

# KAZHDAN–LUSZTIG CELLS IN PLANAR HYPERBOLIC COXETER GROUPS AND AUTOMATA

MIKHAIL V. BELOLIPETSKY, PAUL E. GUNNELLS, AND RICHARD SCOTT

ABSTRACT. Let  $C$  be a one- or two-sided Kazhdan–Lusztig cell in a Coxeter group  $(W, S)$ , and let  $\text{Reduced}(C)$  be the set of reduced expressions of all  $w \in C$ , regarded as a language over the alphabet  $S$ . Casselman has conjectured that  $\text{Reduced}(C)$  is regular. In this paper we give a conjectural description of the cells when  $W$  is the group corresponding to a hyperbolic polygon, and show that our conjectures imply Casselman’s.

## 1. INTRODUCTION

Let  $W$  be a Coxeter group with generating set  $S$ . In their study of representations of Coxeter groups and Hecke algebras, Kazhdan and Lusztig introduced the decomposition of  $W$  into *cells* [16]. The cells are equivalence classes in  $W$  determined by the left and right descent sets of elements of  $W$  and the degrees of the Kazhdan–Lusztig polynomials  $P_{x,y}$  (§2). Today cells are known to have many applications in representation theory; for some references, see the bibliography of [13].

The paper addresses the computability of the cells, in the following sense. Given a cell  $C$ , one can ask for an efficient way to encode its elements. Since elements of  $W$  are easily represented by reduced expressions in the generators  $S$ , it is natural to ask for a solution in terms of such expressions. However, since the definition of the cells involves a complicated equivalence relation, it is certainly not clear that this is possible.

Despite this, W. Casselman has conjectured that cells can be efficiently encoded. To state his conjecture, we need some terminology from the theory of formal languages; for more information see [1].

Let  $A$  be a finite alphabet of characters. By a *language*  $L$  over  $A$  we mean a collection of finite-length ordered words built from elements of  $A$ . A *finite state automaton*  $\mathcal{A}$  with alphabet  $A$  is a finite directed graph on a vertex set  $\mathcal{S}$ , called states, with edges labeled by elements of  $A \cup \{\varepsilon\}$ . Different edges leaving a given vertex are assumed to have different labels. One vertex is defined to be the *initial*

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*state*; a subset of  $\mathcal{S}$  is chosen and defined to be the *accepting states*. A finite state automaton encodes certain words built from  $A$  through path traversal: one starts at the initial state and follows a directed path of any length that terminates at an accepting state. As the path is traversed the vertex labels are concatenated into a word (the symbol  $\varepsilon$  represents a “null-transition;” the word is unaltered if  $\varepsilon$  is read). The collection of words that can be so constructed forms the language recognized by  $\mathcal{A}$ . A language is called *regular* if it can be recognized by a finite state automaton.

Regular languages are the simplest infinite languages one encounters in the hierarchy of formal languages. Many languages in algebra are regular. For instance, via an earlier paper of Davis–Shapiro [10], work of Brink–Howlett implies that the language  $\text{Reduced}(W)$  of all reduced expressions in the generators  $S$  is regular [8]. Any cell  $C$  induces a sub language  $\text{Reduced}(C) \subset \text{Reduced}(W)$ , namely all the reduced expressions of elements in  $C$ . We can now state Casselman’s conjecture:

**1.1. Conjecture.** *For any Coxeter group  $W$  and any (two- or one-sided) cell  $C \subset W$ , the language  $\text{Reduced}(C)$  is regular.*

Casselmann’s conjecture is known to be true for affine Weyl groups from earlier work of one of us (PG) [14]. In this paper we investigate the case that  $(W, S)$  is a Coxeter group corresponding to a hyperbolic polygon. In other words,  $W$  can be realized as the discrete subgroup of isometries of the hyperbolic plane  $\mathfrak{H}$  generated by the reflections through the side of a geodesic polygon. The cells of such groups have been considered earlier by Bédard [2, 3] and one of us (MB) [4]. We state conjectures due to two of us (MB and PG) that describes the Kazhdan–Lusztig cells of  $W$  in terms of reduced expressions. Then we prove (assuming the conjectures) that for any left, right, or 2-sided Kazhdan–Lusztig cell  $C$ , the language  $\text{Reduced}(C)$  is regular. Moreover, when combined with previous work of two of us (MB and PG), the results in this paper prove the regularity of cells for certain Coxeter groups (cf. Remark 5.5). We note that the proofs in this paper use word-hyperbolicity of  $W$  in an essential way, and in particular do not apply to affine Weyl groups.

