FINITENESS OF MINIMAL MODULAR SYMBOLS FOR SL_n

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ABSTRACT. Let $K/\mathbb{Q}$ be a number field with ring of integers $\mathscr{O}_K$, and let $\Gamma \subset SL_n(\mathscr{O}_K)$ be a finite index subgroup. Using a classical construction from the geometry of numbers and the theory of modular symbols, we exhibit a finite spanning set of the highest nonvanishing rational cohomology group of Γ.

1. INTRODUCTION

Let $K/\mathbb{Q}$ be a number field with ring of integers $\mathscr{O}_K$. Let $\Gamma \subset SL_n(\mathscr{O}_K)$ be a finite index subgroup, and let ν be the virtual cohomological dimension of Γ. That is, if $\Gamma' \subset \Gamma$ is any finite index torsion-free subgroup, then $H^i(\Gamma', M) = 0$ for $i > \nu$ and any $\mathbb{Z}\Gamma$-module M.

Let $\mathscr{M}$ be the free abelian group generated by the symbols $[v_1, \ldots, v_n]$, where the v_i are nonzero points in K^n, modulo the following relations:

(1) If τ is a permutation on n letters, then $[v_1, \ldots, v_n] = \operatorname{sgn}(\tau)[\tau(v_1), \ldots, \tau(v_n)]$, where $\operatorname{sgn}(\tau)$ is the sign of τ.
(2) If $q \in K^\times$, then $[qv_1, v_2, \ldots, v_n] = [v_1, \ldots, v_n]$.
(3) If the v_i are linearly dependent, then $[v_1, \ldots, v_n] = 0$.
(4) If $v_0, \ldots, v_n$ are nonzero points in K^n, then $\sum_i (-1)^i [v_0, \ldots, \hat{v_i}, \ldots, v_n] = 0$.

Elements of $\mathscr{M}$ are called *minimal modular symbols*. By a theorem of Ash [1] there is a surjective map of $\mathbb{Z}\Gamma$-modules

$$\mathscr{M} \longrightarrow H^\nu(\Gamma, \mathbb{Q}). \tag{5}$$

Hence $\mathscr{M}$ modulo Γ is a spanning set for $H^\nu(\Gamma, \mathbb{Q})$.

However, this spanning set is not finite. If $\mathscr{O}_K$ is a euclidean domain, Ash and Rudolph [2] define a subset $\mathscr{M}' \subset \mathscr{M}$ with finite image under (5), and give an algorithm to write any $[m] \in \mathscr{M}$ as a sum $[m] = \sum [m_i]$, where each $[m_i] \in \mathscr{M}'$. Our goal in this note is more modest: in Theorem 1 we exhibit a finite spanning set for $H^\nu(\Gamma, \mathbb{Q})$ for all K, not necessarily euclidean, but we provide no reduction algorithm. The proof relies on a classical construction of Minkowski from the geometry of numbers.

2. STATEMENT OF THE RESULT

2.1. Let $(K : \mathbb{Q}) = d$. We use a "restriction of scalars" construction to put an integral structure on $\mathscr{M}$. Recall that the *regular representation* $\mathscr{O}_K \to M_d(\mathbb{Z})$ is defined by $\alpha \mapsto \ell_\alpha$, where ℓ_α is the matrix of the map $x \mapsto \alpha x$ in terms of a fixed $\mathbb{Z}$-basis of $\mathscr{O}_K$. This induces a map $\varphi \colon M_n(\mathscr{O}_K) \to M_{nd}(\mathbb{Z})$, in which matrix entries are taken to $d \times d$ blocks.

Via φ any column vector $v \in \mathscr{O}_K^n$ determines d column vectors $\{v^1, \ldots, v^d\} \subset \mathbb{Z}^{nd}$. Let $L(v) \subset \mathbb{Z}^{nd}$ be the $\mathbb{Z}$-lattice generated by the v^i, and define the *index* of v to be

$$i(v) := \left[(L(v) \otimes \mathbb{Q}) \cap \mathbb{Z}^{nd} : L(v) \cap \mathbb{Z}^{nd} \right].$$

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Definition 1. A nonzero vector $v \in \mathscr{O}_K^n$ is *primitive* if, for all $\alpha \in K^\times$ with $\alpha v \in \mathscr{O}_K^n$, we have $i(v) \leq i(\alpha v)$.

Let $N_{K/\mathbb{Q}}\colon K \to \mathbb{Q}$ be the norm map.

