COHOMOLOGY OF CONGRUENCE SUBGROUPS OF $\mathrm{SL}_4(\mathbb{Z})$

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Abstract. Let $N > 1$ be an integer, and let $\Gamma = \Gamma_0(N) \subset \mathrm{SL}_4(\mathbb{Z})$ be the subgroup of matrices with bottom row congruent to $(0, 0, 0, *) \mod N$. We compute $H^5(\Gamma; \mathbb{C})$ for a range of N, and compute the action of some Hecke operators on many of these groups. We relate the classes we find to classes coming from the boundary of the Borel-Serre compactification, to Eisenstein series, and to classical holomorphic modular forms of weights 2 and 4.

1. Introduction

1.1. Let $n \geq 1$, and let Γ be a congruence subgroup of $\mathrm{SL}_n(\mathbb{Z})$ of level N. Let S_N be the subsemigroup of the integral matrices in $\mathrm{GL}_n(\mathbb{Q})$ such that (Γ, S_N) is a Hecke pair.

We denote by $\mathscr{H}(N)$ the $\mathbb{C}$-algebra of double cosets $\Gamma S_N \Gamma$. This algebra acts on the cohomology and homology of Γ with any coefficient $\mathbb{C}S_N$-module. When a double coset is acting, we call the map defined by its action a Hecke operator. In this paper we will work only with the trivial coefficient module $\mathbb{C}$.

Let l be a prime not dividing N, and let $D(l, k)$ be the diagonal matrix with diagonal $(1, \ldots, 1, l, \ldots, l)$, where the number of l's is k. Then $\mathscr{H}(N)$ contains all double cosets of the form $\Gamma D(l, k)\Gamma$, and we denote the corresponding Hecke operator by $T(l, k)$. Fix a prime p not dividing N, and an embedding of $\mathbb{Q}_p$ into $\mathbb{C}$. Let $G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$.

Definition 1.2. Let $\mathcal{V}$ be an $\mathscr{H}(N)$-module and suppose $\beta \in \mathcal{V}$ is an eigenclass for the action of $\mathscr{H}(N)$. For l prime to N, write $T(l, k)(\beta) = a(l, k)\beta$, where the $a(l, k) \in \mathbb{C}, k = 0, \ldots, n$ are algebraic integers. Let ρ be a continuous semisimple representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_n(\mathbb{Q}_p)$, unramified outside pN, such that

$$\sum_k (-1)^k l^{k(k-1)/2} a(l, k) X^k = \det(I - \rho(\mathrm{Frob}_l)X) \tag{1}$$

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for all l not dividing pN. Then we shall say that ρ is *attached* to β.

For example, theorems of Eichler, Shimura and Deligne imply that if $n = 2$ and $\mathcal{V}$ is the Hecke-module of classical holomorphic modular cuspforms for $\mathrm{SL}_2(\mathbb{Z})$, then there always exists a ρ attached to any Hecke eigenform $\beta \in \mathcal{V}$.

Standard conjectures (for example in [15]) state that if $\mathcal{V}$ is the cuspidal cohomology of Γ with trivial coefficients, then any Hecke eigenclass should have an attached p-adic Galois representation. In fact, one has the stronger conjecture that there should be an attached motive (cf. [15]). We can also extend the conjecture of [15] to include all the cohomology—in principle, the theory of Eisenstein series should allow one to reduce the extended conjecture to the one for cuspidal cohomology.

In a series of papers [6, 8, 1, 11], the first author with a number of coworkers has tested this conjecture and a mod p variant of it when $n = 3$. Other tests for $n = 3$ can be found in the work of van Geemen and Top with van der Kallen and Verberkmoes [27, 28, 30, 29]. The purpose of this paper is to make the first computational tests of this conjecture for $n = 4$.([1])

1.3. We work here with $\Gamma = \Gamma_0(N)$, defined as the subgroup of $\mathrm{SL}_n(\mathbb{Z})$ consisting of the matrices with last row congruent to $(0, \ldots, 0, *)$ modulo N.

Conceptually, there is no big difference in computing the cohomology of Γ between the cases of $n = 3$—or even $n = 2$—and the case of general n. One needs to write down a simplicial complex C homotopic to the chains of a $B\Gamma(1)$-space, and compute the cohomology.

When $n = 2$ or 3, the most interesting part of cohomology, namely the cuspidal part, occurs in the top dimension of C. Therefore to compute it all we have to compute is the cokernel of a coboundary map. However, when $n = 4$ the cuspidal cohomology occurs in dimensions 4 and 5, whereas the virtual cohomological dimension of Γ, and hence smallest possible dimension of C, is 6. So now the cohomology is a subquotient, which adds considerably to the complexity of the computer programs. The cuspidal cohomology in degree 4 is dual to that in degree 5, so we concentrate on computing the latter in this paper.

But the big difference between $n = 2, 3$ and $n = 4$ occurs when we try to compute the Hecke action on the cohomology. In the top dimension we can use the Ash-Rudolph algorithm or its variants [9], as was done for $n = 3$ in the works cited above. However, for $n = 4$, where we look just below the top dimension, a brand new idea was necessary. This is due to the second author, and is the subject of [18]. Thus this paper is also a test of the algorithms proposed in [18], and they pass with flying colors. It is an open question whether the algorithms of [18] terminate in a finite number of steps; in practice, though, they always terminate quickly, and we used them here without problems.

[1] One goal is to find examples of *nonselfdual* cuspidal cohomology classes. On the Galois side, J. Scholten has constructed such representations in the etale cohomology of threefolds [23].

Our computations of $H^5(\Gamma_0(N);\mathbb{C})$, detailed in §5, were made for all $11 \leq N \leq 53$, as well as some additional odd $N \leq 73$.([2]) No cusp forms were discovered, but some interesting phenomenology of the boundary cohomology was observed, as discussed in §6. This leads to some open questions about the cohomology of the boundary, which are discussed there. Since we have not completed the Hecke computations for some high levels, it is possible that we do have a cusp form that we haven't yet identified as such. There is a bound due to Fermigier [16] that states that if $N < 31$, there cannot be any cuspidal cohomology.

For high levels our computations produced large sparse matrices, as large as 110464×30836 for level 48. To perform linear algebra with these matrices, we used a version of the Lanczos algorithm mod p, in the spirit of LaMacchia-Odlyzko [19] (see also [25]).

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2. Background

2.1. Let V be the $\mathbb{R}$-vector space of all symmetric $n \times n$ matrices, and let $C \subset V$ be the cone of positive-definite matrices. Then the group $G = \mathrm{SL}_n(\mathbb{R})$ acts on C on the left by $(g, c) \mapsto g \cdot c \cdot g^t$, and the stabilizer of any given point is isomorphic to SO_n.

Let X be C mod homotheties. The G-action on C commutes with the homotheties and induces a transitive G-action on X. The stabilizer of any given point of X is again isomorphic to SO_n. After choosing a basepoint, we may identify X with the global riemannian symmetric space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SO}_n$, a contractible, noncompact, smooth manifold of real dimension $d = n(n+1)/2 - 1$.

The group $\mathrm{SL}_n(\mathbb{Z})$ acts on X via the G-action, and does so properly discontinuously. Hence if $\Gamma \subset \mathrm{SL}_n(\mathbb{Z})$ is any finite-index subgroup, the quotient $\Gamma\backslash X$ is a real noncompact manifold except for at most finitely many quotient singularities. We may then identify the complex group cohomology $H^*(\Gamma;\mathbb{C})$ with $H^*(\Gamma\backslash X;\mathbb{C})$. Although the dimension of $\Gamma\backslash X$ is d, it can be shown that $H^i(\Gamma\backslash X;\mathbb{C}) = 0$ if $i > d - n + 1$ [12, Theorem 11.4.4]. The number $\nu = d - n + 1$ is called the *virtual cohomological dimension* of Γ.

In this paper we will always take Γ to be the congruence subgroup $\Gamma_0(N)$ of matrices whose last row is congruent to $(0, \dots, 0, *) \mod N$.

2.2. Recall that a point in $\mathbb{Z}^n$ is said to be *primitive* if the greatest common divisor of its coordinates is 1. In particular, a primitive point is nonzero. Let $\mathscr{P} \subset \mathbb{Z}^n$ be the set of primitive points. Any $v \in \mathscr{P}$, written as a column vector, determines a

[2]Actually, to avoid numerical instability in floating-point computations, we replace $\mathbb{C}$ with the finite field $\mathbb{F}_{31991}$ (cf. §5.3).

rank-one symmetric matrix $q(v) \in \bar{C}$ by $q(v) = v \cdot v^t$. The *Voronoĭ polyhedron* Π is the closed convex hull of the points $q(v)$, as v ranges over $\mathscr{P}$.

Note that, by construction, $\mathrm{SL}_n(\mathbb{Z})$ acts on Π. The cones over the faces of Π form a fan $\mathscr{V}$ that induces a Γ-admissible decomposition of C [2, p. 117]. Essentially, this means that Γ acts on $\mathscr{V}$; that each cone is spanned by a *finite* collection of points $q(v)$ where $v \in \mathscr{P}$; and that there are only finitely many Γ-orbits in $\mathscr{V}$. The fan $\mathscr{V}$ provides a reduction theory for C in the following sense: any point $x \in C$ is contained in a unique $\sigma(x) \in \mathscr{V}$, and the set $\{\gamma \in \mathrm{SL}_n(\mathbb{Z}) \mid \gamma \cdot \sigma(x) = \sigma(x)\}$ is finite.

2.3. We summarize facts about the well-rounded retract of [3, 4]. There is a deformation retraction $C \to C$ that is equivariant under the actions of both $\bar{\Gamma} = \mathrm{SL}_n(\mathbb{Z})$ and the homotheties. Its image modulo homotheties is the *well-rounded retract* W in X. The well-rounded retract is contractible, since it is a deformation retract of the contractible space X. Hence the cohomology of $\Gamma\backslash X$ with coefficients in $\mathbb{C}$ is canonically isomorphic to the equivariant cohomology $H^i_\Gamma(W;\mathbb{C})$ where Γ acts trivially on the coefficient module $\mathbb{C}$. This is in turn canonically isomorphic to the complex cohomology $H^i(\Gamma\backslash W;\mathbb{C})$, since $\mathbb{C}$ has characteristic zero, which moreover is isomorphic to $H^i(\Gamma;\mathbb{C})$. We will focus on computation of the equivariant cohomology. The dimension of W equals the virtual cohomological dimension ν, and the quotient $\Gamma\backslash W$ is compact.

The well-rounded retract W is naturally a locally finite cell complex, the cells being convex polytopes in V. The group $\mathrm{SL}_n(\mathbb{Z})$ preserves the cell structure, and the stabilizer of each cell in $\mathrm{SL}_n(\mathbb{Z})$ is finite. The theory of cores and co-cores in [7, Chapter 2] shows that the cells in W are in a one-to-one, inclusion-reversing correspondence with the cones in the Voronoĭ fan $\mathscr{V}$. By abuse of notation, the cell in W corresponding to σ will still be denoted σ.

2.4. One can give a more precise description of the combinatorics of the cones in $\mathscr{V}$ and the cells in W. To each $\sigma \in \mathscr{V}$, we define the set

$$M(\sigma) = \{v \in \mathscr{P} \mid q(v) \text{ is a vertex of the face of } \Pi \text{ generating } \sigma.\}$$

We associate the same set $M(\sigma)$ to the corresponding cell in W, and call $M(\sigma)$ the set of *minimal vectors of* σ (because of how W is constructed in [3, 4]). Since Π is the convex hull of the $q(v)$'s, it is clear that $\mu\colon \sigma \to M(\sigma)$ is an inclusion-preserving (respectively, inclusion-reversing) bijection between the cones in $\mathscr{V}$ (resp., cells in W) and a collection of finite subsets of $\mathscr{P}$. In principle, this reduces the study of the combinatorics of $\mathscr{V}$ and W to the study of the image of μ. For instance, face relations $\tau \subseteq \sigma$ in $\mathscr{V}$ are read off from subset relations $M(\tau) \subseteq M(\sigma)$. In practice, determining the image of μ calls for explicit computations with real quadratic forms, computations whose difficulty grows exponentially as a function of n. The computations have been carried out completely for $n \leq 5$ by various authors.