We now give an overview of the paper. In §2 we give background on Coxeter groups and recall the definition of Kazhdan–Lusztig cells. Section 3 states conjectures for cells in Coxeter groups attached to tessellations of the hyperbolic plane by polygons. In §4 we give background on word hyperbolic groups and state the results we need from geometric group theory. Finally §5 gives our main results.

## 2. DEFINITIONS AND BASIC EXAMPLES

In this section we recall the basics of Coxeter groups and define Kazhdan–Lusztig cells. For more details we refer to [6, 15, 16].

A *Coxeter group*  $W$  is a group generated by a finite subset  $S \subset W$  where the defining relations have the form  $(st)^{m(s,t)} = 1$  for pairs of generators  $s, t \in S$ . The exponents  $m(s, t)$  are taken from  $\mathbb{N} \cup \{\infty\}$ , and we require  $m(s, s) = 1$ , so that each

generator  $s$  is an involution. Let  $I \subset S$  be a subset of the generators. The subgroup of  $W$  generated by  $I$  is called a parabolic subgroup and is denoted  $W_I$ .

Any representation of  $w \in W$  as a product of generators is called an *expression*. An expression is called *reduced* if it cannot be made shorter by applying the defining relations of  $W$ . The length of a shortest expression for  $w$  is denoted  $l(w)$ . For any  $w \in W$ , we define the *left descent set*  $\mathcal{L}(w) \subset S$  to consist of those  $s \in S$  such that  $l(sw) < l(w)$ . We similarly define the *right descent set*  $\mathcal{R}(w)$  to be those  $s$  such that  $l(ws) < l(w)$ .

Given an expression  $s_1 \cdots s_N$ , a *subexpression* is a (possibly empty) expression of the form  $s_{i_1} \cdots s_{i_M}$ , where  $1 \leq i_1 < \cdots < i_M \leq N$ . The *Chevalley–Bruhat order* is the partial order on  $W$  defined by putting  $v \leq w$  if an expression for  $v$  appears as a subexpression of a reduced expression for  $w$ . Given any  $v, w \in W$ , let  $[v, w]$  be the interval between  $v$  and  $w$ , that is  $[v, w] = \{x \in W \mid v \leq x \leq w\}$ .

The Kazhdan–Lusztig polynomials are most easily defined in terms of an auxiliary family of polynomials, the *R-polynomials*. This family  $\{R_{v,w}(q) \in \mathbb{Z}[q] \mid v, w \in W\}$  is defined to be the unique collection of polynomials satisfying the following properties (cf. [6, Theorem 5.1.1]): (i)  $R_{v,w}(q) = 0$  if  $v \not\leq w$ ; (ii)  $R_{v,w}(q) = 1$  if  $v = w$ ; and (iii) if  $s \in \mathcal{R}(w)$ , then  $R_{v,w}(q) = R_{vs,ws}(q)$  if  $s \in \mathcal{R}(v)$ , and is  $qR_{vs,ws}(q) + (q-1)R_{v,ws}(q)$  otherwise. Given the *R-polynomials*, the *Kazhdan–Lusztig polynomials*  $P_{v,w}(q)$  can be described as the unique family of polynomials satisfying (cf. [6, Theorem 5.1.4]) (i)  $P_{v,w}(q) = 0$  if  $v \not\leq w$ ; (ii)  $P_{v,w}(q) = 1$  if  $v = w$ ; (iii)  $\deg P_{v,w}(q) \leq (l(w) - l(v) - 1)/2$  if  $v < w$ ; and (iv)  $q^{l(w)-l(v)}P_{v,w}(q^{-1}) = \sum_{x \in [v,w]} R_{v,x}(q)P_{x,w}(q)$  if  $v \leq w$ . If  $v < w$ , we write  $\mu(v, w)$  for the coefficient of  $q^{(l(w)-l(v)-1)/2}$  in  $P_{v,w}(q)$ . We write  $v \dashrightarrow w$  and  $w \dashrightarrow v$  if  $\mu(v, w) \neq 0$ .

We are finally ready to define cells. The *left W-graph*  $\Gamma_{\mathcal{L}}$  of  $W$  is the directed graph with vertex set  $W$ , and with an arrow from  $v$  to  $w$  if and only if  $v \dashrightarrow w$  and  $\mathcal{L}(v) \not\subset \mathcal{L}(w)$ . The left cells are extracted from  $\Gamma_{\mathcal{L}}$  as follows. Given any directed graph, we say two vertices are in the same *strong connected component* if there exist directed paths from each vertex to the other. Then the *left cells* of  $W$  are exactly the strong connected components of the graph  $\Gamma_{\mathcal{L}}$ . The *right cells* are defined using the analogously constructed *right W-graph*  $\Gamma_{\mathcal{R}}$ . We say  $v, w$  are in the same *two-sided cell* if we can find a sequence  $v = w_1, w_2, \dots, w_k = w$  such that  $w_i, w_{i+1}$  lie in either the same left or right cell.