Proposition 1. *Let v, $v' \in \mathscr{O}_K^n$ be nonzero and primitive, and suppose that $v = \alpha v'$ for some $\alpha \in K^\times$. Then $|N_{K/\mathbb{Q}}(\alpha)| = 1$.*

Proof. Choose points $w_i \in \mathscr{O}_K^n$ so that $m = (v, w_1, \dots, w_{n-1})$ and $m' = (v', w_1, \dots, w_{n-1})$ have nonzero determinant. Then $m = m' \cdot A$, where $A = \operatorname{diag}(\alpha, 1, \dots, 1)$.

Now consider $\varphi(m) \in M_{nd}(\mathbb{Z})$. The index of the $\mathbb{Z}$-lattice generated by the columns of $\varphi(m)$ is equal to $|\det \varphi(m)|$, and since $i(v) = i(v')$, this index must also equal $|\det \varphi(m')|$. This implies $|\det \varphi(A)| = 1$, and the result follows since $\det \varphi(A) = N_{K/\mathbb{Q}}(\alpha)$. $\square$

2.2. Let $[m] = [v_1, \dots v_n]$ be a modular symbol. Applying (2), we may assume each $v_i \in \mathscr{O}_K^n$ is primitive. Define a map

$$\| \ \| \colon \mathscr{M} \longrightarrow \mathbb{Z}^{>0}, \tag{6}$$

$$m \longmapsto |N_{K/\mathbb{Q}}(\det m)|. \tag{7}$$

Proposition 2. *The map $\| \ \|$ is well defined.*

Proof. Let m, $m' \in M_n(\mathscr{O}_K)$ have primitive, nonzero columns, and suppose $[m] = [m']$. Applying (1) if necessary, we may assume $m = m' \cdot A$, where $A = \operatorname{diag}(\alpha_1, \dots, \alpha_n)$. By Proposition 1 we have $|N_{K/\mathbb{Q}}(\alpha_i) = 1|$, which implies $|\det \varphi(A)| = 1$. The result follows since $\|m\| = |\det \varphi(m)|$. $\square$

Definition 2. Given $C \geq 1$, let

$$\mathscr{M}(C) := \bigl\{[m] \in \mathscr{M} \bigm| \|m\| \leq C\bigr\}.$$

Proposition 3. *If $\Gamma \subset SL_n(\mathscr{O}_K)$ is of finite index, then $\Gamma \backslash \mathscr{M}(C)$ is finite for all $C \geq 1$.*

Proof. It suffices to verify the statement for $\Gamma = SL_n(\mathscr{O}_K)$. To simplify notation, we write G for $GL_n(\mathscr{O}_K)$. If $m \in M_n(\mathscr{O}_K)$, then we write $\Lambda(m)$ for the lattice Gm.

Let $M(C)$ denote the set of matrices $\bigl\{m \in M_n(\mathscr{O}_K) \bigm| |N_{K/\mathbb{Q}}(\det m)| \leq C\bigr\}$. We claim that $G\backslash M(C)$ is finite for any $C \geq 1$. Indeed, the set $\bigl\{m \bigm| [\mathscr{O}_K^n : \Lambda(m)] \leq C\bigr\}$ is finite modulo G. Now for any matrix m, the index $[\mathscr{O}_K^n : \Lambda(m)]$ is equal to $\mathscr{N}(\operatorname{ord}(T))$, where $\mathscr{N}$ denotes the ideal norm, and $\operatorname{ord}(T)$ is the *order ideal* of the torsion module $T := \mathscr{O}_K^n/\Lambda(m)$ ([3], §4D). Furthermore, $\operatorname{ord}(T)$ is a principal ideal generated by $\det(f)$, where $f\colon \mathscr{O}_K^n \to \mathscr{O}_K^n$ is any $\mathscr{O}_K$-linear endomorphism with image $\Lambda(m)$. Clearly for f we may take multiplication by m, and thus

$$\begin{aligned}[\mathscr{O}_K^n : \Lambda(m)] &= \mathscr{N}((\det m)) \\ &= |N_{K/\mathbb{Q}}(\det m)|.\end{aligned}$$

This implies $M(C)$ is finite modulo G.

We claim this implies $\mathscr{M}(C)$ is finite modulo Γ. To see this we use the following easily verified fact. Suppose a group A acts on a set S, and $B \lhd A$ is a normal subgroup. Then if A/B is a finite group, and $A\backslash S$ is finite, then $B\backslash S$ is finite.

