G are important for the representation theory of an algebraic group of type dual to G over a local field. This is a piece of conjectures by Arthur as

Date: 25 November 2002.

explained in Vogan [19]. Vogan's reformulation of Arthur's conjecture posits that one-half of a Dynkin element for a group of type dual to the type of G (viewed as a character of a maximal torus T of G) gives rise to a spherical representation of G which is unitary. This has been proved in some cases: see [19] for references. The main theorem of this paper suggests a generalization about where to look for other semi-simple conjugacy classes in $\mathfrak{g}$ with this property. We suspect these elements will be important in representation theory.

Here is a brief outline of the paper. In section 2, we introduce definitions and state the main theorem, Theorem 2.1. In section 3, we give a proof. In section 4, we make a definition of a property relating Dynkin elements to B-stable ideals in the nilradical of a Borel subalgebra. We use this definition to give an elementary proof of the main theorem in type A_n (but we suspect the proof works in all types). In the final section, we speculate on some generalizations of the main theorem to Kazhdan-Lusztig cells and $\tilde{S}$-cells for G of general type.

We thank J. Humphreys, G. Lusztig, I. Mirkovic, A. Ram, and D. Vogan, Jr. for helpful discussions. We gratefully acknowledge the support of NSF grants DMS-0245580 (P.G.) and NSF grants DMS-0201826 and DMS-9729992 (E.S.).

2. Main Theorem

Let $(\Phi, X, \Phi^\vee, Y)$ be the root datum of a simple algebraic group G defined over the complex numbers with respect to a maximal torus T. Let W be the Weyl group. Fix a Borel subgroup B containing T and let Φ^+ be the determined positive roots and Π the simple roots. Let $\mathfrak{g}, \mathfrak{b}, \mathfrak{h}$ be the Lie algebras of G, B, T, respectively. Let $V = Y \otimes \mathbf{R}$. We make V into a Euclidean space by fixing a W-invariant positive-definite inner product $(\ ,\ )$, with the induced norm denoted $\|\cdot\|$. The inner product is unique up to a non-zero scalar, since V is an irreducible representation of W. The closure of the dominant Weyl chamber in V is

$$\mathfrak{C} := \{x \in V \mid \langle \alpha, x\rangle \geq 0 \text{ for all } \alpha \in \Phi^+\},$$

where $\langle\ ,\ \rangle$ is the extension of the pairing of weights and coweights.

Inspired by the work of Shi [14], [15], we partition V into regions, called ST-regions (ST stands for sign type), as follows. Let $x \in V$. For each positive root $\alpha \in \Phi^+$, let X_α equal $+, 0$, or $-$, according to whether $\langle \alpha, x\rangle \geq 1$, $0 \leq \langle \alpha, x\rangle < 1$, or $\langle \alpha, x\rangle < 0$, respectively. The set of values $\{X_\alpha \mid \alpha \in \Phi^+\}$ which arise in this fashion is called an admissible sign type. Now V can be partitioned according to sign type (we drop the word admissible): two elements $x, y \in V$ belong to the same ST-region if they determine the same sign type. It is clear that the interior of an ST-region coincides with a connected component of the hyperplane arrangement studied by Shi [15].

In this note we are interested in the dominant sign types (and corresponding ST-regions). These are the sign types with no minus signs; the corresponding regions lie in $\mathfrak{C}$. The dominant sign types have an important interpretation in terms of the Lie algebra $\mathfrak{g}$ of G. Namely, let $\mathfrak{g}_\alpha \subset \mathfrak{g}$ be the root space for $\alpha \in \Phi$. Then for each dominant sign type $\mathcal{S} = \{X_\alpha\}$ the subspace

$$\mathfrak{n}_{\mathcal{S}} := \bigoplus_{X_\alpha = +} \mathfrak{g}_\alpha$$

is a B-stable ideal in the nilradical $\mathfrak{n}$ of $\mathfrak{b}$; and every such ideal arises in this way. The second statement above can be traced back to Shi's characterization of admissible sign types [15]; more recently, it was proved in a different manner by Cellini-Papi [3].