We need one final ingredient to state our conjecture in the next section: the *a*-function.

Let  $\mathcal{H}$  denote the Hecke algebra of  $W$  over the ring  $\mathcal{A} = \mathbb{Z}[q^{1/2}, q^{-1/2}]$  of Laurent polynomials in  $q^{1/2}$ . This algebra is a free  $\mathcal{A}$ -module with a basis  $\mathcal{T} = \{T_w \mid w \in W\}$  and with multiplication determined by  $T_w T_{w'} = T_{ww'}$  if  $l(ww') = l(w) + l(w')$ , and  $T_s^2 = q + (q-1)T_s$  for  $s \in S$ . Together with the basis  $\mathcal{T}$ , we can define in  $\mathcal{H}$  another basis  $\mathcal{C} = \{C_w \mid w \in W\}$ . The element  $C_w \in \mathcal{C}$  can be expressed in terms of  $\mathcal{T}$  and

the Kazhdan–Lusztig polynomials by

$$C_w = \sum_{y \leq w} (-1)^{l(w)-l(y)} q^{l(w)/2-l(y)} P_{y,w}(q^{-1}) T_y.$$

Now consider the multiplication of the  $\mathcal{C}$ -basis elements in  $\mathcal{H}$ . We can write

$$C_x C_y = \sum_z h_{x,y,z} C_z, \quad h_{x,y,z} \in \mathcal{A}.$$

Let  $a(z)$  be the smallest integer such that  $q^{a(z)/2} h_{x,y,z} \in \mathcal{A}^+$  for all  $x, y \in W$ , where  $\mathcal{A}^+ = \mathbb{Z}[q^{1/2}]$ . It is a standard conjecture that  $\{a(w) \mid w \in W\} \subset \mathbb{Z}$  is bounded for any Coxeter group. The  $a$ -function was introduced by Lusztig in [18], where he proved this conjecture for affine Weyl groups. In [4] it was shown that the  $a$ -function is bounded for right-angled Coxeter groups. N. Xi recently showed that the  $a$ -function is bounded for Coxeter groups with complete Coxeter graphs (i.e. no two generators commute) [20]; this paper has further ramifications for our current article, see Theorem 3.2. P. Zhou has recently proved that the  $a$ -function is bounded if  $W$  has rank 3 [21].

### 3. CONJECTURES ABOUT CELLS OF HYPERBOLIC POLYGON GROUPS

In this paper we take  $W$  to be a hyperbolic polygon group. This means the following. Let  $\mathfrak{H}$  be the hyperbolic plane, and let  $\Delta \subset \mathfrak{H}$  be an  $n$ -sided geodesic polygon with angles  $\alpha_i = \pi/a_i$ ,  $i = 1, \dots, n$ . (We omit the conditions the denominators  $a_i$  satisfy to make  $\Delta$  hyperbolic; we also allow the angles to vanish, in which case the polygon has ideal vertices.) Label the sides of  $\Delta$  by  $\sigma_1, \dots, \sigma_n$ , such that the angle  $\alpha_i$  sits between the sides  $\sigma_i, \sigma_{i+1}$ , and where the subscripts are taken mod  $n$  as necessary. Then the generating set  $S$  of  $W$  has  $n$  elements  $s_1, \dots, s_n$ , corresponding to the sides  $\sigma_i$ . We put  $m(s_i, s_j) = \infty$  unless  $\sigma_i$  and  $\sigma_j$  meet at the angle  $\alpha_k \neq 0$ . In the latter case we put  $m(s_i, s_j) = a_k$ .

It is not hard to see that  $W$  is isomorphic to the discrete subgroup of isometries of  $\mathfrak{H}$  generated by reflections in the lines through the  $\sigma_i$ . Thus there is an action of  $W$  on  $\mathfrak{H}$  by reflections, the polygon  $\Delta$  is a fundamental domain, and the translates  $\{w \cdot \Delta \mid w \in W\}$  form a tessellation of  $\mathfrak{H}$  (note our convention that the reflection action of  $W$  on  $\mathfrak{H}$  is a left action). The correspondence  $w \mapsto w \cdot \Delta$  is a bijection between  $W$  and the tiles in the tessellation. Using this we identify  $W$  with the set of all tiles.