To apply this, let $A = G$ and $B = Z \times \Gamma$, where Z is the center of G. The group $G/(Z \times \Gamma)$ is isomorphic to U/U^n, where $U = \mathscr{O}_K^\times$ and U^n is the subgroup of nth powers. Moreover, $G/(Z \times \Gamma)$ is finite by Dirichlet's unit theorem. Setting $S = M(C)$, we have that $G\backslash M(C)$ finite implies that $(Z \times \Gamma)\backslash M(C)$ is finite. Since $M(C)$ maps surjectively onto $\mathscr{M}(C)$, and Z acts trivially on $\mathscr{M}$, the result follows. $\square$

2.3. Let $V = K \otimes \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$, where $r + 2s = d$. We define the *Minkowski constant*[1] M_K by

$$M_K = \left(\frac{\pi}{4}\right)^s \frac{d^d}{d!}.$$

We now state our main result.

Theorem 1. *$\mathscr{M}(C)$ spans $H^\nu(\Gamma, \mathbb{Q})$ for*

$$C \geq \left\lfloor \left(\frac{\sqrt{|D_K|}}{M_K}\right)^n \right\rfloor.$$

3. Proof

3.1. Fix a modular symbol $[m] = [v_1, \dots v_n]$. We claim that if $\|m\| > (\sqrt{|D_K|}/M_K)^n$, then we can find a nonzero $x \in \mathscr{O}_K^n$ such that $x = \sum q_i v_i$ with $q_i \in K$ and $0 \leq |N_{K/\mathbb{Q}}(q_i)| < 1$. We may then use (4) to construct a relation $[m] = \sum (-1)^i [m_i]$, where $[m_i] := [v_1, \dots, v_{i-1}, \hat{v}_i, x, v_{i+1}, \dots, v_n]$. Clearly $\|m_i\| < \|m\|$. This claim implies the theorem, because by iterating this process we can write any $[m] \in \mathscr{M}$ as a sum of symbols from $\mathscr{M}(C)$.

To prove the claim we use φ. Fix a $\mathbb{Z}$-basis $\omega_1, \dots, \omega_d$ of $\mathscr{O}_K$, and apply φ to m:

$$(v_1, \dots, v_n) \longmapsto (v_1^1, \dots, v_1^d, \dots, v_n^1, \dots, v_n^d) \in M_{nd}(\mathbb{Z}).$$

Then $\varphi(m_i)$ is obtained from $\varphi(m)$ by replacing the columns $v_i^1, \dots, v_i^d$ with $x^1, \dots, x^d$.

For $1 \leq i \leq n$, $1 \leq j \leq d$, let λ_i^j be real variables, and consider the region $S \subset \mathbb{R}^{nd}$ defined by

$$S := \left\{ \sum_{\substack{1 \leq i \leq n \\ 1 \leq j \leq d}} \lambda_i^j v_i^j \;\middle|\; \left| N_{K/\mathbb{Q}}\Big(\sum_j \lambda_i^j \omega_j\Big) \right| < 1, \text{where } 1 \leq i \leq n \right\}.$$

Here we interpret $N_{K/\mathbb{Q}}(\sum_j \lambda_i^j \omega_j)$ to mean the polynomial in $\mathbb{Z}[\lambda_i^j]$ constructed using the norm form.

Lemma 1. *There exists a nonzero $x \in \mathscr{O}_K^n$ such that $x = \sum q_i v_i$ with $q_i \in K$ and $0 \leq |N_{K/\mathbb{Q}}(q_i)| < 1$ if and only if the region S contains a nonzero $\xi \in \mathbb{Z}^{nd}$.*

Proof. Given $x \in \mathscr{O}_K^n$ satisfying the hypotheses, the x^i are nonzero rational integral points in S. Conversely, let $\xi \in S$ be a nonzero rational integral point. Then the definition of S ensures that $\varphi(\omega_i)\xi \in S$ for each ω_i in our $\mathbb{Z}$-basis of $\mathscr{O}_K$. The d vectors $(\varphi(\omega_1)x, \dots, \varphi(\omega_d)x)$ are $\varphi(x)$ for some $x \in \mathscr{O}_K^n$, and this x has the desired properties. □

3.2. Now we will find a bounded symmetric convex body $P \subset S$ and show that if $\|m\| > (\sqrt{|D_K|}/M_K)^n$, then $\operatorname{vol} P > 2^{nd}$. Then by Minkowski's theorem ([4], IV.2.6) P and hence S will contain a nonzero integral point. By Lemma 1 this will imply Theorem 1.

