ON THE COHOMOLOGY OF LINEAR GROUPS OVER IMAGINARY QUADRATIC FIELDS

HERBERT GANGL, PAUL E. GUNNELLS, JONATHAN HANKE, ACHILL SCHÜRMANN, MATHIEU DUTOUR SIKIRIĆ, AND DAN YASAKI

ABSTRACT. Let F be an imaginary quadratic field of discriminant $D < 0$, and let $\mathcal{O} = \mathcal{O}_D$ be its ring of integers. Let Γ be the group $\mathrm{GL}_N(\mathcal{O})$. In this paper we investigate the cohomology of Γ for $N = 3, 4$ and for a selection of discriminants: $D \geq -24$ when $N = 3$, and $D = -3, -4$ when $N = 4$. In particular we compute the integral cohomology of Γ up to p-power torsion for small primes p. Our main tool is the polyhedral reduction theory for Γ developed by Ash [4, Ch. II] and Koecher [18]. Our results extend work of Staffeldt [28], who treated the case $n = 3$, $D = -4$. In a sequel [11] to this paper, we will apply some of these results to the computations with the K-groups $K_4(\mathcal{O}_D)$, when $D = -3, -4$.

1. INTRODUCTION

1.1. Let F be an imaginary quadratic field of discriminant $D < 0$, and let $\mathcal{O} = \mathcal{O}_D$ be its ring of integers. Let Γ be the group $\mathrm{GL}_N(\mathcal{O})$. The homology and cohomology of Γ when $N = 2$ — or rather Γ's close cousin the Bianchi group $\mathrm{PSL}_2(\mathcal{O})$ — have been well studied in the literature. For an (incomplete) selection of results we refer to [5, 10, 21–24, 30]. Today we have a good understanding of a wide range of examples, and one can even compute them for very large discriminants (cf. [32]). For $N > 2$, on the other hand, the group Γ has not received the same attention. The first example known to us is the work of Staffeldt [28]. He treated the case $N = 3$, $D = -4$ with the goal of understanding the 3-torsion in $K_3(\mathbb{Z}[\sqrt{-1}])$. The second example is [8], which investigates the case of the groups $\mathrm{GL}(L)$ for L an $\mathcal{O}$-lattice that is not necessarily a free $\mathcal{O}$-module. This allows the authors to compute the Hermite constants of those rings in case $D \geq -10$ and $\mathrm{rank}(L) \leq 3$. Our methods apply as well to the non-free case, and the corresponding cohomology computations would be useful when investigating automorphic forms over number fields that are not principal ideal domains (cf. [29, Appendix] for more about the connection between cohomology of arithmetic groups and automorphic forms).

In this paper we rectify this situation somewhat by beginning the first systematic computations for higher rank linear groups over $\mathcal{O}$. In particular investigate the cohomology of Γ for $N = 3, 4$ and for a selection of discriminants: $D \geq -24$ when $N = 3$, and $D = -3, -4$ when $N = 4$. We explicitly compute the polyhedral reduction domains arising from Voronoi's theory of perfect forms, as generalized by Ash [4, Ch. II] and Koecher [18]. This allows

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us to compute the integral cohomology of Γ up to p-power torsion for small primes p. In a sequel [11] to this paper, we will apply some of these results to computations with the K-groups $K_4(\mathcal{O}_D)$, when $D = -3, -4$.

1.2. Here is a guide to the paper (which closely follows the structure of the first five sections of [14]). In Section 2 we recall the explicit reduction theory we need to build our chain complexes to compute cohomology. In Section 3 we define the complexes, and explain the relation between what we compute and the cohomology of Γ. In Section 4 we describe a "mass formula" for the cells in our tessellations that provides a non-trivial computational check on the correctness of our constructions. In Section 5 we give an explicit representative for the nontrivial class in the top cohomological degree; this construction is motivated by a similar construction in [13,14]. Finally, in Section 6 we give the results of our computations.

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2. The polyhedral cone

Fix an imaginary quadratic field F of discriminant $D < 0$ with ring of integers $\mathcal{O} = \mathcal{O}_D$. Define $\omega = \omega_D$ as

$$\omega = \begin{cases} \sqrt{D/4} & \text{if } D \equiv 0 \bmod 4 \\ (1+\sqrt{D})/2 & \text{if } D \equiv 1 \bmod 4. \end{cases}$$

Then $F = \mathbb{Q}(\omega)$ and $\mathcal{O} = \mathbb{Z}[\omega]$. Fix a complex embedding $F \hookrightarrow \mathbb{C}$, and identify F with its image in $\mathbb{C}$. We extend this identification to vectors and matrices with coefficients in F.