We can also use this identification to define certain subsets of  $W$ . Recall that  $\mathcal{L}(w)$  denotes the set of left descents of an element  $w$ . Given any subset  $T \subset S$ , we let  $W^T$  be the (possibly empty) set of all  $w \in W$  such that  $\mathcal{L}(w) = T$ . The tessellation allows us to identify the sets  $W^T$  as follows. First,  $W^\emptyset$  consists of  $\Delta$  itself. Next, any edge of  $\Delta$  corresponds to a generator  $s \in S$ . Extending this edge to a line divides the plane  $\mathfrak{H}$  into two half-spaces, one containing  $\Delta$  and one not. The

half-space  $H_s$  not containing  $\Delta$  contains all elements  $w$  such that  $s \in \mathcal{L}(w)$ . Any (non-ideal) vertex of  $\Delta$  corresponds to an order 2 subset  $T$  with  $W^T \neq \emptyset$ . Namely we have  $W^T = H_s \cap H_{s'}$ , where  $T = \{s, s'\}$  and  $s, s'$  label the edges of  $\Delta$  meeting this vertex. Finally, if  $T, T'$  have order 2 and  $T \cap T' = \{s\}$  has order 1, then  $W^{\{s\}}$  consists of  $H_s \setminus W^T \cap W^{T'}$ . These give all subsets  $T$  such that  $W^T \neq \emptyset$ .

We call a subgroup  $D \subset W$  *finite dihedral* if  $D$  is the parabolic subgroup for an order 2 subset  $T$  with  $W^T$  nonempty. Let  $\mathcal{D}$  be the set of finite dihedral subgroups and let  $\mathcal{T}$  be the set of order 2 subsets indexing them. For any  $D \in \mathcal{D}$ , let  $w_D$  be the longest element. Assume that the distinct nonzero exponents are  $e_1 < e_2 < \dots < e_m$ , where  $m \leq n$ . This means there are  $m$  isomorphism classes of dihedral subgroups of  $W$ . We write  $\mathcal{D} = \mathcal{D}_1 \cup \dots \cup \mathcal{D}_m$ , where  $\mathcal{D}_i$  is the set of finite dihedral subgroups of exponent  $e_i$ . We also let  $\mathcal{T} = \mathcal{T}_1 \cup \dots \cup \mathcal{T}_m$  be the corresponding partition of  $\mathcal{T}$ . For  $i = 1, \dots, m$  let  $W_i = \{w_D \mid D \in \mathcal{D}_i\}$  be the sets of longest elements. Hence  $w_D \in W_i$  if and only if  $D$  has exponent  $e_i$ . Finally let  $l_i$  be the common length of the elements in  $W_i$ .

We are now ready to give a conjectural description of the two-sided cells of  $W$ . Let  $C_m, \dots, C_1 \subset W$  be the sequence of subsets defined by

$$w \in C_m \text{ if and only if } w = u.w_D.v \text{ for some } w_D \in W_m,$$

and if  $i < m$ ,

$$w \in C_i \text{ if and only if } w = u.w_D.v \text{ for some } w_D \in W_i,$$

$$\text{and } w \neq u.w_D.v \text{ for any } w_D \in W_k \text{ with } k > i.$$

Here we write  $z = x.y$  in  $W$  if  $z = xy$  and  $l(z) = l(x) + l(y)$ . We also define subsets  $C_{\text{id}} = \{\text{id}\}$  and

$$C_0 = \{w \mid w \text{ has a unique reduced expression}\}.$$

Let  $\mathcal{C}$  be the collection  $\{C_{\text{id}}, C_0, \dots, C_m\}$ .

**3.1. Conjecture.** (1) *The decomposition  $\mathcal{C}$  gives the partition of  $W$  into two-sided cells.*

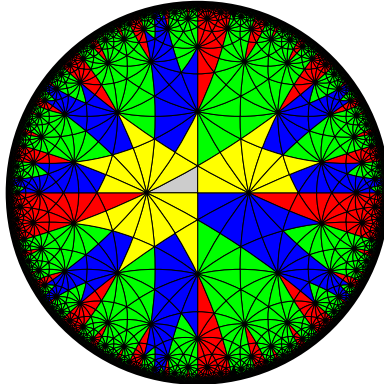
(2) *The  $a$ -function equals  $l_i$  on the two-sided cell  $C_i$ .*

Figure 1 shows an example of the partition  $\mathcal{C}$  for the triangle group

$$W_{237} = \langle r, s, t \mid r^2 = s^2 = t^2 = (rs)^3 = (rt)^2 = (st)^7 = 1 \rangle.$$

There are three finite dihedral subgroups, of orders 4, 6, 14, corresponding to the exponents 2, 3, 7. The subsets  $C_3, C_2, C_1$  are (respectively) the red, blue, and green triangles. Finally the grey triangle is  $C_{\text{id}}$  and the yellow triangles are  $C_0$ .

