WONDERFUL BLOWUPS ASSOCIATED TO GROUP ACTIONS

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ABSTRACT. A group action on a smooth variety provides it with the natural stratification by irreducible components of the fixed point sets of arbitrary subgroups. We show that the corresponding maximal wonderful blowup in the sense of MacPherson-Procesi has only abelian stabilizers. The result is inspired by the abelianization algorithm of Batyrev.

1. INTRODUCTION

1.1. Let $X/\mathbb{C}$ be a smooth variety, and for $n > 1$ consider the product X^n. For any collection $I = \{I_k\}$ of disjoint subsets $I_k \subseteq \{1, \dots, n\}$ with each I_k of order at least two, let Δ_I be the polydiagonal consisting of all $x \in X^n$ with $x_i = x_j$ if and only if $i, j \in I_k$ for some k. The locally closed subvarieties Δ_I provide a stratification of X^n that is stable under the permutation action of the symmetric group S_n. In [4], Fulton and MacPherson constructed a smooth variety $X[n]$ and a birational morphism $\pi \colon X[n] \to X^n$ such that π is an isomorphism over the open set $\Delta_\varnothing$, and such that the stabilizers of the induced S_n action on $X[n]$ are solvable. More recently, Ulyanov [6] constructed a space $X\langle n\rangle$ with a stratified birational morphism to $X[n]$ such that the stablilizers of the induced S_n action are abelian. The variety $X\langle n\rangle$ is constructed by systematically blowing up the closures of the preimages of polydiagonals Δ_I of increasing dimension.

Both $X[n]$ and $X\langle n\rangle$ are special cases of a general construction of MacPherson and Procesi [5]. Suppose that Z is an irreducible smooth variety and that $S = \{S_i\}$ is a set of locally closed subvarieties endowing Z with a stratification. Suppose further that this stratification is *conical*; this means that every point $z \in Z$ has an analytic neighborhood isomorphic to a product of a trivial stratification of a disc and of a restriction to a disc of a $\mathbb{C}^*$-invariant stratification of a vector space. A basic example is $Z = X^n$ and $S = \{\Delta_I\}$. Then MacPherson and Procesi constructed a family of *wonderful* compactifications of the open stratum. These smooth compactifications map birationally onto Z isomorphically away from strata of codimension bigger than one, and have the property that the complement of the open stratum is a normal

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crossing divisor. Moreover, for any S there are canonical minimal and maximal wonderful compactifications. In the example above, these spaces are respectively $X[n]$ and $X\langle n\rangle$.

1.2. In this paper we explain the connection

$$X[n] \Longleftrightarrow S_n \text{ acts with solvable stabilizers}$$
$$X\langle n\rangle \Longleftrightarrow S_n \text{ acts with abelian stabilizers}$$

by investigating wonderful blowups associated to certain stabilizer stratifications of a variety with group action. More precisely, let X be a smooth variety and let G be a finite group acting on X. We define two stratifications of X, the *stabilizer* and Y stratifications, and show that they are conical. Our main results (Corollary 3.3, Theorem 4.3, and Corollary 5.3) are that G acts on the maximal wonderful blowups associated to each of these stratifications with abelian stabilizers, and on the minimal wonderful blowup associated to the Y stratification with solvable stabilizers. The main tool is the abelianization algorithm of Batyrev ([1], corrected in [2]).

1.3. **Notations.** We fix a finite group G for the rest of this paper, and work in the category of smooth G-varieties over $\mathbb{C}$, or smooth complex G-manifolds. We always assume that the action of G is effective, i.e. that only the identity stabilizes all of any G-variety X. If X is a G-variety, then for every point $x \in X$ its stabilizer is denoted by $\mathrm{Stab}(x)$. For every subgroup $H \subseteq G$ the set of fixed points of H is denoted by $\mathrm{Fixed}(H)$. The latter is always a disjoint union of smooth subvarieties of X.

2. Batyrev's abelianization algorithm

2.1. The goal of this section is to describe a version of Batyrev's abelianization algorithm [1, 2]. Our version is a bit easier to formulate but generally results in more blowups.

Let X be a G-variety as above. For every point $x \in X$ and any subgroup H fixing x (H may be a proper subgroup of $\mathrm{Stab}(x)$) consider the decomposition of the tangent space TX_x into irreducible H-modules

$$TX_x = \bigoplus_{i=1}^{k} V_i,$$

where $\dim(V_i) = 1$ for $i \leq k_1$ and $\dim(V_i) > 1$ for $i > k_1$. Each one-dimensional representation V_i gives a character $H \to \mathbb{C}^*$. Denote by H_1 the common kernel of these characters, and let $Y(x, H)$ be the subvariety of X that is the connected component of $\mathrm{Fixed}(H_1)$ passing through x. Observe that the tangent space to $Y(x, H)$ at x is precisely $\oplus_{i=1}^{k_1} V_i$ [2]. Notice that $Y(x, H) = X$ if and only if H is abelian. Otherwise, the codimension of $Y(x, H)$ is at least 2. We will consider the set of the *proper* subvarieties $Y_j \subseteq X$ that are equal to $Y(x, H)$ for at least one pair (x, H).