2.1. **Hermitian forms.** Let $\mathcal{H}^N(\mathbb{C})$ denote the N^2-dimensional real vector space of $N \times N$ Hermitian matrices with complex coefficients. Using the complex embedding of F we can view $\mathcal{H}^N(F)$, the Hermitian matrices with coefficients in F, as a subset of $\mathcal{H}^N(\mathbb{C})$. Moreover, this embedding allows us to view $\mathcal{H}^N(\mathbb{C})$ as a $\mathbb{Q}$-vector space such that the rational points of $\mathcal{H}^N(\mathbb{C})$ are exactly $\mathcal{H}^N(F)$.

Define a map $q\colon \mathcal{O}^N \to \mathcal{H}^N(F)$ by the outer product $q(x) = xx^*$, where $*$ denotes conjugate transpose (with conjugation being the nontrivial conjugation automorphism of F). Each $A \in \mathcal{H}^N(\mathbb{C})$ defines a Hermitian form on $\mathbb{C}^N$ by

$$A[x] = x^*Ax, \quad \text{for } x \in \mathbb{C}^N, \tag{1}$$

where $\cdot^*$ is complex conjugate transpose. Define the non-degenerate bilinear pairing

$$\langle \cdot, \cdot \rangle : \mathcal{H}^N(\mathbb{C}) \times \mathcal{H}^N(\mathbb{C}) \to \mathbb{C}$$

by $\langle A, B\rangle = \mathrm{Tr}(AB)$. For $x \in \mathcal{O}^N$, identified with its image in $\mathbb{C}^N$, one can easily verify that

$$A[x] = \mathrm{Tr}(Aq(x)) = \langle A, q(x)\rangle\,. \tag{2}$$

Let $C_N \subset \mathcal{H}^N(\mathbb{C})$ denote the cone of positive definite Hermitian matrices.

Definition 2.1. For $A \in C_N$, the *minimum of* A, denoted $m_D(A) = m(A)$ is

$$m(A) = \inf_{x \in \mathcal{O}^N \setminus \{0\}} A[x].$$

Note that $m(A) > 0$ since A is positive definite. A vector $v \in \mathcal{O}^N$ is a *minimal vector of* A if $A[v] = m(A)$. Denote the set of minimal vectors of A by $M(A)$.

It should be emphasized that these notions depend on the fixed choice of the imaginary quadratic field F. Since $q(\mathcal{O}^N)$ is discrete in $\mathcal{H}^N(\mathbb{C})$, the minimum for each A is attained by finitely many minimal vectors.

From (2), we see that each vector $v \in \mathcal{O}^N$ gives rise to a linear functional on $\mathcal{H}^N(\mathbb{C})$ defined by $q(v)$.

Definition 2.2. We say that a Hermitian form $A \in C_N$ is a *perfect Hermitian form over* F if

$$\operatorname{span}_{\mathbb{R}} \{q(v) \mid v \in M(A)\} = \mathcal{H}^N(\mathbb{C}).$$

From Definition 2.2, it is clear that a form is perfect if and only if it is uniquely determined by its minimum and its minimal vectors. Equivalently, a form A is perfect when $M(A)$ determines A up to a positive real scalar. It is convenient to normalize a perfect form by requiring that $m(A) = 1$, and we will do so throughout this paper. A priori there is no reason to expect that such an A is actually a rational point in $\mathcal{H}^n(\mathbb{C})$; indeed, when one generalizes these concepts to general number fields this is too much to expect (cf. [15]). However, in our case we can assume this:

Theorem 2.3. [20, Theorem 3.2] *Suppose F is a CM field. Then if $A \in C_N$ is a perfect Hermitian form over F such that $m(A) = 1$, we have $A \in \mathcal{H}^N(F)$.*

We now construct a partial compactification of the cone C_N:

Definition 2.4. A matrix $A \in \mathcal{H}^N(\mathbb{C})$ is said to have an *F-rational kernel* when the kernel of A is spanned by vectors in $F^N \subset \mathbb{C}^N$. Let $C_N^* \subset \mathcal{H}^N(\mathbb{C})$ denote the subset of nonzero positive semi-definite Hermitian forms with F-rational kernel.

Let $\mathbf{G}$ be the reductive group over $\mathbb{Q}$ given by the restriction of scalars $\operatorname{Res}_{F/\mathbb{Q}}(\operatorname{GL}_N)$. Thus $\mathbf{G}(\mathbb{Q}) = \operatorname{GL}_N(F)$ and $\mathbf{G}(\mathbb{Z}) = \operatorname{GL}_N(\mathcal{O})$. The group $\mathbf{G}(\mathbb{R}) = \operatorname{GL}_N(\mathbb{C})$ acts on C_N^* on the left by

$$g \cdot A = gAg^*,$$

where $g \in \operatorname{GL}_N(\mathbb{C})$ and $A \in C_N^*$; one can easily verify that this action preserves C_N. Let $H = R_d\mathbf{G}(\mathbb{R})^0$ be the identity component of the group of real points of the split radical of $\mathbf{G}$. Then $H \simeq \mathbb{R}_+$, and as a subgroup of $\mathbf{G}(\mathbb{R})$ acts on C_N^* by positive real homotheties. Voronoi's work [31], generalized by Ash [4, Ch. II] and Koecher [18] shows that there are only finitely many perfect Hermitian forms over F modulo the action of $\operatorname{GL}_N(\mathcal{O})$ and H.