We remark that both  $C_{\text{id}}$  and  $C_0$  are known to be two-sided cells, the former for trivial reasons and the latter from work of Lusztig [17, §§3.7–3.8]. We also have the following theorem of Xi, which gives confirmation of Conjecture 3.1 for certain  $W$ , not necessarily hyperbolic:

FIGURE 1. The triangle group  $W_{237}$ 

**3.2. Theorem.** [20] *Suppose  $W$  is crystallographic and that no exponent of  $W$  is 2. Then  $\mathcal{C}$  gives the partition of  $W$  into two-sided cells.*

Next we turn to the one-sided cells. Given any  $T \in \mathcal{T}_i$ , define

$$(1) \quad U^T = W^T \setminus \bigcup_{j>i} C_j.$$

In particular if  $i = m$ , we have  $U^T = W^T$ . Let  $\Omega_i = \{w^{-1} \mid w \in C_i\}$ . The one-sided cells will be built from the sets  $w \cdot U^T$ , where  $w \in \Omega_i$  and  $T$  ranges over  $\mathcal{T}_i$ . We put a partial order on  $\Omega_i$  by  $w \preceq y$  if  $l(w) \leq l(y)$  and there exists  $T, T' \in \mathcal{T}_i$  such that  $(w \cdot U^T) \cap (y \cdot U^{T'}) \neq \emptyset$ . We define  $\Omega_i^\circ$  to be the minimal elements in  $\Omega_i$  with respect to this partial order.

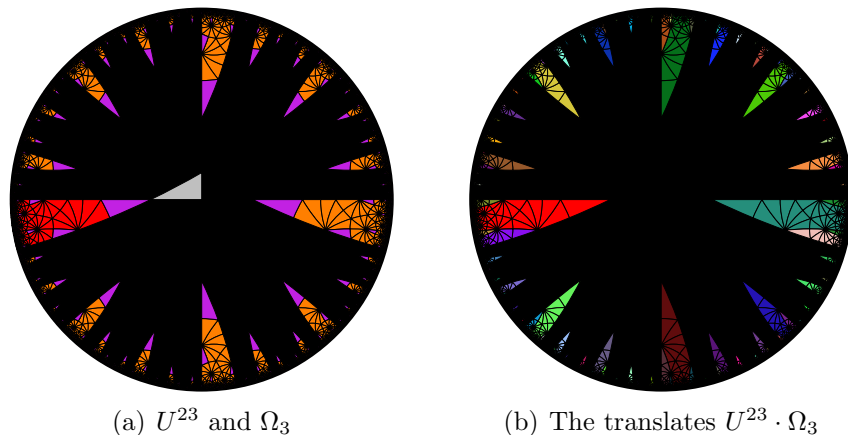
**3.3. Conjecture.** *The subsets  $\{w \cdot U^T \mid w \in \Omega_i^\circ, T \in \mathcal{T}_i\}$  are the right cells in  $C_i$ .*

Figures 2(a)–2(b) illustrate Conjecture 3.3 for  $W_{237}$  and the two-sided cell  $C_3$ . The red region in Figure 2(a), together with the purple triangles inside it, is the subset  $U^T = U^{23}$ , where  $T = \langle s_2, s_3 \rangle$ . This is a one-sided cell in  $C_3$ . The purple triangles are the elements in  $\Omega_3$ . In Figure 2(b) we see the translates  $w \cdot U^{23}$  as  $w$  ranges over  $\Omega_3$ . Note that some regions meet others; indeed, if this happens then one translate of  $U^{23}$  is entirely contained in another. The elements of  $\Omega_3^\circ$  correspond to the purple triangles in Figure 2(a) that lie at the ends of the orange regions. The orange regions, together with the purple triangles in them, are the other one-sided cells in  $C_3$ .

Conjectures 3.3 and 3.1 should be considered as a special case of conjectures from [5] applied to hyperbolic polygon groups.