We now define a partition of $\mathfrak{C}$ into regions called N-regions indexed by the nilpotent orbits in $\mathfrak{g}$; these are closely related to Lusztig's $\tilde{S}$-cells in the affine Weyl group (see the last section). Each N-region will be a union of certain ST-regions. Let $\mathcal{S}$ be a dominant sign type and let $\mathfrak{n}_{\mathcal{S}}$ be the corresponding B-stable ideal. Let $\mathcal{O}_{\mathcal{S}}$ denote the unique nilpotent orbit of $\mathfrak{g}$ such that $\mathcal{O}_{\mathcal{S}} \cap \mathfrak{n}_{\mathcal{S}}$ is dense in $\mathfrak{n}_{\mathcal{S}}$. Given a nilpotent orbit $\mathcal{O}$ we define the N-region indexed by $\mathcal{O}$ to be

$$N_{\mathcal{O}} := \{x \in \mathfrak{C} \mid \mathcal{O}_{\mathcal{S}_x} = \mathcal{O}\}$$

where $\mathcal{S}_x$ is the sign type determined by x.

We now recall the definition of the Dynkin elements of G (with respect to T and B). Given a nilpotent orbit $\mathcal{O}$, let $e \in \mathcal{O}$ and let $\{e, h, f\}$ be an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$ with $h \in \mathfrak{h} = \mathrm{Lie}(T)$. Identifying V with the real subspace of $\mathfrak{h}$ spanned by the coroots $\Phi^\vee$, we always have $h \in V$ (h, in fact, lies in the integral lattice generated by the coroots; this follows from the representation theory of $\mathfrak{g}$ and of $\mathfrak{sl}_2$). By altering our choice of e, we can ensure that $h \in \mathfrak{C}$. Then as noted in the introduction, h depends only on $\mathcal{O}$ and is called the Dynkin element for $\mathcal{O}$.

Our main result is

Theorem 2.1. *Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{g}$. Let $N_{\mathcal{O}}$ be the corresponding N-region in $\mathfrak{C}$. Let x be a point of minimal Euclidean length in the closure of $N_{\mathcal{O}}$. Then $2x$ is the Dynkin element for $\mathcal{O}$ (in particular, $N_{\mathcal{O}}$ is non-empty and x is unique).*

3. Proof

The theorem follows easily from the next two propositions. If $h \in \mathfrak{C}$ is the Dynkin element for the orbit $\mathcal{O}$, we call

$$\mathfrak{n}_h = \oplus_{i \geq 2} \mathfrak{g}_i$$

the **Dynkin ideal** of $\mathcal{O}$, where $\mathfrak{g}_i$ is the i-eigenspace for the action of $\mathrm{ad}(h)$ on $\mathfrak{g}$. This is a B-stable ideal in $\mathfrak{n}$.

Proposition 3.1. *Let $h \in \mathfrak{C}$ be the Dynkin element for the orbit $\mathcal{O}$. Then $\frac{1}{2}h$ lies in $N_{\mathcal{O}}$.*

Proof. Let $\mathcal{S}$ be the sign type for $\frac{1}{2}h$. Then clearly $\mathfrak{n}_{\mathcal{S}} = \mathfrak{n}_h$ from the definitions. The result follows from the well-known fact that $\mathcal{O} \cap \mathfrak{n}_h$ is dense in $\mathfrak{n}_h$ [2]. □

Fix a dominant sign type $\mathcal{S}$ and let $\mathfrak{n}_{\mathcal{S}}$ be the corresponding ideal.

Proposition 3.2. *Let $e \in \mathfrak{n}_{\mathcal{S}}$ and let $h \in \mathfrak{C}$ be the Dynkin element of the G-orbit through e. Then $\|\frac{1}{2}h\| \leq \|x\|$ for any x in the closure of the ST-region corresponding to $\mathcal{S}$.*

Proof. The proof will be a slight modification of an argument due to Barbasch and Vogan ([1] Lemma 5.7). Since all W-invariant positive-definite inner products $(\ , \ )$ on V are unique up to a non-zero scalar, we may as well assume that $(h, \alpha^\vee)$ is an integer for all $\alpha^\vee \in \Phi^\vee$; this is possible since h lies in the coroot lattice. Let $\lambda \in \mathfrak{h}^*$ be the unique element such that $\lambda(x) = (h, x)$ for all $x \in V$. Then λ is integral and it is also dominant since $h \in \mathfrak{C}$.