For the rest of this subsection, we set $n = 4$ and give more details. We state results for the well-rounded retract; these imply their analogues for $\mathscr{V}$. The image of μ was computed independently by [20] and (in essence) [26]. The cells of W fall into eighteen equivalence classes modulo $\mathrm{SL}_n(\mathbb{Z})$. Let T (for "type") be a variable running through these eighteen classes. This partitions the set of cells of W into eighteen pieces called the W_T. Any $\sigma \in W_T$ is said to be *of type* T. In each W_T, we fix one representative cell σ_T, the *standard cell of type* T. The $M(\sigma_T)$'s are written down explicitly in [21].([3]) This determines the image of μ, since the image is the union of the $\mathrm{SL}_n(\mathbb{Z})$-translates of the eighteen $M(\sigma_T)$'s.

2.5. To compute the action of the Hecke operators on cohomology, the well-rounded retract is insufficient, since the operators do not act cellularly. To ameliorate this, we use the sharbly complex. The material in this subsection closely follows [5].

Definition 2.6. [5] The *sharbly complex* is the chain complex $\{S_*, \partial\}$ given by the following data:

1. For $k \geq 0$, S_k is the module of formal $\mathbb{Z}$-linear combinations of basis elements $\mathbf{u} = [v_1, \dots , v_{n+k}]$, where each $v_i \in \mathscr{P}$, mod the relations:
   (a) If τ is a permutation on $(n + k)$ letters, then
   $$[v_1, \dots , v_{n+k}] = \mathrm{sgn}(\tau)[\tau(v_1), \dots , \tau(v_{n+k})],$$
   where $\mathrm{sgn}(\tau)$ is the sign of τ.
   (b) If $q = \pm 1$, then
   $$[qv_1, v_2 \dots , v_{n+k}] = [v_1, \dots , v_{n+k}].$$
   (c) If the rank of the matrix $(v_1, \dots , v_{n+k})$ is less than n, then $\mathbf{u} = 0$.
2. The boundary map $\partial \colon S_k \to S_{k-1}$ is
   $$[v_1, \dots , v_{n+k}] \longmapsto \sum_{i=1}^{n+k} (-1)^i [v_1, \dots , \hat{v}_i, \dots , v_{n+k}].$$

The basis elements $\mathbf{u} = [v_1, \dots, v_{n+k}]$ are called *k-sharblies.* By abuse of notation, we will often use the same symbol $\mathbf{u}$ to denote a k-sharbly and the k-sharbly chain $1 \cdot \mathbf{u}$. The obvious left action of Γ on S_* commutes with ∂.

For any $k \geq 0$, let $(S_k)_\Gamma$ be the module of Γ-coinvariants. This is the quotient of S_k by the relations of the form $\gamma \cdot \mathbf{u} - \mathbf{u}$, where $\gamma \in \Gamma$, $\mathbf{u} \in S_k$. This is also a complex with the induced boundary, which we denote by ∂_Γ. It is known (cf. [18]) that $H^{\nu-k}(\Gamma; \mathbb{C})$ is naturally isomorphic to $H_k((S_*)_\Gamma \otimes \mathbb{C})$.

Let $\mathbf{u} = [v_1, \dots, v_{n+k}]$ be a k-sharbly. Let $\|\mathbf{u}\|$ be
$$\mathrm{Max}\, |\det(v_{i_1}, \dots, v_{i_n})|,$$

[3]Since $v \in M(\sigma) \Leftrightarrow -v \in M(\sigma)$, it is customary to write down only one member of the pair $\pm v$.

where the maximum is taken over all n-fold subsets $\{i_1, \ldots, i_n\} \subset \{1, \ldots, n+k\}$. Note that this quantity is well-defined mod the relations in Definition 2.6. We extend this notion to sharbly chains $\xi = \sum n(\mathbf{u})\mathbf{u}$ by setting $\|\xi\|$ to be the maximum of $\|\mathbf{u}\|$, as $\mathbf{u}$ ranges over all sharblies in the support of ξ. We say that ξ is *reduced* if $\|\xi\| = 1$. It is known (cf. [21]) that for $\Gamma \subset \mathrm{SL}_4(\mathbb{Z})$, the group $H^5(\Gamma; \mathbb{C})$ is spanned by reduced 1-sharbly cycles.

2.7. Since the generators of the sharbly complex are indexed by sets of primitive vectors, it is clear that there is a close relationship between S_* and the chain complex associated to W, although of course S_* is much bigger. Both complexes compute $H^*(\Gamma; \mathbb{C})$. We refer to [18] for a discussion of this, phrased in terms of the fan $\mathscr{V}$. The main advantage of $(S_*)_\Gamma$ is that it admits a Hecke action. Specifically, let $\xi = \sum n(\mathbf{u})\mathbf{u}$ be a sharbly cycle mod Γ, and consider the Hecke operator $T(l, k)$ associated to the double coset $\Gamma D(l, k)\Gamma$ (cf. §1.1). Write

$$\Gamma D(l, k)\Gamma = \coprod_{g \in \Omega} \Gamma g,$$

a finite (disjoint) union. Then

$$T(l, k)(\xi) = \sum_{g \in \Omega, \mathbf{u}} n(\mathbf{u}) g \cdot \mathbf{u}. \tag{2}$$

Since $\Omega \not\subset \mathrm{SL}_n(\mathbb{Z})$ in general, the Hecke-image of a reduced sharbly isn't usually reduced.

3. Implementation details

3.1. We state our results for general n as much as possible, though our main case of interest is $n = 4$. We have working programs for $n \leq 4$. Though we focus on $\mathrm{SL}_n(\mathbb{Z})$, analogous results hold for $\mathrm{GL}_n(\mathbb{Z})$, and we have working programs for both SL and GL.

Section 3 is very technical. The reader may wish to skip to §4 or §5.

3.2. Let $\bar{\Gamma} = \mathrm{SL}_n(\mathbb{Z})$. Recall that W_T is the $\bar{\Gamma}$-orbit of cells of type T in W. Let σ_T be a fixed representative cell in W_T. The stabilizer in $\bar{\Gamma}$ of σ_T is denoted $\bar{\Gamma}_{\sigma_T}$, or $\bar{\Gamma}_T$ for short. This is a finite group that is straightforward to compute, since the minimal vectors $M(\sigma_T)$ are known. Our program maintains a database of the $\bar{\Gamma}_T$.

Standard facts about stabilizers give the following:

Proposition 3.3. *There is a one-to-one correspondence between cells $\sigma \in W_T$ and cosets $\bar{\Gamma}/\bar{\Gamma}_T$, given by $\gamma\sigma_T \leftrightarrow \gamma\bar{\Gamma}_T$ for any $\gamma \in \bar{\Gamma}$ such that $\sigma = \gamma\sigma_T$.*

Under the smaller group Γ, the $\bar{\Gamma}$-orbit W_T breaks up into suborbits. If Γ were a torsion-free group, $\Gamma\backslash W$ would be a finite cell complex, its cells would be given exactly by the Γ-suborbits, and we could compute $H^i(\Gamma\backslash W)$ by the standard methods for cell complexes. In our case, Γ is not torsion-free, but $\Gamma\backslash W$ can be thought of as a finite "orbifold cell" complex, whose elements are the quotients of cells by finite groups; the Γ-suborbits are in one-to-one correspondence with the orbifold cells.

The goal of this subsection is to understand the Γ-suborbits in terms of the actions of the $\bar{\Gamma}_T$ on finite projective spaces. By $\mathbb{P}^{n-1} = \mathbb{P}^{n-1}(\mathbb{Z}/N\mathbb{Z})$, we mean the set of vectors $(x_1, \ldots, x_n) \in (\mathbb{Z}/N\mathbb{Z})^n$ that are primitive in the sense that the ideal $(x_1, \ldots, x_n)$ in $\mathbb{Z}/N\mathbb{Z}$ is (1), modulo the equivalence relation given by scalar multiplication by the units $(\mathbb{Z}/N\mathbb{Z})^\times$ of $\mathbb{Z}/N\mathbb{Z}$. When N is a prime, this is the usual projective space over the field of N elements. As usual, the equivalence class of the vector $(x_1, \ldots, x_n)$ is denoted $\mathbf{a} = [x_1 : \cdots : x_n]$. We view these n-tuples as rows rather than columns; $\bar{\Gamma}$ acts on the right on $\mathbb{P}^{n-1}(\mathbb{Z}/N\mathbb{Z})$ in the obvious way.

We define the *bottom row map* $\mathfrak{b}\colon \bar{\Gamma} \to \mathbb{P}^{n-1}(\mathbb{Z}/N\mathbb{Z})$ as follows. For a matrix $\gamma \in \bar{\Gamma}$, the bottom row of γ is a primitive vector in $\mathbb{Z}^n$. Let $\mathfrak{b}(\gamma)$ be the equivalence class of this image in $\mathbb{P}^{n-1}(\mathbb{Z}/N\mathbb{Z})$.

Lemma 3.4. *The bottom row map $\mathfrak{b} : \bar{\Gamma} \to \mathbb{P}^{n-1}(\mathbb{Z}/N\mathbb{Z})$ induces a bijection between $\Gamma\backslash\bar{\Gamma}$ and $\mathbb{P}^{n-1}(\mathbb{Z}/N\mathbb{Z})$, given by*

$$\Gamma\gamma \mapsto \mathfrak{b}(\gamma).$$

The map is equivariant for the right action of $\bar{\Gamma}$.

Proof. It is a standard fact that a vector in $\mathbb{Z}^n$ is primitive if and only if it is the bottom row of some element of $\bar{\Gamma}$. This implies $\mathfrak{b}$ is surjective and that $\mathfrak{b}^{-1}([0 : \cdots : 0 : 0 : 1]) = \Gamma$. The rest is clear. □

We can now describe the Γ-orbits of cells in each W_T.

Proposition 3.5. *The Γ-orbits of cells in W_T are in one-to-one correspondence with the orbits O of the right $\bar{\Gamma}_T$-action on $\mathbb{P}^{n-1}$.*

Proof. We have $\Gamma\backslash W_T = \Gamma\backslash\bar{\Gamma}/\bar{\Gamma}_T$ by Proposition 3.3, and this equals $\mathbb{P}^{n-1}/\bar{\Gamma}_T$ by Lemma 3.4. □

The first step of our computer program is to determine, for each type T, the decomposition of $\mathbb{P}^{n-1}$ into right $\bar{\Gamma}_T$-orbits. Since we are primarily studying $H^i_\Gamma(W;\mathbb{C})$ for $i = 5, 6$, we only need to work with the T representing cells of dimensions 4, 5 and 6.

We note the following fact, whose proof is immediate.

Lemma 3.6. *Let $\gamma_0 \in \bar{\Gamma}$, and let $\mathbf{a} = \mathfrak{b}(\gamma_0)$ in $\mathbb{P}^{n-1}$. Then the stabilizer of $\mathbf{a}$ under the right action of $\bar{\Gamma}$ is $\gamma_0^{-1}\Gamma\gamma_0$.*

Let $\sigma \in W$ be a cell of type T, with $\sigma = \gamma_0\sigma_T$. Its stabilizer in Γ, denoted Γ_σ, is clearly

$$\Gamma_\sigma = (\gamma_0\bar{\Gamma}_T\gamma_0^{-1}) \cap \Gamma. \tag{3}$$

Lemma 3.6 implies

Lemma 3.7. *The group $\gamma_0^{-1}\Gamma_\sigma\gamma_0$ is the subgroup of $\bar{\Gamma}_T$ that preserves $\mathbf{a} = \mathfrak{b}(\gamma_0)$ under the right action on $\mathbb{P}^{n-1}$.*

3.8. In this subsection, we fix orientations on the cells $\sigma \in W$. It's necessary to be extremely careful—mistakes in orientation are easy to make and will ruin the computations. The price to pay is to sort through the details of the action of $\bar{\Gamma}$.