Let X_N^* denote the quotient $X_N^* = H\backslash C_N^*$, and let $\pi\colon C_N^* \to X_N^*$ denote the projection. Then $X_N = \pi(C_N)$ can be identified with the global Riemannian symmetric space for the reductive group $H\backslash \operatorname{GL}_N(\mathbb{C})$.

2.2. **Two cell complexes.** Let M be a finite subset of $\mathcal{O}^N \setminus \{0\}$. The *perfect cone* of M is the set of nonzero matrices of the form $\sum_{v\in M} \lambda_v q(v)$, where $\lambda_v \in \mathbb{R}_{\geq 0}$; by abuse of language we also call its image by π in X_N^* a perfect cone. For a perfect form A, let $\sigma(A) \subset X_N^*$ be the perfect cone of $M(A)$. One can show [4, 18] that the cells $\sigma(A)$ and their intersections, as A runs over equivalence classes of perfect forms, define a $\operatorname{GL}_N(\mathcal{O})$-invariant cell decomposition of X_N^*. In particular, for a perfect form $A \in C_N$ and an element $\gamma \in \operatorname{GL}_N(\mathcal{O})$, we have

$$\gamma \cdot \sigma(A) = \text{perfect cone of } \{\gamma v \mid v \in M(A)\} = \sigma((\gamma^*)^{-1}A\gamma).$$

Endow X_N^* with the CW-topology [17, Appendix].

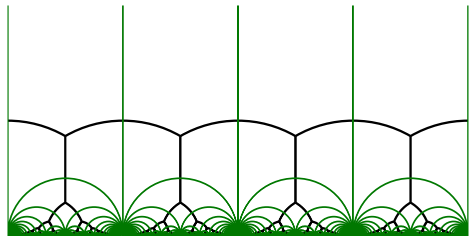

Figure 1. Voronoi tessellation of $\mathfrak{H}$ shown in green, with its dual the well-rounded retract shown in black.

If τ is a closed cell in X_N^* and A is a perfect form with $\tau \subset \sigma(A)$, we let $M(\tau)$ denote the set of vectors $v \in M(A)$ such that $q(v) \in \tau$. The set $M(\tau)$ is independent of the (possible) choice of A. Then τ is the image in X_N^* of the cone C_τ generated by $\{q(v) \mid v \in M(\tau)\}$. For any two closed cells τ and τ' in X_N^*, we have $M(\tau) \cap M(\tau') = M(\tau \cap \tau')$.

Let $\tilde{\Sigma} \subset C_N^*$ be the (infinite) union of all cones C_σ such that $\sigma \in X_N^*$ has nontrivial intersection with X_N. One can verify that the stabilizer of C_σ in $\mathrm{GL}_N(\mathcal{O})$ is equal to the stabilizer of σ. By abuse of notation, we write $M(C_\sigma)$ as $M(\sigma)$.

The collection of cones $\tilde{\Sigma}$ was used by Ash [1, 2] to construct the *well-rounded retract*, a contractible $N^2 - N$ dimensional cell complex W on which $\mathrm{GL}_N(\mathcal{O})$ acts cellularly with finite stabilizers of cells. More precisely, a nonzero finite set $M \subset \mathcal{O}^N$ is called *well-rounded* if the $\mathbb{C}$-span of M is $\mathbb{C}^N$. For a well-rounded subset M, let $\sigma(M)$ denote the set of forms $A \in C_N$ with $M(A) = M$ and $m(A) = 1$. It is easy to prove that if $\sigma(M)$ is non-empty then it is convex and thus topologically a cell. The well-rounded retract is then defined to be

$$W = \bigcup_{M \text{ well-rounded}} \sigma(M).$$

The space W is dual to the decomposition of X_N in a sense made precise in Theorem 2.6 below. For instance when $N = 2$ and $F = \mathbb{Q}$, X can be identified with the upper half-plane $\mathfrak{H} = \{x+iy \mid y > 0\}$. The Voronoi tessellation is the tiling of $\mathfrak{H}$ by with the $\mathrm{SL}_2(\mathbb{Z})$-translates of the ideal geodesic triangle with vertices $\{0, 1, \infty\}$. The well-rounded retract is the dual infinite trivalent tree (see Figure 1).