Let (F_λ, π_λ) be the irreducible finite-dimensional representation of $\mathfrak{g}$ of highest weight λ. The lowest weight of F_λ is $w_0(\lambda)$ where w_0 is the longest element of W (with respect to the simple reflections defined by Π). Let $\mathfrak{s} = \{e', h, f\}$ be an $\mathfrak{sl}_2(\mathbf{C})$-triple with $e' \in \mathcal{O}_e$. Restricting F_λ to $\mathfrak{s}$, we see that the irreducible subrepresentation of $\mathfrak{s}$ of largest dimension has dimension $m+1$ where $m = \lambda(h)$ (as $h \in \mathfrak{C}$). This us allows to determine the index of nilpotency of $\pi_\lambda(e')$ on F_λ: we have

$$\pi_\lambda(e')^m \neq 0, \text{ but } \pi_\lambda(e')^{m+1} = 0.$$

Because e and e' are G-conjugate, the same holds for $\pi_\lambda(e)$. The key point, to be exploited shortly, is that $m = \lambda(h) = \|h\|^2$.

On the other hand, let x be in the ST-region for $\mathcal{S}$ (or in fact, its closure). Viewing $x \in \mathfrak{h}$, we can filter F_λ by the eigenspaces of the semisimple operator $\pi_\lambda(x)$ with corresponding eigenvalues greater than a fixed eigenvalue. The length of this filtration is $\lambda(x) - w_0\lambda(x)$, as $x \in \mathfrak{C}$.

Now if $\alpha \in \Phi^+$ satisfies $\alpha(x) \geq 1$, then any non-zero e_α in $\mathfrak{g}_\alpha$ acts on F_λ by raising degrees by at least 1. Write $e = \sum e_\alpha$ where $e_\alpha \in \mathfrak{g}_\alpha$. The fact that $e \in \mathfrak{n}_\mathcal{S}$ and x belongs to the closure of the ST-region for $\mathcal{S}$ means that $\alpha(x) \geq 1$ for all α appearing in the sum for e. Consequently e acts by raising degrees by at least 1. It follows that $\pi_\lambda(e)^{M+1} = 0$ where $M \leq \lambda(x) - w_0\lambda(x)$. The latter quantity is bounded by $2\|h\|\|x\|$ by Cauchy-Schwartz and the fact that $\|w_0 h\| = \|h\|$.

Hence $m \leq M \leq 2\|h\|\|x\|$. It follows that $\|h\|^2 \leq 2\|h\|\|x\|$ or $\|\frac{1}{2}h\| \leq \|x\|$. $\square$

The proof of the theorem follows: every element in the closure of $N_\mathcal{O}$ has size at least the size of one-half the Dynkin element of $\mathcal{O}$ and this minimum is achieved at one-half the Dynkin element. Uniqueness is a consequence of the fact that the closure of $N_\mathcal{O}$ is convex.

We have the following corollary:

Corollary 3.3. *If $\mathcal{O}'$ lies in the closure of the nilpotent orbit $\mathcal{O}$, then the size of the Dynkin element of $\mathcal{O}'$ is strictly smaller than the size of the Dynkin element of $\mathcal{O}$.*

Proof. Let $\mathfrak{n}_h = \oplus_{i \geq 2}\mathfrak{g}_i$ be the Dynkin ideal for the Dynkin element h of $\mathcal{O}$. The G-saturation of $\mathfrak{n}_h$ is by definition the set of elements $\mathrm{Ad}(g)e$ where $g \in G$ and $e \in \mathfrak{n}_h$; this is exactly the closure of $\mathcal{O}$. Hence $\mathcal{O}'$ intersects $\mathfrak{n}_h$ and the result follows from the previous proposition applied to the sign type determined by $\frac{1}{2}h$. $\square$

4. Property D

We can get another proof of the main theorem if a certain property holds for every B-stable ideal I of $\mathfrak{n}$. Let $\mathfrak{n}_\mathcal{S}$ be the ideal corresponding to the dominant sign-type $\mathcal{S}$. Let $\mathcal{O}_\mathcal{S}$ be the associated nilpotent orbit; that is, the one for which $\mathcal{O}_\mathcal{S} \cap \mathfrak{n}_\mathcal{S}$ is dense in $\mathfrak{n}_\mathcal{S}$.