Recall that $\bar{\Gamma}_T$ is the stabilizer of σ_T in $\bar{\Gamma}$. For each T, there is an orientation character $\bar{\Gamma}_T \to \{\pm 1\}$ indicating whether or not $\gamma \in \bar{\Gamma}_T$ preserves the orientation on σ_T. Our program stores the values of these characters along with $\bar{\Gamma}_T$. Let $\bar{\Gamma}_T^+$ be the subgroup of $\bar{\Gamma}_T$ where the orientation is $+1$.

Remark 3.9. To compute the value of the character at γ, we determine (i) how γ acts on the orientation of the cone C, and divide by (ii) how γ acts on the orientation of the Voronoĭ cone dual to σ_T. Dividing works because C is locally the direct product of the cell and its dual Voronoĭ cone. As for (i), every element of $\bar{\Gamma}$ acts by $+1$ on the orientation of C, since $\bar{\Gamma}$ is a subgroup of the connected group $\mathrm{SL}_n(\mathbb{R})$. In this paper, where $n = 4$ and $\dim \sigma > 0$, it turns out that all the dual Voronoĭ cones are simplicial; the sign in (ii) is the sign of the permutation that γ effects on the bounding rays $q(v)$ of the cone, which is easily computed.

Let O be a right $\bar{\Gamma}_T$-orbit in $\mathbb{P}^{n-1}$. We call O *non-orientable* if for some (which implies every) $\mathbf{a} \in O$, there exists $\gamma \in \bar{\Gamma}_T \smallsetminus \bar{\Gamma}_T^+$ with $\mathbf{a}\gamma = \mathbf{a}$. Otherwise, we call O *orientable*. These notions depend on T, though we usually leave T out of the notation.

If O is orientable, fix some $\mathbf{a}_0 \in O$. Define the *orientation number* of $\mathbf{a} \in O$ to be $+1$ (resp., -1) according as $\mathbf{a} = \mathbf{a}_0\gamma$ for some $\gamma \in \bar{\Gamma}_T^+$ (resp., $\gamma \in \bar{\Gamma}_T \smallsetminus \bar{\Gamma}_T^+$). The orientation number is well-defined precisely because O is orientable. Again, the notions depend on the choice of $\mathbf{a}_0$, though we leave $\mathbf{a}_0$ out of the notation.

Let $\gamma \in \bar{\Gamma}_T$, and let ρ be any cell of W with any given orientation. Since γ acts by diffeomorphisms on C, it carries the orientation on ρ to some orientation on the cell $\gamma\rho$. We write

$$(\gamma)_*(\rho) \tag{4}$$

to denote $\gamma\rho$ together with this orientation. Clearly $(\gamma)_*$ is functorial, and preserves the relative orientation of ρ, τ whenever τ is a codimension-one face of ρ.

Once and for all, fix orientations on the standard cells σ_T. We can now put orientations on all the cells of W.

Definition 3.10. Let σ be a cell in W_T with $\sigma = \gamma_0\sigma_T$. Let O be the right $\bar{\Gamma}_T$-orbit in $\mathbb{P}^{n-1}$ corresponding to σ as in Proposition 3.5. If O is orientable, we give σ the orientation

$$(\text{orientation number of } \mathbf{a}) \cdot (\gamma_0)_*(\sigma_T). \tag{5}$$

If O is non-orientable, we give σ an arbitrary orientation.

Proposition 3.11. *The quantity in* (5) *is well-defined.*

Proof. Let σ, γ_0, O, and $\mathbf{a}_0$ be as in Definition 3.10, with O assumed orientable. Assume $\sigma = \gamma_1\sigma_T$ as well as $\gamma_0\sigma_T$. Let $\mathbf{a}_1 = \mathfrak{b}(\gamma_1)$. Then $\gamma_1^{-1}\gamma_0 \in \bar{\Gamma}_T$. By definition of $\bar{\Gamma}_T^+$, $(\gamma_0)_*(\sigma_T) = (\gamma_1)_*(\sigma_T)$ if and only if $\gamma_1^{-1}\gamma_0 \in \bar{\Gamma}_T^+$. On the other hand, $\mathbf{a}_1\gamma_1^{-1}\gamma_0 = [0 : \cdots : 0 : 1]\gamma_0 = \mathbf{a}$, so $\mathbf{a}$ and $\mathbf{a}_1$ have the same orientation number if and only if $\gamma_1^{-1}\gamma_0 \in \bar{\Gamma}_T^+$. □

We must understand how $\bar{\Gamma}$ and Γ act on the orientations we have just chosen. The following lemma is immediate from (4) and Definition 3.10.

Lemma 3.12. *If $\sigma = \gamma_0\sigma_T$ corresponds to an orientable O, then γ_0 carries the chosen orientation of σ_T to the chosen orientation of σ times the orientation number of* $\mathbf{a}$.

Here is a more general statement.

Proposition 3.13. *Let $\sigma = \gamma_0\sigma_T$ and $\sigma_1 = \gamma_1\sigma$, for some $\gamma_0, \gamma_1 \in \bar{\Gamma}$. Let $\mathbf{a}_0 = \mathfrak{b}(\gamma_0)$ and $\mathbf{a}_1 = \mathfrak{b}(\gamma_1\gamma_0)$. Let O_1, O be the right $\bar{\Gamma}_T$-orbits in $\mathbb{P}^{n-1}$ containing $\mathbf{a}_1, \mathbf{a}_0$; assume these orbits are both orientable. Then γ_1 carries σ to σ_1 while multiplying the orientations by*

$$(\textit{orien. number of } \mathbf{a}) \cdot (\textit{orien. number of } \mathbf{a}_1). \tag{6}$$

Proof. Apply Lemma 3.12 twice. □

Fortunately, (6) becomes trivial when we consider Γ as opposed to $\bar{\Gamma}$.

Proposition 3.14. *Let $\sigma_1 = \gamma_1\sigma$ for $\gamma_1 \in \Gamma$. Let $\mathbf{a}_1, \mathbf{a}, O_1, O$ be as in Proposition 3.13, both orbits being assumed orientable. Then γ_1 carries σ to σ_1 with orientations matching.*

Proof. We have $[0 : \cdots : 0 : 1]\gamma_1 = [0 : \cdots : 0 : 1]$ by the definition of Γ. Hence $\mathbf{a}_1 = ([0 : \cdots : 0 : 1]\gamma_1)\gamma_0 = [0 : \cdots : 0 : 1]\gamma_0 = \mathbf{a}$. Thus the expression in (6) is a square either of $+1$ or of -1. □

3.15. We compute the equivariant cohomology $H^*_\Gamma(W; \mathbb{C})$ using a spectral sequence, following the exposition in [13, VII.7–8]. (The spectral sequence there is for equivariant homology; we make the appropriate modifications for cohomology.)

Let σ be any cell in the well-rounded retract W. Recall that Γ_σ is the stabilizer of σ in Γ. Let $\mathbb{C}_\sigma$ be the Γ_σ-module where $\gamma \in \Gamma_\sigma$ acts by $+1$ if γ preserves the orientation of σ and by -1 if it does not.

For each i, let $W_{(i)}$ be a fixed set of representatives of the Γ-orbits of the cells in W of dimension i. The E_1 term of the spectral sequence is

$$E_1^{i,j} = \bigoplus_{o \in W_{(i)}} H^j(\Gamma_o; \mathbb{C}_o). \tag{7}$$

These cells o (omicron) are in one-to-one correspondence with the O of Proposition 3.5, as T runs through the types of cells of dimension i. Because $\mathbb{C}$ is a field of characteristic zero, all the terms in (7) vanish when $j \neq 0$. In particular, the spectral sequence collapses at E_2. The term $H^0(\Gamma_o; \mathbb{C}_o)$ is the subset of Γ_o-invariants in the module $\mathbb{C}_o$.

Proposition 3.16. *For any cell σ, let T be the type of σ, and let O be the right $\bar{\Gamma}_T$-orbit in $\mathbb{P}^{n-1}$ corresponding to σ as in Proposition 3.5. Then $H^0(\Gamma_\sigma; \mathbb{C})$ is $\mathbb{C}$ if O is orientable, and is 0 if O is non-orientable.*

Proof. When O is orientable, this follows from Proposition 3.14, merely because $\Gamma_\sigma \subset \Gamma$. Now assume O is non-orientable. Let $\sigma = \gamma_0 \sigma_T$, with $\mathbf{a} = \mathfrak{b}(\gamma_0)$. As we have said above, there is some $\gamma_1 \in \bar{\Gamma}_T \smallsetminus \bar{\Gamma}_T^+$ with $\mathbf{a}\gamma_1 = \mathbf{a}$. By (2.2), $\Gamma_\sigma = (\gamma_0 \bar{\Gamma}_T \gamma_0^{-1}) \cap \Gamma$. The element $\gamma_0 \gamma_1 \gamma_0^{-1}$ is in Γ by Lemma 2, so it is in Γ_σ; clearly it carries σ to itself while reversing orientation. Hence Γ_σ acts non-trivially on $\mathbb{C}_\sigma$, meaning $H^0(\Gamma_\sigma; \mathbb{C}_\sigma) = 0$. □

Remark 3.17. The proposition shows that the Γ-orbits of cells coming from non-orientable O contribute nothing to our spectral sequence. We ignore these objects for the rest of the computation, tacitly assuming that all O's mentioned from now on are orientable.

To summarize:

Proposition 3.18. *The $E_1^{i,0}$ term of the equivariant cohomology spectral sequence for $H^i_\Gamma(W; \mathbb{C})$ is a direct sum $\bigoplus_o \mathbb{C}$, where o runs through a set of i-cells in one-to-one correspondence with the orientable right $\bar{\Gamma}_T$-orbits in $\mathbb{P}^{n-1}$, for all types T of cells of dimension i.*

3.19. In §3.21, we will describe the boundary maps d_1 of the spectral sequence. These are the only differentials we need consider, since the sequence collapses at E_2. In this subsection, we give some details concerning how the cells meet at their boundaries.

As usual, a *facet* of a cell σ is any face of σ of codimension one. Let $\mathcal{F}_\sigma$ be the set of facets of σ in W.

We will need to understand how $\mathcal{F}_\sigma$ breaks up into orbits under the action of Γ_σ (and to choose a set $\mathcal{F}'_\sigma$ of representatives of these orbits in $\mathcal{F}_\sigma$). We do this in Proposition 3.20 below. We will start by determining $\mathcal{F}_{\sigma_T}$ for the standard cells σ_T in a form suited to our computation. We will then determine $\mathcal{F}_\sigma$ for any σ.

We make two conventions. (i) If types T and T' occur in the same discussion, it is assumed that a cell of type T has at least some cells of type T' as facets. (ii) If $\sigma = x\sigma_T$ for some $x \in \bar{\Gamma}$, we identify σ with the coset $x\bar{\Gamma}_T$ as in Proposition 3.3. In the expressions of the form

$$\bigcup_{T'} (*),$$

the $(*)$ will be a finite union of cosets, say $x_1\bar{\Gamma}_{T'} \coprod \cdots \coprod x_k\bar{\Gamma}_{T'}$ for $x_1, \ldots, x_k \in \bar{\Gamma}$. It is understood that $(*)$ corresponds to the set of cells $x_1\sigma_{T'}, \ldots, x_k\sigma_{T'}$, as in Proposition 3.3. Even when $(*)$ is a more complicated object, like a double coset, its meaning is that one should decompose it into single cosets by choosing appropriate representatives (which will be the x_i).