Lemma 2.5. *If $C_\sigma \in \tilde{\Sigma}$, then $M(\sigma)$ is well-rounded.*

Proof. This is proved by McConnell in [19, Theorem 2.11] when $F = \mathbb{Q}$, and we mimic his proof to yield the analogous result for imaginary quadratic fields. Let $M = M(C_\sigma)$ denote the spanning vectors of C_σ. Suppose M is not well-rounded. Then M does not span $\mathbb{C}^N$, and so there exists a non-zero vector $w \in \mathbb{C}^N$ such that $vw^* = 0$ for all $v \in M$. Then each $\theta \in C_\sigma$ can be written as a non-zero Hermitian form $\theta = \sum_{v \in M} a_v q(v)$, where $a_v \geq 0$. It follows that

$$\theta[w] = \sum_{v \in M} a_v \langle q(v), q(w) \rangle = \sum_{v \in M} a_v |vw^*|^2 = 0,$$

where $|\cdot|$ is the usual norm on $\mathbb{C}^N$. In particular, θ is not positive definite, contradicting the assumption that $C_\sigma \in \tilde{\Sigma}$. □

Theorem 2.6. *Let $M \subset \mathcal{O}^N$ be well-rounded and let $\sigma(M) \in W$. Then there is a unique cone $C_\sigma \in \tilde{\Sigma}$ such that $M(C_\sigma) = M$. The map $\sigma(M) \mapsto C_\sigma$ is a canonical bijection $W \to \tilde{\Sigma}$ that is inclusion-reversing on the face relations.*

Proof. It suffices to prove that a well-rounded subset $M \subset \mathcal{O}^N$ has $\sigma(M) \neq \emptyset$ if and only if there is a cone $C_\sigma \in \tilde{\Sigma}$ such that $M(\sigma) = M$.

Suppose $C_\sigma \in \tilde{\Sigma}$. By Lemma 2.5, $M = M(C_\sigma)$ is well-rounded. Thus it remains to show that $\sigma(M)$ is non-empty. We do so by constructing a form $B \in \sigma(M)$ with $M(B) = M(C_\sigma)$. Since $C_\sigma \in \tilde{\Sigma}$, there is a perfect form A such that C_σ is a face of the cone $S_A = \pi^{-1}(\sigma(A))$. Furthermore, C_σ can be described as the intersection of S_A with some supporting hyperplane $\{\theta \mid \langle H, \theta\rangle = 0\}$ for some $H \in \mathcal{H}^N(\mathbb{C})$. It follows that

$$\langle H, q(v)\rangle = 0 \quad \text{if } v \in M \quad \text{and} \quad \langle H, q(v)\rangle > 0 \quad \text{if } v \in M(A) \setminus M.$$

Let $B = A + \rho H$. Since $\langle H, q(v)\rangle = B[v]$, a standard argument for Hermitian forms shows that for sufficiently small positive ρ, B is positive definite, $B[v] = m(A)$ for $v \in M$ and $B[v] > m(A)$ for $v \in M(A) \setminus M$. Thus $M(B) = M$ and so $B \in \sigma(M)$. In particular, $\sigma(M)$ is non-empty.

Conversely, suppose M is a well-rounded subset of $\mathcal{O}^N$ with $\sigma(M)$ non-empty. Choose $B \in \sigma(M)$ so that $M(B) = M$. If B is perfect, then $M = M(B)$ and we are done. Otherwise, we can use the generalization of an algorithm of Voronoi [12] to find a perfect form A such that $M(B) \subset M(A)$. Let $H = B - A$. Then

$$\langle H, q(v)\rangle = 0 \quad \text{if } v \in M \quad \text{and} \quad \langle H, q(v)\rangle > 0 \quad \text{if } v \in M(A) \setminus M.$$

Thus the hyperplane $\{\theta \mid \langle H, \theta\rangle = 0\}$ is a supporting hyperplane for the subset of $S_A = \pi^{-1}(\sigma(A))$ spanned by $\{q(v) \mid v \in M\}$. Therefore M defines a face of S_A in $\tilde{\Sigma}$ as desired. ☐

Remark 2.7. Let Σ_n^* denote a set of representatives, modulo the action of $\mathrm{GL}_N(\mathcal{O})$, of n-dimensional cells of X_N^* that meet X_N. Let $\Sigma^* = \cup_n \Sigma_n^*$.

The well-rounded retract W is a proper, contractible $\mathrm{GL}_N(\mathcal{O})$-complex. Modulo $\mathrm{GL}_N(\mathcal{O})$, the cells in W are in bijection with cells in Σ^*, and the isomorphism classes of the stabilizers are preserved under this bijection. To see this, let $\sigma(M)$ be a cell in W. Then the forms $B \in \sigma(M)$ have $M(B) = M$ and $m(B) = 1$. Under the identification in Theorem 2.6, $\sigma(M)$ corresponds to the cone C'_σ with spanning vectors $\{q(v) \mid v \in M\}$. Let σ' be the corresponding cell. There is a cell $\sigma \in \Sigma^*$ that is Γ-equivalent to σ'. If $\gamma \in \mathrm{GL}_N(\mathcal{O})$, then $\gamma \cdot \sigma(M) = \sigma(\{(\gamma^*)^{-1} v \mid v \in M\})$. Thus if $\gamma \in \mathrm{Stab}(\sigma(M))$, then $(\gamma^*)^{-1} \in \mathrm{Stab}(C'_\sigma)$. It is therefore clear that $\mathrm{Stab}(C'_\sigma) = \mathrm{Stab}(\sigma') \simeq \mathrm{Stab}(\sigma)$.