Definition 4.1. We say $\mathfrak{n}_\mathcal{S}$ has **property D** if there exists $e \in \mathcal{O}_\mathcal{S} \cap \mathfrak{n}_\mathcal{S}$ and $h \in \mathfrak{b}$ such that e and h can be extended to an $\mathfrak{s}l_2(\mathbf{C})$-triple $\{e, h, f\}$. We say $\mathfrak{n}_\mathcal{S}$ has **strong property D** if the above holds for any orbit $\mathcal{O}$ which intersects $\mathfrak{n}_\mathcal{S}$.

We suspect that every ideal possesses strong property D. Note that it is not true in general that for every element $e \in \mathfrak{n}_\mathcal{S}$ there exists $h \in \mathfrak{b}$ such that e and h can be extended to an $\mathfrak{s}l_2(\mathbf{C})$-triple.

Lemma 4.2. *Property D holds for every Dynkin ideal.*

Proof. Let h be the Dynkin element for the orbit $\mathcal{O}$ and $\mathfrak{g} = \oplus \mathfrak{g}_i$ be the decomposition of $\mathfrak{g}$ into eigenspaces for $\mathrm{ad}(h)$. Then $\mathfrak{n}_h$ is the nilradical of the parabolic subalgebra $\mathfrak{p} = \oplus_{i \geq 0} \mathfrak{g}_i$ with corresponding subgroup denoted P. Clearly there is $e \in \mathfrak{n}_h \cap \mathcal{O}$ and $h \in \mathfrak{p}$ which extend to an $\mathfrak{sl}_2(\mathbf{C})$-triple. By the conjugacy of Cartan subalgebras, there exists $p \in P$ such that $\mathrm{Ad}(p)h \in \mathfrak{h}$. Since $\mathfrak{n}_h$ is stable under the action of P, $\mathrm{Ad}(p)e$ and $\mathrm{Ad}(p)h$ do the trick. □

Let I and I' be two B-stable ideals in $\mathfrak{n}$. Suppose that $I = I' \oplus \mathfrak{g}_\beta$ for some $\beta \in \Phi^+$. For $\alpha \in \Pi$, let P_α denote the minimal parabolic subgroup containing B whose Lie algebra contains $\mathfrak{g}_{-\alpha}$. Assume that I is invariant under the action of P_α. It is not hard to see that $\langle \beta, \alpha^\vee \rangle \leq 0$.

Lemma 4.3. *Assume that $\langle \beta, \alpha^\vee \rangle < 0$ above. Then if property D (or the strong version) holds for I, it also holds I'.*

Proof. Let $e \in I$ and $h \in \mathfrak{b}$ satisfy the conditions for strong property D for the orbit $\mathcal{O}_e$. Since h lies in some Cartan subalgebra of $\mathfrak{b}$, we may assume $h \in \mathfrak{h}$ since all Cartan subalgebras of $\mathfrak{b}$ are conjugate under B and I is B-stable.

Let L_α be the Levi subgroup of P_α containing T. Consider the representation of L_α on I and let M be the irreducible constituent containing $\mathfrak{g}_\beta$: it has lowest weight β and highest weight $\beta - \langle \beta, \alpha^\vee \rangle \alpha$ (and these are distinct by the hypothesis).