The boundary of σ_T is a union of (the closures of) various cells of type T':

$$\mathcal{F}_{\sigma_T} = \bigcup_{T'} \beta_{T',1}\bar{\Gamma}_{T'} \coprod \cdots \coprod \beta_{T',k}\bar{\Gamma}_{T'} \tag{8}$$

for some $\beta_{T',i} \in \bar{\Gamma}$. However, (8) is invariant under the left action of σ_T's stabilizer $\bar{\Gamma}_T$. Hence there must be finitely many $\alpha_{(T,T',\iota)} \in \bar{\Gamma}$ such that

$$\mathcal{F}_{\sigma_T} = \bigcup_{T'} \bar{\Gamma}_T\alpha_{(T,T',1)}\bar{\Gamma}_{T'} \coprod \cdots \coprod \bar{\Gamma}_T\alpha_{(T,T',k)}\bar{\Gamma}_{T'}. \tag{9}$$

We have computed the $\alpha_{(T,T',\iota)}$ by hand for $n \leq 4$. In our cases of interest ($n = 4$, $\dim\sigma \geq 4$), one finds there is only one ι—that is, the right-hand side of (9) is actually just one double coset. In fact, we find that we may take $\alpha_{(T,T',\iota)}$ to be the identity except in one case, where $(T, T') = (5\text{b}, 4\text{b})$ in the notation of [21] and

$$\alpha_{5\text{b},4\text{b},1} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 \end{pmatrix}.$$

From now on, we will write α for $\alpha_{(T,T',1)}$, the dependence on T and T' being understood.

To find an expression for $\mathcal{F}_\sigma$ for $\sigma = \gamma_0\sigma_T$, we multiply (9) by γ_0 to obtain

$$\mathcal{F}_\sigma = \bigcup_{T'} \gamma_0\bar{\Gamma}_T\alpha\bar{\Gamma}_{T'}. \tag{10}$$

To exhibit this as a union of cosets of the form $x_i\bar{\Gamma}_{T'}$, we rewrite it as

$$\mathcal{F}_\sigma = \bigcup_{T'} \gamma_0\alpha(\alpha^{-1}\bar{\Gamma}_T\alpha)\bar{\Gamma}_{T'} \tag{11}$$

$$= \bigcup_{T'} \gamma_0\alpha(\alpha^{-1}\bar{\Gamma}_T\alpha/(\alpha^{-1}\bar{\Gamma}_T\alpha \cap \bar{\Gamma}_{T'}))\bar{\Gamma}_{T'}, \tag{12}$$

The last formula (for each T') is a disjoint union of single cosets, in one-to-one correspondence with a set of representatives of the cosets in $\alpha^{-1}\bar{\Gamma}_T\alpha/(\alpha^{-1}\bar{\Gamma}_T\alpha \cap \bar{\Gamma}_{T'})$.

It is a matter of formal manipulation to get an expression for $\Gamma_\sigma\backslash\mathcal{F}_\sigma$:

$$\Gamma_\sigma\backslash\mathcal{F}_\sigma = \bigcup_{T'} (\gamma_0\bar\Gamma_T\gamma_0^{-1}\cap\Gamma)\backslash\gamma_0\alpha(\alpha^{-1}\bar\Gamma_T\alpha/(\alpha^{-1}\bar\Gamma_T\alpha\cap\bar\Gamma_{T'}))\bar\Gamma_{T'} \tag{13}$$

$$= \bigcup_{T'} \gamma_0(\bar\Gamma_T\cap\gamma_0^{-1}\Gamma\gamma_0)\backslash\alpha(\alpha^{-1}\bar\Gamma_T\alpha/(\alpha^{-1}\bar\Gamma_T\alpha\cap\bar\Gamma_{T'}))\bar\Gamma_{T'} \tag{14}$$

$$= \bigcup_{T'} \gamma_0\alpha[\underbrace{\underbrace{\alpha^{-1}(\bar\Gamma_T\cap\gamma_0^{-1}\Gamma\gamma_0)\alpha}_{A}\backslash\alpha^{-1}\bar\Gamma_T\alpha}_{B}\,/\,\underbrace{(\alpha^{-1}\bar\Gamma_T\alpha\cap\bar\Gamma_{T'})}_{C}]\bar\Gamma_{T'} \tag{15}$$

where for each T' the right-hand side of (15) is again expressed as a disjoint union of $\bar\Gamma_{T'}$-cosets, in one-to-one correspondence with a set of representatives of the double coset expression in square brackets.

We must interpret (11)–(12) in terms of the $\bar\Gamma_{T'}$-orbits in $\mathbb{P}^{n-1}$. As usual, let O be the right $\bar\Gamma_T$-orbit corresponding to σ, with $\mathbf{a} = \mathfrak{b}(\gamma_0)$. Let $\mathbf{c} = \mathfrak{b}(\gamma_0\alpha)$, so that $\mathbf{c} = \mathbf{a}\alpha$. Now the quantity A in (15) is exactly the subgroup of $\alpha^{-1}\bar\Gamma_T\alpha$ that preserves $\mathbf{c}$. Hence any set of coset representatives for B in (15) is a set of matrices whose bottom rows, in $\mathbb{P}^{n-1}$, are the members of the right $(\alpha^{-1}\bar\Gamma_T\alpha)$-orbit of $\mathbf{c}$. Equivalently, any set of representatives for B is a set of matrices whose bottom rows are exactly the members of $O\cdot\alpha$. The group denoted C in (15) acts on this orbit $O\cdot\alpha$, decomposing it into suborbits; for given T', the disjoint $\bar\Gamma_{T'}$-cosets in (11)–(12) are in one-to-one correspondence with these suborbits. We summarize this result as follows:

Proposition 3.20. *Let T, T' and α be as introduced in this subsection. Let σ be a cell of type T, represented by the $\bar\Gamma_T$-orbit O in the manner of §3.8. Decompose the orbit $O\cdot\alpha$ into its suborbits $O_1,\dots,O_k$ under the group $C = \alpha^{-1}\bar\Gamma_T\alpha\cap\bar\Gamma_{T'}$. Let $\mathbf{a}_j\in O_j$. Let $\gamma_j\in\bar\Gamma$ be chosen so that $\mathfrak{b}(\gamma_j) = \mathbf{a}_j$. (We may, in fact, take γ_j to be of the form $\gamma_0\hat\gamma_j\alpha$ for some $\hat\gamma_j\in\Gamma_T$.) Then the union over all T' of the cells*

$$\{\gamma_1\sigma_{T'},\dots,\gamma_k\sigma_{T'}\}$$

is a set $\mathcal{F}'_\sigma$ of representatives for $\Gamma_\sigma\backslash\mathcal{F}_\sigma$.

For all pairs T, T', we store the intersection $C = \alpha^{-1}\bar\Gamma_T\alpha\cap\bar\Gamma_{T'}$ in our program.

3.21. We can now determine the boundary maps d_1 in the spectral sequence. Recall that $W_{(i)}$ is a fixed set of representatives of the Γ-orbits of the cells in W of dimension i. Superseding the use of o in §3.15, we let σ run through $W_{(i+1)}$, and let τ run through the set $\mathcal{F}'_\sigma$ of representatives of the facets of σ. To compute $H^i_\Gamma(W;\mathbb{C})$ for subgroups of $\mathrm{SL}_4(\mathbb{Z})$ for $i = 5, 6$, we must work out the map $d_1^{i,0}$ for $i = 4$ and 5.

We follow [13, VII.8], taking the dual to turn homology into cohomology. The map

$$d_1^{i,0} : \bigoplus_{\tau_0 \in W_{(i)}} H^0(\Gamma_{\tau_0}; \mathbb{C}_{\tau_0}) \to \bigoplus_{\sigma \in W_{(i+1)}} H^0(\Gamma_\sigma; \mathbb{C}_\sigma) \tag{16}$$

is a sum of terms, one for each pair (σ, τ); here τ_0 is the fixed representative in $W_{(i)}$ that is Γ-equivalent to τ. From now on, we focus on a single pair σ, τ. Call their types T and T', respectively. Let O be the right $\bar{\Gamma}_T$-orbit in $\mathbb{P}^{n-1}$ associated to σ by Proposition 3.5. If $\sigma = \gamma_0 \sigma_T$, let $\mathbf{a} = \mathfrak{b}(\gamma_0) \in O$. Let $\Gamma_{\sigma\tau} = \Gamma_\sigma \cap \Gamma_\tau$.

As in [13, p. 176], the term for (σ, τ) in $d_1^{i,0}$ is the composition

$$H^0(\Gamma_{\tau_0}; \mathbb{C}_{\tau_0}) \xrightarrow{v_\tau} H^0(\Gamma_\tau; \mathbb{C}_\tau) \xrightarrow{u_{\sigma\tau}} H^0(\Gamma_{\sigma\tau}; \mathbb{C}_\sigma) \xrightarrow{t_{\sigma\tau}} H^0(\Gamma_\sigma; \mathbb{C}_\sigma) . \tag{17}$$

We now give the definition of these maps and show how to compute them. Note that all the coefficient modules are copies of $\mathbb{C}$ on which the groups act trivially.

The map $t_{\sigma\tau}$ is the transfer map $\mathbb{C} \to \mathbb{C}$ given by multiplication by the scalar $[\Gamma_\sigma : \Gamma_{\sigma\tau}]$. We evaluate this scalar as follows. First of all, $\Gamma_\sigma = (\gamma_0 \bar{\Gamma}_T \gamma_0^{-1}) \cap \Gamma$, which has the same cardinality as $\bar{\Gamma}_T \cap (\gamma_0^{-1} \Gamma \gamma_0)$. The latter is the subgroup of $\bar{\Gamma}_T$ that fixes $\mathbf{a}$. Thus

$$\#(\Gamma_\sigma) = \frac{\#(\bar{\Gamma}_T)}{\#(\bar{\Gamma}_T\text{-orbit of } \mathbf{a})} = \frac{\#(\bar{\Gamma}_T)}{\#(O)}. \tag{18}$$

The numerator is known because we stored $\bar{\Gamma}_T$. The denominator is easily recovered from the computer's lists of orbits.

We now evaluate the order of $\Gamma_{\sigma\tau}$. As before, $\Gamma_\sigma = \gamma_0 \bar{\Gamma}_T \gamma_0^{-1} \cap \Gamma$. By Proposition 3.20, $\tau = \gamma_0 \hat{\gamma} \alpha \sigma_{T'}$ for some $\hat{\gamma} \in \bar{\Gamma}_T$. Hence

$$\Gamma_\tau = ((\gamma_0 \hat{\gamma} \alpha) \bar{\Gamma}_{T'} (\alpha^{-1} \hat{\gamma}^{-1} \gamma_0^{-1})) \cap \Gamma.$$

Writing out $\Gamma_{\sigma\tau} = \Gamma_\sigma \cap \Gamma_\tau$ and conjugating by $(\gamma_0 \hat{\gamma} \alpha)$, we find that $\Gamma_{\sigma\tau}$ has the same cardinality as

$$\underbrace{(\alpha^{-1} \bar{\Gamma}_T \alpha \cap \bar{\Gamma}_{T'})}_{D} \cap (\alpha^{-1} \hat{\gamma}^{-1} \gamma_0^{-1} \Gamma \gamma_0 \hat{\gamma} \alpha) \tag{19}$$

The group D is the same as the group C from (15), and is one of the groups we've stored. And the whole group in (19) is exactly the subgroup of D that fixes the point $\mathbf{a}_j$ of Proposition 3.20. So

$$\#(\Gamma_{\sigma\tau}) = \frac{\#(\alpha^{-1} \bar{\Gamma}_T \alpha \cap \bar{\Gamma}_{T'})}{\#(O_j)}. \tag{20}$$

Again, the numerator is known from what's stored, and the denominator is easy to evaluate.

We have

$$t_{\sigma\tau} = \frac{(18)}{(20)}. \tag{21}$$

As a consistency check, the program signals an error if the computed value of (21) is not an integer.