The map $g \mapsto (g^*)^{-1}$ is an isomorphism of groups, so the duality between W mod Γ and Σ^* in Theorem 2.6 allows us to work with Σ^* and its stabilizer subgroups as if we were working with a proper, contractible $\mathrm{GL}_N(\mathcal{O})$-complex. This fact will be used later to compute the mass formula (§4).

3. The cohomology and homology

The material in this section follows [14, §3], [27, §2], and [3], all of which rely on [9]. Recall that $\Gamma = \mathrm{GL}_N(\mathcal{O})$. In this section, we introduce a complex $\mathrm{Vor}_{N,D} = (V_*(\Gamma), d_*)$ of $\mathbb{Z}[\Gamma]$-modules whose homology is isomorphic to the group cohomology of Γ modulo small primes. More precisely, for any positive integer n let $\mathcal{S}_n$ be the Serre class of finite abelian groups with orders only divisible by primes less than or equal to n [25]. Then the main result of this section (Theorem 3.7) is that, if $n = n(N, D)$ is larger than all the primes dividing the orders of finite subgroups of Γ, then modulo $\mathcal{S}_n$ the homology of $\mathrm{Vor}_{N,D}$ is isomorphic to the group cohomology of Γ.

As above let $\Sigma_n^* = \Sigma_n^*(\Gamma)$ denote a finite set of representatives, modulo the action of Γ, of n-dimensional cells of X_N^* which meet X_N. A cell σ is called orientable if every element in

$\mathrm{Stab}(\sigma)$ preserves the orientation of σ. By $\Sigma_n = \Sigma_n(\Gamma)$ we denote the set of orientable cells in $\Sigma_n^*(\Gamma)$ Let $\Sigma^* = \cup_n \Sigma_n^*$ and let $\Sigma = \cup_n \Sigma_n$.

3.1. **Steinberg homology.** The arithmetic group Γ—like any arithmetic group—is a virtual duality group [6, Thm 11.4.4]. This means that there is a $\mathbb{Z}[\Gamma]$-module I that plays the role of an "orientation module" for an analogue of Poincaré duality: for all coefficient modules M there is an isomorphism between the cohomology of Γ' with coefficients in M and the homology of Γ' with coefficients in $M \otimes I$, for any torsion-free finite index subgroup $\Gamma' \subset \Gamma$. For more details see [6, §11]. The module I is called the *dualizing module* for Γ. Borel and Serre prove that I is isomorphic to the *Steinberg module* $\mathrm{St}_\Gamma = H^\nu(\Gamma, \mathbb{Z}[\Gamma])$, where $\nu = \mathrm{vcd}\,\Gamma$ is the *virtual cohomological dimension* of Γ (cf. §4); note that for $\Gamma = \mathrm{GL}_N(\mathcal{O})$, we have $\nu = N^2 - N$.

For any Γ-module M, the *Steinberg homology* [9, p. 279], denoted $H_*^{\mathrm{St}}(\Gamma, M)$, is defined as $H_*^{\mathrm{St}}(\Gamma, M) = H_*(\Gamma, \mathrm{St}_\Gamma \otimes M)$. Let $\hat{H}^*$ denote the *Farrell cohomology* $\hat{H}^*(\Gamma, M)$ of Γ with coefficients in M (cf. [9, X.3]). There is a long exact sequence

$$\cdots \to H_{\nu-i}^{\mathrm{St}} \to H^i \to \hat{H}^i \to H_{\nu-(i+1)}^{\mathrm{St}} \to H^{i+1} \to \hat{H}^{i+1} \to \cdots \tag{3}$$

that implies that we can understand the group cohomology by understanding the Steinberg homology and the Farrell cohomology. In particular, if the Farrell cohomology vanishes, (which is the case when the torsion primes in Γ are invertible in M, see [9, IX.9 et seq.]), then the Steinberg homology is exactly the group cohomology $H_{\nu-k}^{\mathrm{St}}(\Gamma, M) \simeq H^k(\Gamma, M)$.

We now specialize to the case $M = \mathbb{Z}$ with trivial Γ-action, and in the remainder of this section omit the coefficients from homology and cohomology groups. Note that the well-rounded retract W is an $(N^2 - N)$-dimensional proper and contractible Γ-complex.

Proposition 3.1. *Let b be an upper bound on the torsion primes for $\Gamma = \mathrm{GL}_N(\mathcal{O})$. Then modulo the Serre class $\mathcal{S}_b$, we have*

$$H_{\nu-k}^{\mathrm{St}}(\Gamma) \simeq H^k(\Gamma).$$

3.2. **The Voronoi complex.** Let $V_n(\Gamma)$ denote the free abelian group generated by $\Sigma_n(\Gamma)$. Let $d_n \colon V_n(\Gamma) \to V_{n-1}(\Gamma)$ be the map defined in [14, §3.1], and denote the complex $(V_*(\Gamma), d_*)$ by $\mathrm{Vor}_{N,D}$.