Write $e = \sum e_\gamma$, a sum of non-zero root vectors. If $\gamma = \beta$ does not appear in the sum, we are done (since then $e \in I'$). If $\gamma = \beta - \langle \beta, \alpha^\vee \rangle \alpha$ does not appear in the sum, we proceed by letting $n \in N_G(T)$ represent the simple reflection $s_\alpha \in W$. Then $\mathrm{Ad}(n)e \in I'$ and $\mathrm{Ad}(n)h \in \mathfrak{h}$ and we are done. Finally if both the highest weight and lowest weight for M appear in the sum for e, we must have $\alpha(h) = 0$ (as $[h, e] = 2e$) and thus h lies in the center of the Lie algebra of L_α. From the representation theory of $SL_2(\mathbf{C})$ (with respect to the representation M), there exists $g \in L_\alpha$ such that $\mathrm{Ad}(g)e \in I'$. In this case, $\mathrm{Ad}(g)h = h$ and the proof is complete. □

The proof, in particular, implies that I and I' have the same associated orbit. The idea for considering two ideals related in the above fashion is closely connected to an equivalence relation on ideals defined in [18].

Proposition 4.4. *When G is of type A_n, strong property D holds for every B-stable ideal in $\mathfrak{n}$.*

Proof. Let P be a parabolic subgroup containing B and $\mathfrak{n}_P$ the nilradical of its Lie algebra. It is not difficult (G is of type A) to show that $\mathfrak{n}_P$ has strong property D by exhibiting explicit elements $e \in \mathcal{O}$ in a standard form for each orbit $\mathcal{O}$ in the closure of the Richardson orbit for P. Of course, this uses the classification of nilpotent orbits in type A and knowledge of the closure relations.

Then the first theorem in [18] and the previous lemma imply that strong property D holds for every B-stable ideal in $\mathfrak{n}$. If we wanted only property D, Lemma 4.2 would allow us to avoid the classification of nilpotent orbits. □

We have also checked that property D (and strong property D) holds in many cases (including G_2, F_4, E_6, and E_7). We suspect strong property D holds in all types. Our purpose in introducing and studying property D is the next proposition.

Proposition 4.5. *Let $\mathcal{S}$ be a dominant sign type. If $\mathfrak{n}_{\mathcal{S}}$ possesses strong property D, then Proposition 3.2 has another proof.*

Proof. Let $e \in \mathfrak{n}_{\mathcal{S}}$ and $h \in \mathfrak{b}$ satisfy the conditions of property D for the orbit $\mathcal{O}_e$; let $f \in \mathfrak{g}$ be such that $\{e, h, f\}$ is an $\mathfrak{sl}_2(\mathbf{C})$-triple. We may assume $h \in \mathfrak{h}$ as in Lemma 4.3.

Write $e = \sum e_\alpha$ where $e_\alpha \in \mathfrak{g}_\alpha$ and denote by $\mathrm{supp}(e)$ the set of α which appear in this sum. Clearly, $\alpha(\frac{1}{2}h) = 1$ for all $\alpha \in \mathrm{supp}(e)$.

First, we claim that h lies in the (complex) span of the coroots $\alpha^\vee$ such that $\alpha \in \mathrm{supp}(e)$. This is easy: write $f = \sum e_\beta$. Then $[e, f] = h$ implies that only the terms in f with $\beta = -\alpha$ for some $\alpha \in \mathrm{supp}(e)$ yield non-zero brackets with terms in e and these terms are multiples of $\alpha^\vee$, as desired. It follows that h lies in the real span of these coroots, since h in fact lies in the lattice generated by the coroots, as noted earlier.

Define the following convex region in V:

$$R_e = \{v \in V \mid \alpha(v) \geq 1 \text{ for all } \alpha \in \mathrm{supp}(e)\}.$$

Since $e \in \mathfrak{n}_{\mathcal{S}}$, R_e contains the closure of the ST-region for $\mathcal{S}$. Hence if we prove the inequality in Proposition 3.2 for R_e, then we are done.

Since R_e is convex, it is clear that any point of minimal length must satisfy $\alpha(x) = 1$ for all $\alpha \in \mathrm{supp}(e)$. Namely, connect a putative minimum point x by a line to the origin; it must cross the hyperplane $H_{\alpha,1}$ where α takes the value 1. We will deduce that $\frac{1}{2}h$ is the point of minimal length of R_e by Lagrange multipliers with several constraints (it is clear that $\frac{1}{2}h \in R_e$).