Brown's $u_{\sigma\tau} : \mathbb{C} \to \mathbb{C}$ is the composition

$$H^0(\Gamma_\tau; \mathbb{C}_\tau) \to H^0(\Gamma_{\sigma\tau}; \mathbb{C}_\tau) \to H^0(\Gamma_{\sigma\tau}; \mathbb{C}_\sigma).$$

The first arrow is induced from the inclusion $\Gamma_{\sigma\tau} \hookrightarrow \Gamma_\tau$, and is easily seen to be the identity. The second arrow is induced by the $\Gamma_{\sigma\tau}$-map $\partial_{\sigma\tau} : \mathbb{C}_\sigma \to \mathbb{C}_\tau$, namely the (σ,τ)-component of the cellular boundary operator on W. Let $[\sigma : \tau]$ be ± 1 depending on whether the orientation on σ from Definition 3.10 does or does not induce the orientation on the facet τ from Definition 3.10. Then $\partial_{\sigma\tau}$ is the map $\mathbb{C} \to \mathbb{C}$ given by the scalar $[\sigma : \tau]$. Thus $u_{\sigma\tau} = [\sigma : \tau]$.

To evaluate $[\sigma : \tau]$, we reduce the problem to evaluating the boundary operator on a small list of pairs of cells. Write $\tau = \gamma_0 \hat{\gamma} \alpha \sigma_{T'}$. Then $\sigma = \gamma_0 \hat{\gamma} \sigma_T$, since $\hat{\gamma} \in \bar{\Gamma}_T$. If $\mathbf{x} \in \mathbb{P}^{n-1}$ is part of any oriented $\bar{\Gamma}_T$-orbit (for any T), write $\mathrm{sgn}_T(\mathbf{x})$ for the orientation number of $\mathbf{x}$ with respect to $\bar{\Gamma}_T$. By Definition 3.10, σ and τ—together with their standard orientations—are given as follows:

$$\sigma = (\gamma_0\hat{\gamma})_*(\sigma_T) \cdot \mathrm{sgn}_T(\mathfrak{b}(\gamma_0\hat{\gamma})) \tag{22}$$

$$\tau = (\gamma_0\hat{\gamma}\alpha)_*(\sigma_{T'}) \cdot \mathrm{sgn}_{T'}(\mathfrak{b}(\gamma_0\hat{\gamma}\alpha)). \tag{23}$$

Because $(\dots)_*$ is functorial,

$$\tau = (\gamma_0\hat{\gamma})_*(\alpha)_*(\sigma_{T'}) \cdot \mathrm{sgn}_{T'}(\mathfrak{b}(\gamma_0\hat{\gamma}\alpha)).$$

Since $(\dots)_*$ preserves the $[\ :\ ]$ relation, we may cancel out $(\gamma_0\hat{\gamma})_*$'s, obtaining

$$[\sigma : \tau] = \underbrace{\mathrm{sgn}_T(\mathfrak{b}(\gamma_0\hat{\gamma}))}_{E} \cdot \underbrace{\mathrm{sgn}_{T'}(\mathfrak{b}(\gamma_0\hat{\gamma}\alpha))}_{F} \cdot [\sigma_T : (\alpha)_*(\sigma_{T'})]. \tag{24}$$

We evaluate each factor in (24) in turn. In F, $\mathfrak{b}(\gamma_0\hat{\gamma}\alpha)$ is the point $\mathbf{a}_j$ of Proposition 3.20, and $\mathrm{sgn}_{T'}(\mathbf{a}_j)$ is simply the orientation number of $\mathbf{a}_j$ in its $\bar{\Gamma}_{T'}$-orbit. Similarly, E is the orientation number of $\mathfrak{b}(\gamma_0\hat{\gamma})$, which is a $\bar{\Gamma}_T$-translate of $\mathbf{a}$, in its $\bar{\Gamma}_T$-orbit. In practice, though, we do not know $\hat{\gamma}$ or $\mathfrak{b}(\gamma_0\hat{\gamma})$ explicitly, so it is easier to evaluate E by another method. Note that $\mathfrak{b}(\gamma_0\hat{\gamma}) \cdot \alpha = \mathfrak{b}(\gamma_0\hat{\gamma}\alpha) = \mathbf{a}_j$, an element of the $(\alpha^{-1}\bar{\Gamma}_T\alpha)$-orbit $O \cdot \alpha$ of Proposition 3.20. It is easy to compute all the points in $O \cdot \alpha$, using the group $\alpha^{-1}\bar{\Gamma}_T\alpha$ (which was stored). For some $\mathbf{x}$, we will have the equation $\mathbf{a}_j = \mathbf{x} \cdot \alpha$. Then E will be the orientation number for this $\mathbf{x}$.

The quantities $[\sigma_T : (\alpha)_*(\sigma_{T'})]$ are evaluated by hand, for all T, T' occurring in (9). Several issues arise. First, we must find $[\sigma_T : \sigma_{T'}]$ whenever the standard cell $\sigma_{T'}$ is a facet of σ_T. No matter how we choose the orientations at the start, it is in general impossible to arrange our choices so that all the relative orientations are positive. As a general illustration, if a 0-cell, two 1-cells, and a 2-cell meet locally in a picture like the first quadrant of $\mathbb{R}^2$, and if both 1-cells are oriented to point away from the origin, then the pair (2-cell, x-axis) must have relative orientation opposite to that of the pair (2-cell, y-axis), no matter how we orient the 2-cell. In the $\mathrm{SL}_4(\mathbb{Z})$ case, one can draw a schematic picture of how the standard cells meet and can read off all the relative orientations $[\sigma_T : \sigma_{T'}]$.

Second, we must compute $(\alpha)_*(\sigma_{T'})$ when α isn't the identity, knowing the orientation on $\sigma_{T'}$. This involves the same techniques as in Remark 3.9.

Third, we must find some facet v of σ_T whose orientation is known, and must compare $[\sigma_T : v]$ to $[\sigma_T : (\alpha)_*(\sigma_{T'})]$. In practice, if $\dim \sigma_T = k$, this means finding a chain of $(k-1)$-cells between v and $(\alpha)_*(\sigma_{T'})$ such that consecutive members of the chain meet in faces of dimension $k-2$, and comparing the orientations of v and $(\alpha)_*(\sigma_{T'})$ across the $(k-2)$-faces.

Finally, we must compute v_τ, but this is easy. It is induced by the conjugation action of the element of Γ that carries τ to τ_0. But by Proposition 3.14, any element of Γ preserves the orientations on the cells. Hence one finds $v_\tau = +1$.

4. Hecke operators

4.1. We identify the cochain complex in (16) with the complex of cellular cochains on W by identifying an i-cell σ with a generator of $H^0(\Gamma_\sigma; \mathbb{C}_\sigma)$, taking care to make the signs match. Formulas like $\sum n(\sigma)\sigma$ will denote the corresponding cocycles in either complex, and will be referred to as *W-cocycles*

Let $\beta \in H^5(\Gamma; \mathbb{C})$ be a class, and let $u = \sum n(\sigma)\sigma$ be a representative for β in terms of the previous paragraph. Let $T(l,k)$ be a Hecke operator. To compute the action of $T(l,k)$ on u, we do the following:

1. Convert u to a reduced 1-sharbly cycle ξ, and then compute $T(l,k)(\xi)$ using (2) in §2.7.
2. Use the algorithm from [18] to write $T(l,k)(\xi)$ as a sum of reduced 1-sharbly cycles.
3. Convert these reduced 1-sharbly cycles to W-cocycles.

Step 2 is described in detail in [18], and we refer the reader to that article. In this section, we focus on steps 1 and 3 in the context of §3.

We begin with a definition from [18]:

Definition 4.2. [18, Definition 5.3] Let $\mathbf{u}$ be a basis element of the k-sharblies S_k. Then a *lift* for $\mathbf{u}$ is an $n \times (n+k)$ integral matrix M with primitive columns such that $[M_1, \ldots, M_{n+k}] = \mathbf{u}$, where M_i is the ith column of M.

Modulo the action of $\mathrm{GL}_4(\mathbb{Z})$, there is only one orbit of reduced basis 0-sharblies and only three orbits of reduced basis 1-sharblies. The identity matrix serves as a lift for a member of the first orbit, and lifts representing elements of the latter three orbits are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

We call these the *standard* 0- and 1-sharblies. The sets of primitive vectors indexing the standard 6 and 5-cells in W coincide with the sets of column vectors of these matrices. By abuse of language we will speak of the "standard sharbly of type T," and will use the notation $\mathbf{u}_T$.

4.3. Given a sharbly cycle ξ, we denote by $\operatorname{supp}\xi$ the support of ξ. Suppose that Γ is torsion-free. Then according to [18], a 1-sharbly cycle ξ mod Γ with coefficients in a ring R can be encoded by a collection of 4-tuples $(\mathbf{u}, n(\mathbf{u}), \{\mathbf{v}\}, \{L(\mathbf{v})\})$, where

1. $\mathbf{u} \in \operatorname{supp}\xi$,
2. $n(\mathbf{u}) \in R$,
3. $\{\mathbf{v}\} = \operatorname{supp}\partial\mathbf{u}$, and
4. $\{L(\mathbf{v})\}$ is a Γ-equivariant set of lifts for $\{\mathbf{v}\}$.

The Γ-equivariance condition in 4 is the following. Suppose that for $\mathbf{u}, \mathbf{u}' \in \operatorname{supp}\xi$ there exist $\mathbf{v} \in \operatorname{supp}(\partial\mathbf{u})$ and $\mathbf{v}' \in \operatorname{supp}(\partial\mathbf{u}')$ such that $\mathbf{v} = \gamma \cdot \mathbf{v}'$ for some $\gamma \in \Gamma$. Then we require $L(\mathbf{v}) = \gamma L(\mathbf{v}')$.

In the case under study, Γ is *not* torsion-free, and the above data needs to be modified. Suppose that a 0-sharbly $\mathbf{v} \in \operatorname{supp}\partial\mathbf{u}$ has a nontrivial stabilizer $\Gamma(\mathbf{v}) \subset \Gamma$, and let m be any lift of $\mathbf{v}$. Then in the cycle ξ we replace $n(\mathbf{u})\mathbf{u}$ with

$$\sum_{\gamma\in\Gamma(\mathbf{v})} \frac{n(\mathbf{u})}{\#\Gamma(\mathbf{v})}\mathbf{u}_\gamma,$$

where $\mathbf{u}_\gamma$ has the same data as $\mathbf{u}$, except that we give $\mathbf{v}$ the lift γm. (Note that this is possible in our case since the coefficient ring $R = \mathbb{C}$ is divisible.)

4.4. Now we describe how to construct the data in §4.3 to produce a 1-sharbly chain ξ corresponding to the W-cocycle u. There are two steps.

First, choose σ such that $n(\sigma) \neq 0$ in u. According to Proposition 3.3, the 5-cell $\sigma \in \Gamma\backslash W$ is encoded as a coset $[\gamma\Gamma_T]$, where $\gamma \in \Gamma$ and Γ_T is the stabilizer of the standard cell of type T. Moreover, the coset $[\gamma\Gamma_T]$ is encoded as the orbit O, which can be regarded a set of triples $\{(\mathbf{a}, \pm 1, \gamma_{\mathbf{a}})\}$, where $\mathbf{a} \in \mathbb{P}^3(\mathbb{Z}/N\mathbb{Z})$, $\gamma_{\mathbf{a}} \in \Gamma_T$, and ± 1 is the orientation number (§3.8). From O we arbitrarily choose a triple with orientation number 1, and then using Hermite normal form construct a matrix $\gamma \in \mathrm{SL}_4(\mathbb{Z})$ with bottom row equal to $\mathbf{a}$. Then the contribution of σ to the 1-sharbly chain is

$$n(\sigma)\gamma\mathbf{u}_T.$$

We do this for all of $\operatorname{supp} u$ and sum to produce ξ. For each $\mathbf{u} \in \operatorname{supp} \xi$, we write $\mathbf{u}(\gamma)$ if we want to indicate the element γ used in the construction of $\mathbf{u}$.