Theorem 2.2. [1, 2] *In the above setup, denote by $r(X)$ the maximum number of different subvarieties Y_j with non-trivial intersection. Denote by Z the set of points $x \in X$ that are contained in $r(X)$ different Y_j. Then Z is smooth and G-equivariant. We let X_1 be the blowup of X along Z, and iterate the procedure: for $i \geq 1$ we compute $r(X_i)$ and Z_i, and define X_{i+1} to be the blowup of X_i along Z_i. We claim that this process terminates after at most $r(X)$ steps, and that the stabilizer of every point of the resulting variety is abelian.*

Proof. It is straightforward to see that the intersection of any number of the Y_j is smooth. This implies that Z is smooth, and it is clearly G-equivariant. It will now suffice to prove that $r(X_1) < r(X)$.

We claim that every $Y_{j,1}$ for X_1 is a proper preimage of some Y_j for X that is not contained in Z and vice versa. Let $Y_{j,1} = Y(x_1, H)$ be a proper subvariety of X_1. Denote the image of x_1 under the blowdown map by x, and consider $Y(x, H)$. If $x \notin Z$, there is nothing to prove. If $x \in Z$, notice that Z is a subvariety of $Y = Y(x, H)$, so the tangent space TZ_x is an H-submodule of

$$TY_x = \bigoplus_{i=1}^{k_1} V_i.$$

If the connected component of Z passing through x equals Y, then the fiber of the blowup over x, which equals $\mathbb{P}(\oplus_{i>k_1} V_i)$, has no H-fixed points. Therefore the tangent space to Z is a proper submodule of TY_x, which we assume to equal $\oplus_{i=1}^{k_2} V_i$. The point x_1 then corresponds to a one-dimensional submodule of TY_x/TZ_x, which we will identify with V_{k_2+1}. The tangent space $(TX_1)_{x_1}$ is isomorphic as an H-module to

$$\bigoplus_{i=1}^{k_2+1} V_i \bigoplus_{i=k_2+2}^{k} (V_{k_2+1} \otimes V_i).$$

Therefore, the tangent space to $Y_{j,1}$ at x_1 is

$$\bigoplus_{i=1}^{k_2+1} V_i \bigoplus_{i=k_2+2}^{k_1} (V_{k_2+1} \otimes V_i),$$

which is the tangent space of the proper preimage of Y. The statement in the opposite direction is proved similarly.

To show that $r(X_1) < r(X)$, suppose that x_1 is contained in $r(X)$ different proper preimages of the Y_j. Then x_1 corresponds to a normal direction to Z which is contained in all these Y_j's, which leads to a contradiction. □

3. Batyrev's abelianization as a maximal wonderful blowup

3.1. The proof of Theorem 2.2 shows that once the Y_j are defined for X, their exact nature is not important for the further blowups. Indeed, at every step one blows up the intersection set of the maximum number of the birational proper preimages of the Y_j. This allows us to identify the result of Batyrev's algorithm with the *maximal wonderful blowup* of the MacPherson-Procesi family of blowups [5] associated to the stratification defined by the Y_j.

Definition 3.2. The Y *stratification* of X is the stratification induced by the Y_j as follows. For every subset of $\{Y_j\}$ one defines a stratum that consists of all points that lie in all Y_j from the subset, but do not lie in any other Y_j. The empty strata are then ignored.

Corollary 3.3. *The Y stratification of X is conical. The maximal wonderful blowup coincides with the result of Batyrev's algorithm above. In particular, it has only abelian stabilizers.*

Proof. First, it is easy to see that the Y stratification is conical. Moreover, this can be said about any stratification induced by some connected components of fixed point sets of some subgroups of G. Indeed, for any $x \in X$ there exists a $\mathrm{Stab}(x)$-equivariant isomorphism of a neighborhood $U \ni x$ and a neighborhood of the origin in TX_x. Under this isomorphism all strata map to linear subspaces, so the stratification is conical.