To do this, apply $\varphi(m^{-1})$ to S. This carries $\{v_i^j\}$ onto the standard basis of $\mathbb{R}^{nd}$. We can then write $\varphi(m^{-1})(S)$ as the n-fold product T^n, where $T \subset \mathbb{R}^d$ is the region

$$T := \left\{ (y_1, \dots, y_d) \;\middle|\; \left| N_{K/\mathbb{Q}}\Big(\sum y_i \omega_i\Big) \right| \leq 1 \right\}.$$

This region can be transformed further as follows. The vector space V contains $\mathscr{O}_K$, embedded by $\alpha \mapsto \alpha \otimes 1$. Let $\mu \colon \mathbb{R}^d \to \mathbb{R}^d$ be the linear map taking the standard basis of $\mathbb{Z}^d$ to $\{\omega_1 \otimes 1, \dots, \omega_d \otimes 1\}$. Then

$$\mu(T) = \left\{ (x_1, \dots, x_r, z_1, \dots, z_s) \in \mathbb{R}^r \times \mathbb{C}^s \;\middle|\; |x_1 \cdots x_r| z_1 \bar{z}_1 \cdots z_s \bar{z}_s < 1 \right\}.$$

[1] A classical result of Minkowski bounds the discriminant D_K of K: if $K \otimes \mathbb{R} \cong \mathbb{R}^r \times \mathbb{C}^s$, then $D_K^2 \geq M_K$.

Now we construct the *generalized octahedron* of Minkowski. This will be a bounded, symmetric, convex body in $\mu(T)$. Take polar coordinates (ρ_i, θ_i) for the z_i, and let $V^+ \subset V$ be the subset

$$V^+ := \bigl\{(x_1, \ldots, x_r, \rho_1, \theta_1, \ldots, \rho_s, \theta_s) \bigm| x_i \geq 0,\ \rho_i \geq 0, \text{ and } \theta_i = 0\bigr\}.$$

Definition 3. Given a point $p \in \mu(T) \cap V^+$, let $Q(p)$ be the subset of $\mu(T)$ constructed as follows:

1. Construct the tangent hyperplane to $\mu(T) \cap V^+$ at p.
2. Use this hyperplane and the bounding hyperplanes of V^+ to cut out a $(2r)$-simplex Δ in V^+.
3. Apply the motions $x_i \mapsto -x_i$ and $\theta_i \mapsto \theta_i + \beta$, $0 \leq \beta \leq 2\pi$ to Δ to sweep out $Q(p)$ (cf. Figure 1).

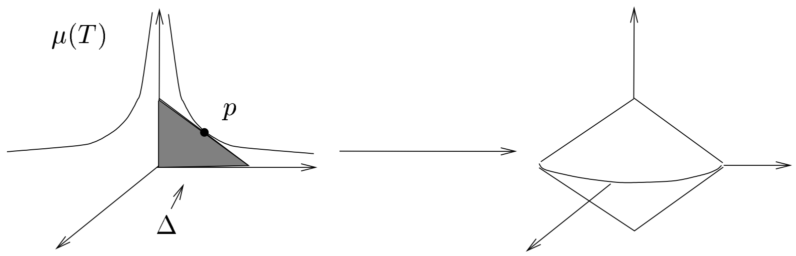

FIGURE 1. The generalized octahedron for $(r, s) = (1, 1)$.

Lemma 2. *$Q(p)$ is bounded, symmetric, and convex. The volume of $Q(p)$ is independent of p, and is*

$$2^{r+s} M_K. \tag{8}$$

Proof. All statements are standard results from the geometry of numbers ([4], IV.2), except for the independence of p. However, this is easy to verify. $\square$

3.3. We now complete the proof of the theorem. We choose $p \in \mu(T) \cap V^+$ and abbreviate $Q(p)$ to Q. Define $P \subset S$ by

$$P := \varphi(m)\bigl(\mu^{-1}Q \times \cdots \times \mu^{-1}Q\bigr).$$

P is symmetric, bounded, and convex, and

$$\operatorname{vol} P = |\det \varphi(m)| \left(\frac{\operatorname{vol} Q}{|\det \mu|}\right)^n. \tag{9}$$

Now in (9) we apply Lemma 2, and substitute $|\det \mu| = 2^{-s}\sqrt{|D_K|}$ and $\|m\| = |\det \varphi(m)|$. To ensure that $\operatorname{vol} P > 2^{nd}$, we require

$$\|m\| > \left(\frac{\sqrt{|D_K|}}{M_K}\right)^n,$$

as desired.

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