4.5. At this stage, we have a 1-sharbly chain, and we need to choose lifts to reflect the cycle structure of ξ. This we do as follows. In the spirit of §3.19, for each type T we choose a set of matrices $\Omega_T \subset \mathrm{SL}_4(\mathbb{Z})$ such that $\partial\colon S_1 \to S_0$ can be written as

$$\partial\colon \mathbf{u}_T \longmapsto \sum_{\omega \in \Omega_T} \omega \mathbf{v},$$

where $\mathbf{v}$ is the standard basis 0-sharbly. Note the absence of signs in this map—the signs in the boundary map in Definition 2.6 have been encoded in the ω's, which may nontrivially permute the column vectors of $\mathbf{v}$.

Form the 0-sharbly chain

$$\sum_{\mathbf{u}(\gamma), \omega \in \Omega_T} \gamma \omega \mathbf{v}, \tag{25}$$

where $\mathbf{v}$ is the standard 0-sharbly, we sum over all $\mathbf{u}(\gamma) \in \operatorname{supp} \xi$, and T is the type of $\mathbf{u}(\gamma)$. (In (25) we have abbreviated $\alpha_{(T,T',i)}$ to α_i, since T is determined by $\mathbf{u}(\gamma)$, and there is only one type of reduced 0-sharbly mod $\mathrm{SL}_4(\mathbb{Z})$.) Note that this sum is in S_*, not $(S_*)_\Gamma$; the only relations we apply are those in the sharbly complex.

4.6. After summing, we find that some 0-sharblies cancel, and some remain. For those that canceled, we can choose any lifts we like, as long as we choose the same lifts for all terms that cancel each other.

The remaining 0-sharblies form a chain η that vanishes in $(S_0)_\Gamma \otimes \mathbb{C}$, and we must choose nontrivial lifts for them. To do this, first arbitrarily choose lifts for each 0-sharbly in $\operatorname{supp} \eta$. The data we computed in §3.2 allows us to easily compute the distinct orbits of Γ in $\operatorname{supp} \eta$. We do this and order each orbit.

Suppose $\mathbf{v}_0$ is the first 0-sharbly in one of these orbits, and that it corresponds to the triple $(\mathbf{a}_0, \pm 1, \gamma_{\mathbf{a}_0})$. Let $\mathbf{v}_0(\gamma)$ be its lift. Then if $\mathbf{v}$ is any other 0-sharbly in $\mathbf{v}_0$'s orbit, corresponding to the triple $(\mathbf{a}, \pm 1, \gamma_{\mathbf{a}})$, we replace the lift $\mathbf{v}(\gamma)$ by

$$\mathbf{v}(\gamma) \longleftarrow \mathbf{v}(\gamma)\gamma_{\mathbf{a}}^{-1}\gamma_{\mathbf{a}_0}$$

After these lifts are constructed, the cycle ξ is ready for input in the Hecke operator program.

4.7. Upon completion, the Hecke operator program returns a reduced 1-sharbly cycle, which we must convert to a W-cocycle. So let $\xi = \sum n(\mathbf{u})\mathbf{u}$ be a reduced 1-sharbly cycle, and let $\mathbf{u} \in \operatorname{supp} \xi$. First we determine which of the three types T of standard reduced 1-sharblies $\mathbf{u}$ has. Then we must find a matrix $\gamma(\mathbf{u}) \in \mathrm{SL}_4(\mathbb{Z})$ such that

$$\gamma(\mathbf{u}) \cdot \mathbf{u}_T = \mathbf{u},$$

where $\mathbf{u}_T$ is the standard 1-sharbly of type T. This is straightforward, although one must be careful to incorporate the orientation number of $\gamma(\mathbf{u})$.

In practice, the main step is the following. Let $\mathbf{u}$ be a reduced 1-sharbly basis element with lift M. We choose a nonsingular 4×4 minor m of M and construct m^{-1}. Then $m^{-1} \cdot \mathbf{u}$ will be a 1-sharbly with lift $m^{-1}M$, and will be standard except possibly for one column vector. By multiplying $m^{-1}M$ on the left by elements of the stabilizer of the standard 0-sharbly, we can eventually produce the standard 1-sharbly with same type as $\mathbf{u}$. This allows us to construct $\gamma(\mathbf{u})$.

5. Numerical results

5.1. In this section we present numerical data from our experiments. As mentioned in §1, to avoid floating point problems with $\mathbb{C}$-coefficients we usually work with $\mathbb{F} = \mathbb{F}_{31991}$, the finite field with 31991 elements, and in some cases work with $\mathbb{Z}$ or $\mathbb{Q}$. The computations were carried out on a variety of Unix machines at Columbia and Oklahoma State. The code for the cohomology of W (§3) was written in Common Lisp. The Hecke operator code (§4) was written in C++, and used the LiDIA library [14]. Perl scripts patched together the outputs of the various programs and produced the tables in §5.5.

5.2. We first describe how we performed linear algebra on the large sparse matrices that arise in our computations. Fix the level N. We use the notation of §3.21, working over $\mathbb{C}$ at first, and letting $V_{(i)}$ be the domain of the map $d_1^{i,0}$ in (16). To compute $H^5(\Gamma;\mathbb{C})$, we must find the kernel of $d_1^{5,0}$ modulo the image of $d_1^{4,0}$. We prefer to find the kernel of a single matrix. Regard the $d_1^{i,0}$ as matrices acting on the left on column vectors, and let $\mathfrak{D}_\mathbb{Z}$ be the matrix where $d_1^{5,0}$ is stacked on top of the transpose of $d_1^{4,0}$:

$$\mathfrak{D}_\mathbb{Z} = \begin{pmatrix} d_1^{5,0} \\ (d_1^{4,0})^{tr} \end{pmatrix}$$

This matrix defines a map $V_{(5)} \to V_{(6)} \oplus V_{(4)}$, where we use the standard inner product to identify the transpose of $d_1^{4,0}$ with its adjoint. The kernel of this map is the space of *harmonic 5-cocycles*; it is isomorphic to $H^5(\Gamma;\mathbb{C})$. Let $s = \dim V_{(5)}$, the number of columns of $\mathfrak{D}_\mathbb{Z}$.

The matrix $\mathfrak{D}_\mathbb{Z}$ has coefficients in $\mathbb{Z}$. For any ring R, set $\mathfrak{D}_R = \mathfrak{D}_\mathbb{Z} \otimes R$, and set $\mathfrak{D} = \mathfrak{D}_\mathbb{F}$. In the tables in §5.4, the value of "rank" we report in the rows labeled R is a number almost certainly equal to $\dim \ker \mathfrak{D}_R$, whose computation is explained below. This number is also almost certainly equal to $\dim H^5(\Gamma;\mathbb{C})$.

5.3. When $\mathfrak{D}$ is very large, we could not have found its kernel without a sparse version of the Lanczos algorithm. This algorithm is usually used with real or complex matrices, particularly for eigenvalue problems. Following ideas in [19], we translated it into the mod-p setting. Let $\mathfrak{E} = \mathfrak{D}^{tr} \cdot \mathfrak{D}$, a symmetrized version of $\mathfrak{D}$. We choose a random non-zero seed vector $\mathbf{v}$ with coefficients in $\mathbb{F}$ and consider the sequence $\mathbf{v}, \mathfrak{E}\mathbf{v}, \mathfrak{E}^2\mathbf{v}, \ldots$. The Lanczos algorithm shows us how to compute not this sequence, but the sequence $\mathbf{v} = \mathbf{q}_0, \mathbf{q}_1, \ldots$ resulting from it by the Gram-Schmidt orthogonalization process. We perform the Gram-Schmidt process mod p in the naive way, using $\sum x_j y_j$ for the inner product. This means we don't have the usual guarantee that $\sum x_j^2$ will be non-zero when $(x_1, x_2, \ldots) \neq (0, 0, \ldots)$. If the inner product is ever 0 for non-trivial (x_j), we simply abort and choose another random seed $\mathbf{v}$; even for large $\mathfrak{D}$, these aborts happen less than half the time.

The strength of the algorithm is that the RAM only has to hold the sparse matrix $\mathfrak{D}$ and a few vectors of storage. It does not have to hold $\mathfrak{E}$, which is dense in general. The $\mathbf{q}$'s form a dense matrix, but they may be stored on the disk, not in RAM. (In our implementation, $\mathfrak{D}^{tr}$ was stored in RAM along with $\mathfrak{D}$.)

Let k be the largest value for which the set $\{\mathbf{q}_0, \ldots, \mathbf{q}_{s-k}\}$ is linearly dependent. Reading the $\mathbf{q}$'s back in from disk, the algorithm allows us to backsolve for a non-zero vector $\mathbf{y} \in \ker \mathfrak{E}$. What we want is an element of $\ker \mathfrak{D}$. A priori, we only know $\ker \mathfrak{E} \supseteq \ker \mathfrak{D}$, and we will see that the containment is not always an equality (though it would be over $\mathbb{R}$ or $\mathbb{C}$). However, by checking $\mathfrak{D} \cdot \mathbf{y} = 0$ directly, we always find in practice that the $\mathbf{y}$ we compute lie in $\ker \mathfrak{D}$.

Thus each successful run of our algorithm produces one kernel vector for $\mathfrak{D}$. It also produces k. One can easily show $k \geq \dim \ker \mathfrak{E}$. We run Lanczos up to 30 or 40 times with different random seeds, and we find k is independent of the random seed used.([4]) We conclude that $\dim \ker \mathfrak{E} = k$; though we have not proved this, the computational evidence seems conclusive. It is clear that $\dim \ker \mathfrak{E} \geq \dim \ker \mathfrak{D}$.

To find a basis of $\ker \mathfrak{D}$, we run Lanczos many times until we have a set S of $k+10$ elements of $\ker \mathfrak{D}$. We use mod p Gram-Schmidt on subsets $S' \subseteq S$ to find maximal linearly independent subsets of S. We start with several different S''s. In all the cases we checked, we found that maximal linearly independent sets in S had a common cardinality k', that any k'-element subset of S was linearly independent, and that any subset of S with more than k' elements was dependent. We conclude that k' is the value of $\dim \ker \mathfrak{D}$; again, the computational evidence is convincing, though all we have proved is $k' \leq \dim \ker \mathfrak{D}$. In the rows marked "Lanczos" in Table 1, what we report as "rank" is k'. In the rows marked $\mathbb{Z}$ or $\mathbb{Q}$, what we report (namely k') is provably the rank of $H^5(\Gamma, \mathbb{C})$. We do find $k' < k$ sometimes in practice.

5.4. In Table 1, we give the results of our Betti number computations. *Gauß* means that ordinary Gaussian elimination was used to find the kernel of $\mathfrak{D}$, and *Lanczos* means the algorithm of §5.3 was used. The entries marked with $*$ are those for which

[4]For one level, random seeds produced a certain value of k, while one random seed produced a value that was greater by 1. The Lanczos method behaves this way when $E = \{\mathfrak{E}\mathbf{v}, \mathfrak{E}^2\mathbf{v}, \mathfrak{E}^3\mathbf{v}, \ldots\}$ does not span the image of $\mathfrak{E}$, but only a proper subspace of the image. Since $\mathbf{v}$ is chosen randomly, it is extremely rare for E to span less than the full image; our data bears this statement out.

$k = k'$. The entries marked with ? are those for which we only used Lanczos to compute the probable rank k, and didn't apply Gram-Schmidt to compute k'.