Let ∂X_N^* denote the cells in X_N^* that do not meet X. Then ∂X_N^* is a Γ-invariant subcomplex of X_N^*. Let $H_*^\Gamma(X_N^*, \partial X_N^*)$ denote the relative equivariant homology of the pair $(X_N^*, \partial X_N^*)$ with integral coefficients (cf. [9, VII.7]).

Proposition 3.2. *Let b be an upper bound on the torsion primes for $\Gamma = \mathrm{GL}_N(\mathcal{O})$. Modulo the Serre class $\mathcal{S}_b$,*

$$H_n(\mathrm{Vor}_{N,D}) \simeq H_n^\Gamma(X_N^*, \partial X_N^*).$$

Proof. This result follows from [27, Proposition 2]. The argument is explained in detail for $F = \mathbb{Q}$ in [14, §3.2] and extends to F imaginary quadratic. For the convenience of the reader we recall the argument.

There is a spectral sequence E_{pq}^r converging to the equivariant homology groups

$$H_{p+q}^\Gamma(X_N^*, \partial X_N^*)$$

of the homology pair $(X_N^*, \partial X_N^*)$ such that

$$E_{pq}^1 = \bigoplus_{\sigma \in \Sigma_p^*} H_q(\mathrm{Stab}(\sigma), \mathbb{Z}_\sigma) \Rightarrow H_{p+q}^\Gamma(X_N^*, \partial X_N^*), \tag{4}$$

where $\mathbb{Z}_\sigma$ is the orientation module for σ.

When σ is not orientable the homology $H_0(\mathrm{Stab}(\sigma), \mathbb{Z}_\sigma)$ is killed by 2. Otherwise, $H_0(\mathrm{Stab}(\sigma), \mathbb{Z}_\sigma) \simeq \mathbb{Z}_\sigma$. Therefore, modulo $\mathcal{S}_2$, we have

$$E^1_{n,0} = \bigoplus_{\sigma \in \Sigma_n} \mathbb{Z}_\sigma.$$

Furthermore, when $q > 0$, $H_q(\mathrm{Stab}(\sigma), \mathbb{Z}_\sigma)$ lies in the Serre class $\mathcal{S}_b$, where b is the upper bound on the torsion primes for Γ. In other words, modulo $\mathcal{S}_b$, the E^1 page of the spectral sequence (4) is concentrated in the bottom row, and there is an identification of the bottom row groups with the groups $V_n(\Gamma)$ defined above. Finally, [27, Proposition 2] shows that $d^1_n = d_n$. □

Remark 3.3. If we do not work modulo $\mathcal{S}_b$, then there are other entries in the spectral sequence to consider before we get $H^\Gamma_n(X^*_N, \partial X^*_N)$. In particular, the small torsion in $H_*(\mathrm{Vor}_{N,D})$ may not agree in general with the small torsion in $H^\Gamma_{p+q}(X^*_N, \partial X^*_N)$.

3.3. **Equivariant relative homology to Steinberg homology.** For $V \subset F^N$ a proper subspace, let $C(V)$ be the set of matrices $A \in C^*_N$ such that the kernel of A is $V \otimes_\mathbb{Q} \mathbb{R}$, and let $X(V) = \pi(C(V))$. The closure $\overline{C(V)}$ of $C(V)$ in the usual topology induced from $\mathcal{H}^N(\mathbb{C})$ consists of matrices whose kernel contains V. Let $\overline{X(V)} = \pi(\overline{C(V)})$.

The following Lemma shows that $\overline{X(V)}$ is contractible. Soulé proves this for $F = \mathbb{Q}$ in [27, Lemma 1] and [26, Lemma 2], and the same proof with simple modifications applies to our setting:

Lemma 3.4. *For any proper subspace $V \subset F^N$, the CW-complex $\overline{X(V)}$ is contractible.*

Proof. Let A be a perfect Hermitian form over F. Then $\sigma(A) \cap \overline{X(V)}$ is the perfect cone of the intersection $M(A) \cap V^\perp$, where $V^\perp$ is the orthogonal complement to V in F^N. It follows that that $\overline{X(V)}$ is a sub-CW-complex of X^*_N.

Now, we show that $\overline{X(V)}$ is contractible. Since $\overline{X(V)}$ has the same homotopy type as $X(V)$, we prove that the latter is contractible. For this, it suffices to prove that the CW-topology on $X(V)$ coincides with its usual topology, since $X(V)$ in this topology is clearly contractible (it is convex). And to prove this, we argue that the covering of $X(V)$ by the closed sets of the form $\sigma(A) \cap X(V)$, where A is perfect, is locally finite.