We have reduced the problem to minimizing the function $g(x) = \|x\|$, subject to the constraint that x lies on $H_{\alpha,1}$ for $\alpha \in \mathrm{supp}(e)$. Now the gradient of g is a constant multiple of x and the gradient of the linear equation defining $H_{\alpha,1}$ is a constant multiple of $\alpha^\vee$. Hence by Lagrange multipliers, the minimum is achieved when x lies in the real span of the $\alpha^\vee$ with $\alpha \in \mathrm{supp}(e)$ and x satisfies $\alpha(x) = 1$ for all $\alpha \in \mathrm{supp}(e)$. Since $\frac{1}{2}h$ has these properties, $\frac{1}{2}h$ is a point of minimal length (it is unique, since R_e is convex). ☐

If we knew only that property D was true for all B-stable ideals of $\mathfrak{n}$, then we would have another proof of the main theorem although not its corollary.

5. Relation to cells and generalizations

Our main result can be re-phrased in type A_n by the work of Shi and Lusztig. Let $\mathcal{C}$ be a canonical left Kazhdan-Lusztig cell in the affine Weyl group W_a [13]. These cells are indexed by nilpotent orbits for the group dual to G; this is Lusztig's parametrization of two-sided cells [11].

Let θ be the highest root of Φ^+. Let A be the fundamental alcove for W_a defined by

$$A := \{x \in V \mid \alpha(x) > 0 \text{ for } \alpha \in \Pi,\ \theta(x) < 1\}.$$

Define $\bar{\mathcal{C}}$ to be the closure of

$$\{w(A) \mid w \in \mathcal{C}\}$$

where the action of W_a on alcoves is the one in [9]. In particular, $\bar{\mathcal{C}} \subset \mathfrak{C}$.

By the classification of left Kazhdan-Lusztig cells in type A_n (due to Shi [14] and Lusztig [10]) and by a comment of Lusztig in [10] (proved by Lawton in [8]), $\bar{\mathcal{C}}$ exactly coincides with the closure of $N_{\mathcal{O}}$ where $\mathcal{O}$ is the orbit dual to the one parametrizing $\mathcal{C}$.

We remark that the Lawton-Lusztig result is closely related to knowing that property D holds in type A_n and that, in fact, property D can be deduced from it.

Our main theorem now becomes:

Theorem 5.1. *Let $\mathcal{O}$ be a nilpotent orbit in $\mathfrak{sl}_n(\mathbf{C})$ and let $\mathcal{C}$ be the canonical left cell corresponding to the dual orbit. The element of minimal length in $\bar{\mathcal{C}}$ is exactly one-half the Dynkin element for $\mathcal{O}$.*

Already in B_2 and G_2 when $\mathcal{C}$ is a canonical left cell, it is no longer true that $\bar{\mathcal{C}}$ is a union of the closures of some ST-regions. Nevertheless, the elements of minimal length in $\bar{\mathcal{C}}$ are interesting. We can find them in rank 2 by looking at the pictures in [9].

Example 5.2. In type B_2, the elements are (written as a weighted diagram– see the introduction – with the first simple root being long): $(0,0)$, $(0,\frac{1}{2})$, $(1,\frac{1}{2})$, $(1,1)$, coming from the cell attached by Lusztig to the regular, the subregular, the minimal, and the zero orbit, respectively. Only the third element is not one-half a Dynkin element. By passing to the dual side, we can consider this element as a character for a dual maximal torus (as in the introduction); it yields an irreducible highest weight module whose annihilator is the Joseph ideal [6].

Example 5.3. In type G_2, the elements are (written as a weighted diagram, with the first simple root being long): $(0,0)$, $(\frac{1}{2},0)$, $(\frac{1}{2},\frac{1}{2})$, $(1,\frac{1}{3})$, $(1,1)$, coming from the cell attached to the regular, the subregular, the middle, the minimal, and the zero orbit respectively. The third and fourth elements are not one-half of a Dynkin element. Passing to the dual group, the fourth element yields an irreducible highest weight module whose annihilator is the Joseph ideal [6]. In a similar fashion, the third element yields a primitive ideal which Joseph showed was completely prime and whose associated variety was the middle orbit [7].