Level	Coefficients	rank	remarks	Level	Coefficients	rank	remarks
11	$\mathbb{Z}$	2	Gauß	35	$\mathbb{F}$	7*	Lanczos
12	$\mathbb{F}$	0	Gauß	36	$\mathbb{F}$	24	Lanczos
13	$\mathbb{Z}$	1	Gauß	37	$\mathbb{F}$	8	Gauß
14	$\mathbb{F}$	2	Gauß	38	$\mathbb{F}$	14*	Lanczos
15	$\mathbb{F}$	2	Gauß	39	$\mathbb{F}$	10*	Lanczos
16	$\mathbb{F}$	3	Gauß	40	$\mathbb{F}$	9*	Lanczos
17	$\mathbb{Z}$	3	Gauß	41	$\mathbb{F}$	9*	Lanczos
18	$\mathbb{F}$	9	Gauß	42	$\mathbb{F}$	17	Lanczos
19	$\mathbb{Z}$	3	Gauß	43	$\mathbb{F}$	10*	Lanczos
20	$\mathbb{F}$	2	Gauß	44	$\mathbb{F}$	18	Lanczos
21	$\mathbb{F}$	3	Gauß	45	$\mathbb{F}$	27	Lanczos
22	$\mathbb{F}$	7	Gauß	46	$\mathbb{F}$	19*	Lanczos
23	$\mathbb{Z}$	5	Gauß	47	$\mathbb{F}$	11*	Lanczos
24	$\mathbb{F}$	2	Gauß	48	$\mathbb{F}$	26	Lanczos
25	$\mathbb{F}$	7	Gauß	49	$\mathbb{F}$	33*	Lanczos
26	$\mathbb{F}$	7	Gauß	50	$\mathbb{F}$	34	Lanczos
27	$\mathbb{F}$	12	Gauß	51	$\mathbb{F}$	19*	Lanczos
28	$\mathbb{F}$	7	Gauß	52	$\mathbb{F}$	21*	Lanczos
29	$\mathbb{Q}$	6	Gauß	53	$\mathbb{F}$	17*	Lanczos
30	$\mathbb{F}$	8	Lanczos	55	$\mathbb{F}$	15?	Lanczos
31	$\mathbb{F}$	6	Gauß	59	$\mathbb{F}$	14*	Lanczos
32	$\mathbb{F}$	12	Lanczos	61	$\mathbb{F}$	20*	Lanczos
33	$\mathbb{F}$	10*	Lanczos	67	$\mathbb{F}$	17?	Lanczos
34	$\mathbb{F}$	12*	Lanczos	71	$\mathbb{F}$	17?	Lanczos
				73	$\mathbb{F}$	20*	Lanczos

TABLE 1. Probable Betti numbers for $H^5(\Gamma_0(N); \mathbb{C})$.

5.5. Next, we present the Hecke data we computed. These computations are much more arduous than computing Betti numbers, and grow in complexity very fast as a function of the number of cells of the retract W mod Γ. Hence we were able to compute only a few Hecke operators, usually only $T(2,*)$ and $T(3,*)$. Beyond level 20, it becomes infeasible to compute $T(3,*)$; hence most of our data at large levels is only for $T(2,*)$. Happily this is usually sufficient to guess persuasively what is happening with the cohomology.

We use the following conventions and abbreviations. All polynomials should be considered as elements of $\bar{\mathbb{F}}[X]$, where the bar denotes algebraic closure. We denote the p-adic cyclotomic character of $G_{\mathbb{Q}}$ by ϵ, so that $\epsilon(\mathrm{Frob}_l) = l$ for any $l \neq p$. The symbol IIa (resp. IIb, IV) denotes a Galois representation of the form $\epsilon^a\sigma_k \oplus \epsilon^b \oplus \epsilon^c$, where (k, a, b, c) is $(2, 0, 2, 3)$ (resp. $(2, 2, 0, 1)$, $(4, 0, 1, 2)$), if σ_k is the Galois representation associated to a weight k classical holomorphic cuspidal newform $f \in S_k^{\mathrm{new}}(N')$, where N' divides N. The same symbol prefixed with an "E" denotes a Galois representation of the same form except that σ_k is the Galois representation attached to an Eisenstein series of weight k for $\Gamma_1(N')$.

The individual tables are organized as follows. For each level we give the rank, as defined at the end of §5.3. Then each block gives data for the Hecke eigenspaces. The first column gives the type of Galois representation seemingly attached to this eigenspace. The second column gives the dimension of this eigenspace. The third column gives the index l of the Hecke operator, and the fourth column gives the corresponding factored *Hecke polynomial.* This is the polynomial defined on the right of (1) in Definition 1.2; it succinctly encodes the Hecke action on any vector in the eigenspace. At the bottom of each table, we indicate the $\mathbb{Q}$-splitting of the spaces of newforms $S_k = S_k^{\mathrm{new}}(N)$ under the action of the Hecke operators; this data is from [24]. After each table, we comment of the eigenclasses.

Level **11**. rank = 2.			
IIa	1	T_2	$(1-4X)(1-8X)(1+2X+2X^2)$
		T_3	$(1-9X)(1-27X)(1+X+3X^2)$
		T_5	$(1-25X)(1-125X)(1-X+5X^2)$
		T_7	$(1-49X)(1-343X)(1+2X+7X^2)$
IIb	1	T_2	$(1-X)(1-2X)(1+8X+32X^2)$
		T_3	$(1-X)(1-3X)(1+9X+243X^2)$
		T_5	$(1-X)(1-5X)(1-25X+3125X^2)$
		T_7	$(1-X)(1-7X)(1+98X+16807X^2)$
$\dim S_2(11) = 1$, $\dim S_4(11) = 2$			

The weight 4 newform doesn't lift.

Level **13**. rank = 1.			
IV	1	T_2	$(1-2X)(1-4X)(1+5X+8X^2)$
		T_3	$(1-3X)(1-9X)(1+7X+27X^2)$
		T_5	$(1-5X)(1-25X)(1+7X+125X^2)$
		T_7	$(1-7X)(1-49X)(1+13X+343X^2)$
$\dim S_2(13) = 0$, $\dim S_4(13) = 1+2$			

Only the rational weight 4 newform lifts.

Level **14**. rank = 2.			
IIa	1	T_3	$(1-9X)(1-27X)(1+2X+3X^2)$
IIb	1	T_3	$(1-X)(1-3X)(1+18X+243X^2)$
$\dim S_2(14)=1$, $\dim S_4(14)=1+1$			

No weight four newforms lift.

Level **15**. rank = 2.			
IIa	1	T_2	$(1-4X)(1-8X)(1+X+2X^2)$
IIb	1	T_2	$(1-X)(1-2X)(1+4X+32X^2)$
$\dim S_2(15)=1$, $\dim S_4(15)=1+1$			

No weight four newforms lift.

Level **17**. rank = 2.			
IIa	1	T_2	$(1-4X)(1-8X)(1+X+2X^2)$
		T_3	$(1-9X)(1-27X)(1+3X^2)$
		T_5	$(1-25X)(1-125X)(1+2X+5X^2)$
		T_7	$(1-49X)(1-343X)(1-4X+7X^2)$
IIb	1	T_2	$(1-X)(1-2X)(1+4X+32X^2)$
		T_3	$(1-X)(1-3X)(1+243X^2)$
		T_5	$(1-X)(1-5X)(1+50X+3125X^2)$
		T_7	$(1-X)(1-7X)(1-196X+16807X^2)$
$\dim S_2(17)=1$, $\dim S_4(17)=1+3$			

No weight four newforms lift.

Level **19**. rank = 3.			
IIa	1	T_2	$(1-4X)(1-8X)(1+2X^2)$
		T_3	$(1-9X)(1-27X)(1+2X+3X^2)$
		T_5	$(1-25X)(1-125X)(1-3X+5X^2)$
IIb	1	T_2	$(1-X)(1-2X)(1+32X^2)$
		T_3	$(1-X)(1-3X)(1+18X+243X^2)$
		T_5	$(1-X)(1-5X)(1-75X+3125X^2)$
IV	1	T_2	$(1-2X)(1-4X)(1+3X+8X^2)$
		T_3	$(1-3X)(1-9X)(1+5X+27X^2)$
		T_5	$(1-5X)(1-25X)(1+12X+125X^2)$
$\dim S_2(19)=1$, $\dim S_4(19)=1+3$			

The rational weight four newform lifts.

Level **20**. rank = 2.			
IIa	1	T_3	$(1-9X)(1-27X)(1+2X+3X^2)$
IIb	1	T_3	$(1-X)(1-3X)(1+18X+243X^2)$
$\dim S_2(20)=1$, $\dim S_4(20)=1$			

The rational weight four newform doesn't lift.

Level **21**. rank = 3.			
IIa	1	T_2	$(1-4X)(1-8X)(1+X+2X^2)$
IIb	1	T_2	$(1-X)(1-2X)(1+4X+32X^2)$
IV	1	T_2	$(1-2X)(1-4X)(1+3X+8X^2)$
$\dim S_2(21)=1$, $\dim S_4(21)=1+1+2$			

Of the weight four rational newforms, only one lifts.

Level **23**. rank = 5.			
IIa	2	T_2	$(1-4X)(1-8X)(1-\alpha X+2X^2)$
IIb	2	T_2	$(1-X)(1-2X)(1-4\alpha X+32X^2)$
IV	1	T_2	$(1-2X)(1-4X)(1+2X+8X^2)$
$\dim S_2(23)=2$, $\dim S_4(23)=1+4$			

Here α satisfies $\alpha^2-\alpha+1=0$. The rational weight four newform lifts.

Level **25**. rank = 7.			
IV	1	T_2	$(1-2X)(1-4X)(1+X+8X^2)$
EIIa	2	T_2	$(1-4X)(1-8X)(1+\alpha X+2X^2)$
EIV	2	T_2	$(1-2X)(1-4X)(1+\beta X+8X^2)$
EIIb	2	T_2	$(1-X)(1-2X)(1+\gamma X+32X^2)$
$\dim S_2(25)=0$, $\dim S_4(25)=1+1+1$			

Of the three weight four rational newforms, only one lifts. Here α satisfies $\alpha^2+1=0$, β satisfies $\beta^2+49=0$, and γ satisfies $\gamma^2+16=0$.

Level **27**. rank = 12.			
IIa	1	T_2	$(1-4X)(1-8X)(1+2X^2)$
IIb	1	T_2	$(1-X)(1-2X)(1+32X^2)$
IV	1	T_2	$(1-2X)(1-4X)(1+3X+8X^2)$
EIIa	3	T_2	$(1-4X)(1-8X)(1+X)(1+2X)$
EIV	3	T_2	$(1-2X)(1-4X)(1+X)(1+8X)$
EIIb	3	T_2	$(1-X)(1-2X)(1+4X)(1+8X)$
$\dim S_2(27)=1$, $\dim S_4(27)=1+1+2$			

Of the two weight four rational newforms, only one lifts.

Level **29**. rank = 6.			
IIa	2	T_2 T_3	$(1-4X)(1-8X)(1-\alpha X+2X^2)$ $(1-9X)(1-27X)(1+\alpha X+3X^2)$
IIb	2	T_2 T_3	$(1-X)(1-2X)(1-4\alpha X+32X^2)$ $(1-X)(1-3X)(1+9\alpha X+243X^2)$
IV	2	T_2 T_3	$(1-2X)(1-4X)(1-\alpha X+8X^2)$ $(1-3X)(1-9X)(1-(3\alpha+8)X+27X^2)$
$\dim S_2(29)=2$, $\dim S_4(29)=2+5$			

Here α satisfies $\alpha^2+2\alpha-1=0$. In this example, the weight two and weight four newforms that lift are defined over the same quadratic extension of $\mathbb{Q}$.

Level **31**. rank = 6.			
IIa	2	T_2 T_3	$(1-4X)(1-8X)(1-\alpha X+2X^2)$ $(1-9X)(1-27X)(1+2\alpha X+3X^2)$
IIb	2	T_2 T_3	$(1-X)(1-2X)(1-4\alpha X+32X^2)$ $(1-X)(1-3X)(1+18\alpha X+243X^2)$
IV	2	T_2 T_3	$(1-2X)(1-4X)(1-\beta X+8X^2)$ $(1-3X)(1-9X)(1+(2\beta+6)X+27X^2)$
$\dim S_2(31)=2$, $\dim S_4(31)=2+5$			

Here α satisfies $\alpha^2-\alpha-1=0$, and β satisfies $\beta^2+5\beta+2=0$.