#### 4. WORD HYPERBOLIC GROUPS AND AUTOMATA

In this section we prove that certain languages in word hyperbolic groups are regular. We will then apply these results to the languages  $\text{Reduced}(C)$  where  $C$  is a



Kazhdan–Lusztig cell in a hyperbolic polygon group. First we define word hyperbolic groups and recall some of the standard facts we shall need. Details and additional properties can be found, for example, in [7, 12].

Let  $(X, d)$  be a geodesic metric space. A *geodesic triangle* consists of 3 points in  $X$  together with geodesics joining each pair of points. A geodesic triangle is called  $\delta$ -thin ( $\delta \in \mathbb{R}_{>0}$ ) if every side is in a  $\delta$ -neighborhood of the other two sides. The metric space  $X$  is called  $\delta$ -hyperbolic if every geodesic triangle is  $\delta$ -thin.

Given a group  $W$  and a generating set  $S$ , we let  $\text{Cay}(W, S)$  denote the corresponding Cayley graph, which we regard as a geodesic metric space by identifying each edge with a unit length interval. Note that the metric restricts to the *word metric*  $d_S : W \times W \rightarrow \mathbb{Z}_{\geq 0}$  on the vertices of the Cayley graph; that is, for any  $u, v \in W$  the distance  $d_S(u, v)$  is the minimal length of a geodesic from  $u$  to  $v$  in  $\text{Cay}(W, S)$ . We define the *length* of an element  $w \in W$  by  $l(w) = d_S(1, w)$ .

**4.1. Definition.** A finitely generated group  $W$  is *word hyperbolic* if for some (equivalently, any) finite generating set  $S$ , there exists a  $\delta$  such that  $\text{Cay}(W, S)$  is  $\delta$ -hyperbolic.

It is known that a word hyperbolic group cannot contain a subgroup isomorphic to  $\mathbb{Z} \times \mathbb{Z}$ . For Coxeter groups, this condition is also sufficient.

**4.2. Proposition.** [9, Corollary 12.6.3] *A Coxeter group  $W$  is word hyperbolic if and only if it contains no subgroup isomorphic to  $\mathbb{Z} \times \mathbb{Z}$ .*

In particular, if a Coxeter group  $W$  is a lattice in the isometry group of  $\mathfrak{H}$  (for example, a hyperbolic polygon group), then  $W$  is word hyperbolic.

The key property of hyperbolic groups that we shall need is the *fellow-traveler property*. For a group  $W$  with generating set  $S$ , we let  $S^*$  denote the language of all words over the alphabet  $S$ . Any word  $\alpha \in S^*$  determines a path in  $\text{Cay}(W, S)$  that starts at the identity vertex  $1 \in W$ . We let  $|\alpha|$  denote the length of this path and

$\bar{\alpha} \in W$  denote the terminal vertex. Keeping the terminology for Coxeter groups, we say that a word  $\alpha \in S^*$  is an *expression for*  $w \in W$  if  $\bar{\alpha} = w$ . (Note that  $S = S^{-1}$ , so that every element of  $W$  is represented by some  $\alpha \in S^*$ , i.e. so that the map  $\alpha \mapsto \bar{\alpha}$  from  $S^*$  to  $W$  is surjective.) An expression  $\alpha$  for  $w$  is a *reduced expression for*  $w$  if the corresponding path in  $\text{Cay}(W, S)$  is a minimal length geodesic between 1 and  $w$ . In other words,  $\alpha$  satisfies  $|\alpha| = d_S(1, w) = l(w)$  and  $\bar{\alpha} = w$ .

A subset  $L \subseteq S^*$  is called a *normal form* for  $W$  if the map  $\alpha \mapsto \bar{\alpha}$  from  $L$  to  $W$  is surjective. The normal form we are interested in most is the geodesic normal form, denoted by  $\text{Reduced}(W)$ , consisting of all reduced expressions for all elements in  $W$ . More generally, for any subset  $X \subseteq W$ , we define  $\text{Reduced}(X)$  to be the set of all reduced expressions for elements of  $X$ .

Two words  $\alpha, \beta \in S^*$  with  $|\alpha| \leq |\beta|$ , can be written uniquely as  $\alpha = s_1 s_2 \cdots s_n$  and  $\beta = t_1 t_2 \cdots t_{n+p}$  where  $s_i, t_j \in S$ . We say that  $\alpha$  and  $\beta$  are (*synchronous*) *k-fellow-travelers* if  $d_S(\overline{s_1 \cdots s_i}, \overline{t_1 \cdots t_i}) \leq k$  for all  $i = 1, \dots, n$  and  $d_S(\overline{s_1 \cdots s_n}, \overline{t_1 \cdots t_{n+i}}) \leq k$  for  $i = 1, \dots, p$ . In other words, the corresponding paths for  $\alpha$  and  $\beta$  in the Cayley graph are never more than  $k$ -apart.