Given any positive definite Hermitian form A, let $A^\perp$ be its restriction to the real space $V^\perp_\mathbb{R} = V^\perp \otimes_\mathbb{Q} \mathbb{R}$. Then $X(V)$ is isomorphic to the symmetric space for the group $\mathrm{Aut}(V^\perp_\mathbb{R})$ via the map $A \mapsto A^\perp$. If $\gamma \in \Gamma$ satisfies $\gamma \cdot X(V) \cap X(V) \neq \emptyset$, then in fact γ stabilizes V. Let $P \subset \Gamma$ be the stabilizer of V and let $\alpha \colon P \to \Gamma' = \mathrm{Aut}(V^\perp_\mathbb{R} \cap L)$ be the projection map. Then the set of cells $\sigma(A) \cap X(V)$, as A ranges over the perfect Hermitian forms, is finite modulo Γ': if $\sigma(A) \cap X(V) \cap (\gamma \cdot \sigma(A)) \cap X(V) \neq \emptyset$, then $\gamma \in P$, and thus $(\gamma \cdot \sigma(A) \cap X(V)) = \alpha(\gamma)(\sigma(A) \cap X(V))$.

To conclude the argument one uses Siegel sets; we refer to [7] for their definition and to [4, Ch. II] for their properties in our setting. Given any point $x \in X(V)$, one can find an open set $U \ni x$ and a Siegel set $\mathfrak{S}$ such that $\mathfrak{S} \supset U$. Furthermore, any cell of the form $\sigma(A) \cap X(V)$ is itself contained in another Siegel set $\mathfrak{S}'$. Thus if U meets $\gamma(\sigma(A) \cap X(V))$ for $\gamma \in \Gamma'$, we must have that $\gamma \cdot \mathfrak{S}'$ meets $\mathfrak{S}$. But this is only possible for finitely many γ by standard properties of Siegel sets. Thus the covering of $X(V)$ by the closed sets of the form $\sigma(A) \cap X(V)$ (A perfect) is locally finite, which completes the proof. □

Proposition 3.5. *For every $n \geq 0$, there are canonical isomorphisms of Γ-modules*

$$H_n(X_N^*, \partial X_N^*) \simeq \begin{cases} \mathrm{St}_\Gamma & \text{if } n = N-1, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Since X_N^* is contractible, the long exact sequence for the pair $(X_N^*, \partial X_N^*)$

$$\cdots \to H_n(\partial X_N^*) \to H_n(X_N^*) \to H_n(X_N^*, \partial X_N^*) \to H_{n-1}(\partial X_N^*) \to H_{n-1}(X_N^*) \to \cdots$$

implies $H_n(X_N^*, \partial X_N^*) \simeq \tilde{H}_{n-1}(\partial X_N^*)$ for $n \geq 1$, where the tilde denotes reduced homology.

From properties of Hermitian forms, it is clear that $\overline{X(V)} \cap \overline{X(W)} = \overline{X(V \cap W)}$ for each pair of proper, non-zero subspaces $V, W \subset F^N$. Thus the nerve of the covering $\mathcal{U}$ of ∂X_N^* by $\overline{X(V)}$ as V ranges over non-zero, proper subspaces of F^N is the spherical Tits building $T_{F,N}$. Since each $\overline{X(V)}$ in $\mathcal{U}$ is contractible by Lemma 3.4, the cover $\mathcal{U}$ is a good cover (i.e., nonempty finite intersections are diffeomorphic to $\mathbb{R}^d$ for some d). It follows that the relative homology groups are isomorphic $\tilde{H}_n(\partial X_N^*) \simeq \tilde{H}_n(T_{F,N})$. By the Solomon-Tits theorem, the latter is isomorphic to St_Γ if $n = N-2$ and is trivial otherwise. □

Proposition 3.6. *For all n, we have*

$$H_n^\Gamma(X_N^*, \partial X_N^*) = H_{n-(N-1)}^{\mathrm{St}}(\Gamma).$$

Proof. There is a spectral sequence [27, equation (2)] computing the relative equivariant homology

$$E_{pq}^2 = H_p(\Gamma, H_q(X_N^*, \partial X_N^*)) \Rightarrow H_{p+q}^\Gamma(X_N^*, \partial X_N^*).$$

Proposition 3.5 implies the E^2 page of the spectral sequence is concentrated in the $q = N-1$ column. Then

$$H_{p+(N-1)}^\Gamma(X_N^*, \partial X_N^*) \simeq H_p(\Gamma, H_q(X_N^*, \partial X_n^*)) = H_p(\Gamma, \mathrm{St}_\Gamma) = H_p^{\mathrm{St}}(\Gamma),$$

and the result follows. □

Theorem 3.7. *Let b be an upper bound on the torsion primes for $\mathrm{GL}_N(\mathcal{O})$. Modulo the Serre class $\mathcal{S}_b$,*

$$H_n(\mathrm{Vor}_{N,D}) \simeq H^{N^2-1-n}(\mathrm{GL}_N(\mathcal{O})).$$

Proof. Let $\Gamma = \mathrm{GL}_N(\mathcal{O})$. Modulo $\mathcal{S}_b$, Propositions 3.2, 3.6, and 3.1 imply

$$H_n(\mathrm{Vor}_{N,D}) \simeq H_n^\Gamma(X_N^*, \partial X_N^*) \simeq H_{n-(N-1)}^{\mathrm{St}}(\Gamma) \simeq H^{N^2-1-n}(\Gamma).$$

□

3.4. **Torsion elements in Γ.** To finish this section we discuss the possible torsion that can arise in the stabilizer subgroups of cells in our complexes. This allows us to make the bound in Theorem 3.7 effective.