Level **33**. rank = 10.			
IIa	1	T_2	$(1-4X)(1-8X)(1-X+2X^2)$
IIb	1	T_2	$(1-X)(1-2X)(1-4X+32X^2)$
IV	1	T_2	$(1-2X)(1-4X)(1+5X+8X^2)$
IV	1	T_2	$(1-2X)(1-4X)(1+X+8X^2)$
IIa	3	T_2	$(1-4X)(1-8X)(1+2X+2X^2)$
IIb	3	T_2	$(1-X)(1-2X)(1+8X+32X^2)$
$\dim S_2(33)=1$, $\dim S_4(33)=1+1+2+2$			

The three dimensional eigenspaces are lifts of the weight two newform from level 11.

Level **35**. rank = 7.			
IIa	1	T_2	$(1-4X)(1-8X)(1+2X^2)$
IIb	1	T_2	$(1-X)(1-2X)(1+32X^2)$
IIa	2	T_2	$(1-4X)(1-8X)(1-\alpha X+2X^2)$
IIb	2	T_2	$(1-X)(1-2X)(1-4\alpha X+32X^2)$
IV	1	T_2	$(1-2X)(1-4X)(1-X+8X^2)$
$\dim S_2(35)=1+2$, $\dim S_4(35)=1+2+3$			

Here α satisfies $\alpha^2+\alpha-4=0$.

Level **37**. rank $=8$.			
IIa	1	T_2	$(1-4X)(1-8X)(1+2X+2X^2)$
IIb	1	T_2	$(1-X)(1-2X)(1+8X+32X^2)$
IIa	1	T_2	$(1-4X)(1-8X)(1+2X^2)$
IIb	1	T_2	$(1-X)(1-2X)(1+32X^2)$
IV	4	T_2	$(1-2X)(1-4X)(1-\alpha X+8X^2)$
$\dim S_2(37)=1+1$, $\dim S_4(37)=4+5$			

Here α satisfies $\alpha^4+6\alpha^3-\alpha^2-16\alpha+6=0$.

Level **39**. rank $=10$.			
IIa	1	T_2	$(1-4X)(1-8X)(1-X+2X^2)$
IIb	1	T_2	$(1-X)(1-2X)(1-4X+32X^2)$
IIa	2	T_2	$(1-4X)(1-8X)(1-\alpha X+2X^2)$
IIb	2	T_2	$(1-X)(1-2X)(1-4\alpha X+32X^2)$
IV	1	T_2	$(1-2X)(1-4X)(1+8X^2)$
IV	3	T_2	$(1-2X)(1-4X)(1+5X+8X^2)$
$\dim S_2(39)=1+2$, $\dim S_4(39)=1+2+2$			

Here α satisfies $\alpha^2+2\alpha-6=0$. The three dimensional eigenspaces are lifts of the weight four newform from level 13.

Level **41**. rank $=9$.			
IIa	3	T_2	$(1-4X)(1-8X)(1-\alpha X+2X^2)$
IIb	3	T_2	$(1-X)(1-2X)(1-4\alpha X+32X^2)$
IV	3	T_2	$(1-2X)(1-4X)(1-\beta X+8X^2)$
$\dim S_2(41)=3$, $\dim S_4(41)=3+7$			

Here α satisfies $\alpha^3+\alpha^2-5\alpha-1=0$, and β satisfies $\beta^3+3\beta^2-5\beta-3=0$.

Level **43**. rank $=10$.			
IIa	1	T_2	$(1-4X)(1-8X)(1+2X+2X^2)$
IIb	1	T_2	$(1-X)(1-2X)(1+8X+32X^2)$
IIa	2	T_2	$(1-4X)(1-8X)(1-\alpha X+2X^2)$
IIb	2	T_2	$(1-X)(1-2X)(1-4\alpha X+32X^2)$
IV	4	T_2	$(1-2X)(1-4X)(1-\beta X+8X^2)$
$\dim S_2(43)=1+2$, $\dim S_4(43)=4+6$			

Here α satisfies $\alpha^2-2=0$, and β satisfies $\beta^4+4\beta^3-9\beta^2-14\beta+2=0$.

Level **47**. rank = 11.			
IIa	4	T_2	$(1-4X)(1-8X)(1-\alpha X+2X^2)$
IIb	4	T_2	$(1-X)(1-2X)(1-4\alpha X+32X^2)$
IIa	3	T_2	$(1-4X)(1-8X)(1-\beta X+2X^2)$
$\dim S_2(47)=4$, $\dim S_4(47)=3+8$			

Here α satisfies $\alpha^4-\alpha^3-5\alpha^2+5\alpha-1=0$, and β satisfies $\beta^3+5\beta^2-2\beta-12=0$.

6. Interpretation of the numerical results

6.1. The first step, for each Hecke eigenvector $\beta \in H_5(\Gamma, \mathbb{C})$ and for each prime l, is to write down and factor the Hecke polynomial $P_l(X)$.

We then see that for the data computed so far, β has one of the following Galois representations attached. As above, p is a prime not dividing N or any of the l's we are looking at.

Let $k=2$ or 4 and consider the continuous semisimple representation $\sigma_k : G_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{Q}_p)$ unramified outside pN attached to a classical Hecke eigenform f of weight k and level N' dividing N. If N is prime, we also assume f has trivial nebentypus. Let $\rho = \epsilon^a\sigma_k \oplus \epsilon^b \oplus \epsilon^c$ where $(k,a,b,c)=(2,0,2,3)$ or $(2,2,0,1)$, or $(4,0,1,2)$. If f is an Eisenstein series, then $\rho = \chi_0 \oplus \chi_1\epsilon \oplus \chi_2\epsilon^2 \oplus \chi_3\epsilon^3$, where the χ_i are Dirichlet characters of conductor dividing N with values in a finite extension of $\mathbb{F}$, at least two of which are trivial.

Then for any β there is some choice of such ρ which is apparently attached to β, in the sense that the Hecke polynomial at l equals the characteristic polynomial of Frob_l for all l for which we computed the Hecke eigenvalues.

Thus it appears that none of our computed classes so far is cuspidal. Therefore we should be able to relate them to cohomology of the boundary, either geometrically or in terms of Eisenstein series. We cannot do this thoroughly, because neither the cohomology of the Borel-Serre boundary nor the theory of Eisenstein cohomology has been sufficiently worked out for $\mathrm{GL}_4/\mathbb{Q}$. This is not an easy task. We can give the following indications.

From results of Moeglin-Waldspurger [22], we don't expect any of our classes to be residues of Eisenstein series. In the framework of [17] we can guess that our classes lie either in the part of the cohomology indexed by the associate class of parabolic subgroups of GL_4 of type $(2,1,1)$ or in the part indexed by the Borel subgroup. Here the cuspidal data on the GL_2-factor of the Levi component of the first parabolic comes from the appropriate classical cuspform of weight 2 or 4, and we use the appropriate power of the determinant on the GL_1 factors.

6.2. Geometrically, we make the following comments. Let M be the quotient of the symmetric space for $\mathrm{SL}_4(\mathbb{R})$ by Γ and let ∂M be the boundary of its Borel-Serre compactification. The covering of ∂M by its faces gives a spectral sequence for its cohomology. The E_2 page has for its (i,j)-th term $H^i(\text{Tits building}/\Gamma, H^j(\text{Fiber}))$. An element of that is an assignment: to every face $e'(P)$ of codimension i we assign an element of the cohomology in degree j of $P \cap \Gamma$. These assignments when restricted to a common face of codimension $i+1$ must add up to 0. Such an assignment gives a class in $E_2^{i,j}(\partial M)$. If it persists in the spectral sequence to E_∞, it will contribute to the cohomology $H^{i+j}(\partial M, \mathbb{C})$. There remains the question as to whether this contribution is the restriction of a class in $H^{i+j}(M, \mathbb{C})$.

Note that if $P = LU$ is a Levi decomposition of P then the spectral sequence of the fibration for the cohomology of $P \cap \Gamma$ corresponding to this decomposition is known to degenerate at $E_2^{p,q} = H^p(\Gamma_L, H^q(U \cap \Gamma)$ where Γ_L is the projection of Γ to L.

The classes we have computed so far we expect to be coming in this way from ∂M. From the shape of the apparently associated Galois representations, here is what we believe is their origin. We only sketch the constructions, since a detailed description would require a thorough investigation of the cohomology of ∂M for arbitrary congruence subgroups of $\mathrm{SL}_4(\mathbb{Z})$. First assume f is a cuspform.

6.3. The case where σ has weight 2: By the Eichler-Shimura theorem, the cuspform f that has σ attached shows up as a class α in $H^1(\Delta, \mathbb{C})$, where Δ is the classical $\Gamma_0(N') \subset \mathrm{SL}_2(\mathbb{Z})$. First suppose $N' = N$. Consider the following element of $E_2^{0,5}(\partial M)$: On the standard parabolic subgroup of type $(2,2)$ which is the stabilizer of the span of (e_1, e_2) in 4-space, we put the cohomology class $\alpha \times 1$ on the Levi component where we view the trivial coefficients of α as the module $H^4(U \cap \Gamma)$. One sees that there is a unique class in the appropriate face corresponding to a $(3,1)$-type parabolic subgroup that has the same restriction to the type $(2,1,1)$ face they have in common and restricts to 0 on the other faces. Hence these two glue up to give a class in the boundary. The same construction with the transposed parabolic subgroup also gives a class, and these seem to account for all our β's falling under this case. If $N' \neq N$, we choose a $(2,2)$- parabolic subgroup P such that Γ_L has level N'. Then we imitate the construction above.

6.4. The case where σ has weight 4: Here not every class we construct on the boundary seems to lift to $H^5(M)$, but only some of them. We don't know the reason for this. The construction here creates a class in $E_2^{1,4}(\partial M)$, and if this correctly describes what we have computed, our computed classes of this type would be ghost classes. That is, the corresponding cohomology class in $H^5(M)$ restricts nontrivially to the boundary of M, but it restricts to 0 on each face of the boundary, since it is coming from a class in $E_2^{i,j}(\partial M)$ with $i > 0$.

For this construction, one first chooses a parabolic subgroup P of type $(2,1,1)$. Note that $H^3((U \cap \Gamma), \mathbb{C})$ contains a Γ_L-submodule isomorphic to $\mathcal{V}_2$, the homogeneous complex polynomials of degree 2 on 2-space, after we identify Γ_L with a subgroup of $\mathrm{GL}_2(\mathbb{Z}) \times \mathrm{GL}_1(\mathbb{Z}) \times \mathrm{GL}_1(\mathbb{Z})$. By the Eichler-Shimura theorem, the weight 4 cuspform f

that has σ attached shows up in $H^1(\Delta, \mathcal{V}_2)$. Thus we can view it as in $H^1(\Gamma_L, H^3(U \cap \Gamma))$.

We can do this on three P's which are not conjugate to each other in such a way that they, together with certain classes on $(3,1)$-type parabolic subgroups, all glue together to give a class in $E_2^{1,4}(\partial M)$. The details are left for the reader.

6.5. Finally, the case where σ is an Eisenstein series is harder to understand. We haven't worked out exactly how the gluing process goes in this case, so we're not sure what stratum of the spectral sequence is occupied by the corresponding boundary classes.

We note that in every case, ρ restricted to an inertia subgroup of $G_{\mathbb{Q}}$ at p has the form $1 \oplus \epsilon \oplus \epsilon^2 \oplus \epsilon^3$, which is consistent with the conjecture of Ash-Sinnott [10], since the coefficient module of the cohomology classes we consider is the trivial module.

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