Second, the maximal wonderful blowup is defined by successively blowing up proper preimages of all strata, starting with the strata of smallest dimension. This is equivalent to blowing up strata in any order that is compatible with the partial ordering on the strata, and Batyrev's algorithm clearly provides that. □

4. The stabilizer stratification

4.1. The Y stratification is not local, in the sense that its construction does not commute with G-equivariant open embeddings. A more natural stratification of X is induced by the set of all connected components of $\mathrm{Fixed}(H)$ for all subgroups $H \subseteq G$. We will call it *stabilizer stratification* of X. It is also conical by the argument presented in the proof of Corollary 3.3.

Proposition 4.2. *Let X^Y be the maximal wonderful blowup associated to the Y stratification of X, and let X^{stab} be the maximal wonderful blowup associated to the stabilizer stratification of X. Then there exists a natural G-equivariant map*

$$X^{stab} \longrightarrow X^Y.$$

Proof. The statement is clearly local in X, so it could be assumed that X is a neighborhood of the origin in a G-vector space V. Denote by S_Y and S_{stab} the sets of strata in the Y and stabilizer stratifications respectively. According to [3, Definition p.461, Proposition p.470] the maximal blowups X^Y and X^{stab} can be described as closures of the images of an open subset of V under the map to

$$V \times \prod_{k \in S} \mathbb{P}(V/V_k).$$

Here V_k is the tangent space to the stratum, and S is either S_Y or S_{stab}. The functoriality is then obvious. □

As a corollary we get the following theorem, which is the main result of this paper.

Theorem 4.3. *The stabilizer of every point of the maximal wonderful blowup associated to the stabilizer stratification is abelian.*

Proof. Combine Corollary 3.3 and Proposition 4.2. □

5. Solvable stabilizers

5.1. It is generally much easier to ensure that a blowup of a G-variety has only *solvable* stabilizers, due to the following observation.

Proposition 5.2. *Let $\pi\colon X_1 \to X$ be a G-equivariant birational morphism of G-varieties. Then all stabilizers of X_1 are solvable if and only if the image of the exceptional divisors of π contains all points of X with non-solvable stabilizers.*

Proof. Suppose X_1 has points with non-solvable stabilizers. Consider the point $x_1 \in X_1$ with a non-solvable stabilizer H that lies in the minimum number of exceptional divisors of π. If this minimum number is zero, then X has a point with non-solvable stabilizer that lies outside the image of the exceptional divisors. Otherwise, let E be an exceptional divisor of π that passes through x. The tangent space TX_{x_1} splits as an H-module into TE_{x_1} and a one-dimensional module. This one-dimensional module corresponds to a character $\chi\colon H \to \mathbb{C}^*$. The kernel of this character is again non-solvable; however its fixed point set contains points in the neighborhood of x_1 not lying in E. This contradicts the minimality of x_1 and proves the "if" part. The "only if" part is obvious. □

Corollary 5.3. *The minimal wonderful blowup associated to the Y stratification has only solvable stabilizers.*

Proof. In view of Proposition 5.2, one needs to show that every point $x \in X$ with non-solvable stabilizer is contained in an irreducible stratum of codimension bigger than one. Since all Y_j have codimension at least two, it remains to observe that each point x with a non-solvable stabilizers is contained in $Y = Y(x, \mathrm{Stab}(x)) \neq X$. □

Question 5.4. Is it true that all stabilizers of the minimal wonderful blowup associated with the stabilizer stratification are solvable?

This is really a question of whether there exists a non-solvable group G acting on $\mathbb{C}^d$ with a fixed basis, such that for every $g \in G$ the space $\mathrm{Ker}(g-1)$ is a coordinate subspace.

6. Examples

Example 6.1. As in Section 1, let G be the symmetric group S_n acting on the product X^n of n copies of a variety X. Then the Y stratification differs from the stabilizer stratification only in that it does not distinguish general points on the large diagonals (the conjugates of the preimages of the diagonals in $X \times X$) from the general points on X^n. As the result, the maximal wonderful blowup associated to the stabilizer stratification [6] is obtained from the maximal wonderful blowup for the Y stratification by blowing up disjoint proper preimages of the large diagonals. Theorem 4.3 implies the result of Ulyanov [6] that the stabilizer of the maximal blowup is abelian. It is also easy to see that the discussion of Section 5 implies that the stabilizers of the minimal wonderful blowups are solvable, which was first observed in [4].

It is natural to ask whether for any finite group action of G on X there exists a *minimum abelianization*, defined to be the G-equivariant birational morphism $\pi\colon X_1 \to X$ such that X_1 has abelian stabilizers and such that every other such map factors through π. Unfortunately, the next example shows that this is not the case.

Example 6.2. Let $X = \mathbb{C}^4$ and let $G = S_3$ act on X by linear transformations such that X is a sum of two irreducible 2-dimensional modules of G. Then a blowup at the origin or a blowup along any 2-dimensional submodule of X has only abelian stabilizers, but there is clearly no minimum abelianization.

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