In the rest of the paper, G returns to being of general type. The above examples lead us to conjecture

Conjecture 5.4. *The element of minimal length in $\bar{\mathcal{C}}$ is interesting where $\mathcal{C}$ is a canonical left Kazhdan-Lusztig cell. In particular, on the dual side it yields a completely prime maximal primitive ideal whose associated variety is the orbit which indexes $\mathcal{C}$.*

In particular when G is not of type A_n, the highest weight yielding the Joseph ideal arises as the point of minimal length in the canonical left cell region corresponding to the minimal orbit.

It also seems likely that the element of minimal length in $\bar{\mathcal{C}}$ is one-half a Dynkin element if and only if the orbit ${}^L\mathcal{O}$ indexing $\mathcal{C}$ is special; in which case, it seems likely that the minimal element should be one-half the Dynkin element for the smallest orbit in $\mathfrak{g}$ which maps under Lusztig-Spaltenstein duality to ${}^L\mathcal{O}$.

In [12], Lusztig defined the notion of $\tilde{S}$-cells in W_a. They are parametrized by pairs $(\mathcal{O}, C)$ where $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}$ and C is a conjugacy class in $A(\mathcal{O})$ (the fundamental group of $\mathcal{O}$). Let $\mathcal{D}$ be an $\tilde{S}$-cell and let $\bar{\mathcal{D}} \subset V$ be the closure of

$$\{w(A) \mid w \in \mathcal{D} \text{ and } w(A) \subset \mathfrak{C}\}.$$

We call $\bar{\mathcal{D}}$ a (canonical left) $\tilde{S}$-region. It is not too hard to see that the closure of an N-region is a union of $\tilde{S}$-regions. More explicitly, if $\mathcal{O}$ is a nilpotent orbit in $\mathfrak{g}$,

$$N_{\mathcal{O}} = \bigcup \bar{\mathcal{D}}_{(\mathcal{O},C)}$$

where the union is over all conjugacy classes C in $A(\mathcal{O})$. It follows from our main theorem that one-half the Dynkin element of $\mathcal{O}$ is the minimal element of $\bar{\mathcal{D}}_{(\mathcal{O},C)}$ for some conjugacy class $C \subset A(\mathcal{O})$ (in fact, $C = 1$ but we omit the proof). This leads us to conjecture

Conjecture 5.5. *The element of minimal length in a canonical left $\tilde{S}$-region yields (on the dual side) a completely prime maximal primitive ideal.*

In types B_2 and G_2 the only elements we obtain by looking at $\tilde{S}$-regions which are not one-half a Dynkin element are the elements listed above coming from the Kazhdan-Lusztig cells (we found this from the B_2-picture in [12] and calculating the canonical left $\tilde{S}$-cells in G_2). This leads us to wonder if the elements from the first conjecture are a subset of the elements coming from the second conjecture. This would follow from a conjecture of the second author [17] which we now describe.

Let $\mathcal{N}_{o,c}$ denote the set of all pairs $(\mathcal{O}, C)$ as above. Let ${}^L\mathcal{N}_o$ denote the nilpotent orbits for the dual group ${}^LG$ in ${}^L\mathfrak{g}$. Let $d : \mathcal{N}_{o,c} \to {}^L\mathcal{N}_o$ be the map from [16].

Conjecture 5.6. *Let $\mathcal{C}$ be the canonical left Kazhdan-Lusztig cell indexed by the orbit ${}^L\mathcal{O} \subset {}^L\mathfrak{g}$. Then*

$$\bar{\mathcal{C}} = \bigcup \bar{\mathcal{D}}_{(\mathcal{O},C)}$$

where the union is over all pairs $(\mathcal{O}, C)$ such that $d_{(\mathcal{O},C)} = {}^L\mathcal{O}$.

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UNIVERSITY OF MASSACHUSETTS—AMHERST, AMHERST, MA 01003
E-mail address: gunnells@math.umass.edu

E-mail address: esommers@math.umass.edu