**4.3. Definition.** Given a group  $W$  with generating set  $S$ , a normal form  $L \subseteq S^*$  is said to have the *fellow-traveler property* (respectively, *two-sided fellow-traveler property*) if there exists a  $k > 0$  such that for any  $\alpha, \beta \in L$  with  $\bar{\alpha} = \bar{\beta}t$  for some  $t \in S \cup \{1\}$  (resp.,  $\bar{\alpha} = \overline{s\alpha t}$  for some  $s, t \in S \cup \{1\}$ ), the words  $\alpha$  and  $\beta$  are *k-fellow-travelers*.

**4.4. Remark.** The (two-sided) fellow-traveler property for a normal form  $L$  is known to be equivalent to  $W$  having an *automatic structure* (resp., *biautomatic structure*) with respect to  $L$  in the sense of [11]. In particular, such a normal form must be recognized by a finite-state automaton, hence is a regular language. Obviously, biautomatic implies automatic.

The key fact we shall need is that word hyperbolic groups are biautomatic *with respect to the geodesic normal form*.

**4.5. Proposition.** *If  $W$  is word hyperbolic, and  $S$  is any finite generating set, then  $\text{Reduced}(W)$  has the two-sided fellow-traveler property.*

*Proof.* The two-sided fellow-traveler property is equivalent to both the normal form and its inverse language having the (one-sided) fellow-traveler property [11, Definition 2.5.4 and Lemma 2.5.5]. Since the geodesic language is closed under taking inverses, it is enough to show that  $\text{Reduced}(W)$  satisfies the fellow-traveler property, and this is well-known [11, Theorem 3.4.5].  $\square$

## 5. HYPERBOLIC POLYGON CELLS AND REGULAR LANGUAGES

In this final section we prove our main results, Theorems 5.1 and 5.3. The first uses only the one-sided fellow traveler property, but the second requires the stronger



two-sided property. We then combine them with Conjectures 3.1 and 3.3 to deduce the regularity of certain Kazhdan–Lusztig cells.

Given a group  $W$  and finite generating set  $S$ , let  $\mu$  be any reduced word in  $S^*$ . As above, we write  $w = u.v$  in  $W$  if  $w = uv$  and  $l(w) = l(u) + l(v)$ . We then define the subset  $X_\mu$  of  $W$  by

$$X_\mu = \{w \in W \mid w = u.\bar{\mu}.v \text{ for some } u, v \in W\}.$$

In other words,  $X_\mu$  consists of all elements of  $W$  that have *some* reduced expression containing  $\mu$  as a (consecutive) subword. The language  $\text{Reduced}(X_\mu)$  therefore consists of all reduced expressions that are equivalent to a reduced expression containing  $\mu$  as a subword.

**5.1. Theorem.** *Let  $W$  be a word hyperbolic group, let  $S$  be any finite generating set  $S$  satisfying  $S = S^{-1}$ , and let  $\mu$  be any word in  $\text{Reduced}(W)$ . Then  $\text{Reduced}(X_\mu)$  is a regular language.*

*Proof.* Since  $W$  is word hyperbolic,  $\text{Reduced}(W)$  is a regular language. The sublanguage  $\text{Reduced}_\mu(W)$  consisting of all reduced words that contain  $\mu$  as a subword (i.e., that match the regular expression  $. * \mu . *$ ) is also a regular language.

Now let  $A$  be a finite state automaton accepting  $\text{Reduced}(W)$ , and let  $k$  be a positive integer such that  $\text{Reduced}(W)$  has the  $k$ -fellow-traveler property. Let  $N_k$  be the set of all reduced expressions in  $S^*$  with length  $\leq k$ . Then the standard automaton  $M_\epsilon$  based on  $(A, N_k)$  (see [11, Definition 2.3.3]) accepts the language

$$L = \{(\alpha, \beta) \in \text{Reduced}(W)^2 \mid \bar{\alpha} = \bar{\beta} \text{ and } \alpha \text{ and } \beta \text{ are } k\text{-fellow travelers}\},$$

which is therefore regular. But since  $\text{Reduced}(W)$  satisfies the  $k$ -fellow-traveler property, this language consists precisely of pairs of reduced expressions having the same image in  $W$ , i.e.,

$$L = \{(\alpha, \beta) \in \text{Reduced}_S(W) \times \text{Reduced}_S(W) \mid \bar{\alpha} = \bar{\beta}\}.$$

The language  $\text{Reduced}(X_\mu)$  is obtained by intersecting  $L$  with the language  $\text{Reduced}(W) \times \text{Reduced}_\mu(W)$  and then projecting onto the first factor. By standard predicate calculus for regular languages (see, e.g., [11, Theorem 1.4.6]), the language  $\text{Reduced}(X_\mu)$  is therefore regular.  $\square$

**5.2. Corollary.** *Let  $W$  be a hyperbolic polygon group, and let  $C$  be a conjectural two-sided cell in the decomposition  $\mathcal{C}$  of Conjecture 3.1. Then the language  $\text{Reduced}(C)$  is regular.*

*Proof.* For each finite dihedral subgroup  $D$ , let  $\mu_D$  be any reduced expression for the longest element  $w_D$ . Then

$$\text{Reduced}(C_m) = \bigcup_{\bar{\mu}_D \in W_m} \text{Reduced}(X_{\mu_D}),$$

and for  $1 \leq i < m$ ,

$$\text{Reduced}(C_i) = \bigcup_{\bar{\mu}_D \in W_i} \text{Reduced}(X_{\mu_D}) \setminus \bigcup_{i < j \leq m} \text{Reduced}(C_j).$$

Since these are all obtained using finite unions, complements, and intersections of regular languages, they are regular. For  $\text{Reduced}(C_{\text{id}})$  and  $\text{Reduced}(C_0)$ , we note that the former is finite, and the latter is the complement of  $\text{Reduced}(C_{\text{id}}) \cup \text{Reduced}(C_1) \cup \dots \cup \text{Reduced}(C_m)$  in  $\text{Reduced}(W)$ . It follows that both are regular as well.  $\square$

**5.3. Theorem.** *Let  $W$  be a word hyperbolic group, and let  $S$  be any generating set. Suppose  $X \subseteq W$  is such that  $\text{Reduced}(X)$  is a regular language. Then for any  $w \in W$ , the language  $\text{Reduced}(w \cdot X)$  is also regular.*

*Proof.* In fact, the theorem holds for any normal form on a group that satisfies the two-sided fellow-traveler property (i.e., is biautomatic). The proof is fairly immediate from the definitions; a reference is [19, Lemma 1.2].  $\square$

**5.4. Corollary.** *Let  $W$  be a hyperbolic polygon group and let  $w \cdot U^T$  (for  $w \in \Omega_i^\circ$  and  $T \in \mathcal{T}_i$ ) be one of the conjectured one-sided cells in  $C_i$ . Then  $\text{Reduced}(w \cdot U^T)$  is regular.*

*Proof.* First, we claim that the language  $\text{Reduced}(W^T)$  is regular. This is easily seen using the *canonical automaton*  $\mathcal{A}_{\text{can}}$  that accepts  $\text{Reduced}(W)$  [6, Theorem 4.8.3]. The states of this automaton, all of which are accepting, are given by the regions that are the connected components of the complement of the hyperplane arrangement determined by the *small roots* [6, §4.7]. Since the simple roots are small, the subset of the tessellation of  $\mathfrak{H}$  corresponding to  $W^T$  is given by a union of states of  $\mathcal{A}_{\text{can}}$ . Hence we can make an automaton accepting  $\text{Reduced}(W^T)$  by starting with  $\mathcal{A}_{\text{can}}$  and only making certain states accepting. Thus  $\text{Reduced}(W^T)$  is regular. Since all of the  $\text{Reduced}(C_j)$  are regular, it follows that  $\text{Reduced}(U^T)$  is regular. Hence, by Theorem 5.3,  $\text{Reduced}(w \cdot U^T)$  is also regular.  $\square$

**5.5. Remark.** Conjectures 3.3 and 3.1 are true for right-angled polygon groups by [4], and more generally for polygons with equal angles that satisfy the crystallographic condition by [5, §4]. Thus we can apply the results of this section to show regularity of the languages attached to the cells for those groups.

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IMPA, ESTRADA DONA CASTORINA 110, 22460-320 RIO DE JANEIRO, BRAZIL  
*E-mail address:* mbel@impa.br

UNIVERSITY OF MASSACHUSETTS AMHERST, AMHERST, MA 01003  
*E-mail address:* gunnells@math.umass.edu

DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, SANTA CLARA UNIVERSITY, SANTA CLARA, CA 95053  
*E-mail address:* rscott@math.scu.edu