Lemma 3.8. *Let p be an odd prime, and let $F/\mathbb{Q}$ be a quadratic field. Let $\Phi_p = x^{p-1} + x^{p-2} + \cdots + x + 1$ be the p^{th} cyclotomic polynomial. Then Φ_p factors over F as a product of irreducible polynomials of degree $(p-1)/2$ if $F = \mathbb{Q}(\sqrt{p^*})$, where $p^* = (-1)^{(p-1)/2}p$, and is irreducible otherwise.*

Proof. Let ζ_p denote a primitive p^{th} root of unity. Consider the diagram of Galois extensions below:

$$\begin{tikzcd}[column sep=small] & F(\zeta_p) \arrow[dl, dash] \arrow[dr, dash] & \\ F \arrow[dr, dash] & & \mathbb{Q}(\zeta_p) \arrow[dl, dash] \\ & F \cap \mathbb{Q}(\zeta_p) \arrow[d, dash] & \\ & \mathbb{Q} & \end{tikzcd}$$

If $F = \mathbb{Q}(\sqrt{p^*})$, then F is the unique quadratic subfield of $\mathbb{Q}(\zeta_p)$. It follows that Φ_p factors over F as a product of irreducible polynomials of degree $(p-1)/2$. If $F \neq \mathbb{Q}(\sqrt{p^*})$, then $F \cap \mathbb{Q}(\zeta_p) = \mathbb{Q}$. It follows that $[F(\zeta_p) : \mathbb{Q}(\zeta_p)] = [F : \mathbb{Q}] = 2$ and so $[F(\zeta_p) : F] = p-1$. Thus Φ_p is irreducible over F if $F \neq \mathbb{Q}(\sqrt{p^*})$. $\square$

Lemma 3.9. *Let p be an odd prime, and let $F/\mathbb{Q}$ be a imaginary quadratic field. If $g \in \mathrm{GL}_N(F)$ has order p, then*

$$p \leq \begin{cases} N+1 & \text{if } p \equiv 1 \bmod 4, \\ 2N+1 & \text{otherwise.} \end{cases}$$

Proof. If $p \equiv 1 \bmod 4$, then $p^* > 0$. In particular, $F \neq \mathbb{Q}(\sqrt{p^*})$ and so by Lemma 3.8, Φ_p is irreducible over F. Then the minimal polynomial of g is Φ_p. By the Cayley-Hamilton Theorem, Φ_p divides the characteristic polynomial of g. Therefore $p-1 \leq N$. Similarly, if $p \not\equiv 1 \bmod 4$, then $(p-1)/2 \leq N$. $\square$

Lemmas 3.8 and 3.9 immediately imply the following:

Proposition 3.10. *If $g \in \mathrm{GL}_3(\mathcal{O}_F)$ has prime order q, then $q \in \{2,3,7\}$ for $F = \mathbb{Q}(\sqrt{-7})$ and $q \in \{2,3\}$ otherwise. If $g \in \mathrm{GL}_4(\mathcal{O}_F)$ has prime order q, then $q \in \{2,3,5,7\}$ for $F = \mathbb{Q}(\sqrt{-7})$ and $q \in \{2,3,5\}$ otherwise.*

In Tables 1–12 we give the factorizations of the orders of the stabilizers of the cells in Σ^*.

4. A mass formula for the Voronoi complex

The computation of the cell complex is a relatively difficult task and there are many ways in which the computation can turn out to be wrong. Hence it is important to have checks that allow us to give strong evidence for the correctness of the computations. One is that the complexes we construct actually are chain complexes, namely that their differentials square to zero. Another is the *mass formula*, stated in Theorem 4.6. According to this formula, the alternating sum over the cells of Σ^* of the inverse orders of the stabilizer subgroups must vanish. A good reference for this section is [9, Ch. IX, §§6–7], and we follow it closely. The main theorem underlying this computation is due to Harder [16].

4.1. **Euler characteristics.** We begin by recalling some definitions. The *cohomological dimension* $\operatorname{cd}\Gamma$ of a group Γ is the largest $n \in \mathbb{Z} \cup \{\infty\}$ such that there exists a $\mathbb{Z}\Gamma$-module M with $H^n(\Gamma; M) \neq 0$. The *virtual cohomological dimension* $\operatorname{vcd}\Gamma$ of Γ is defined to be the cohomological dimension of any torsion-free finite index subgroup of Γ (one can show that this is well-defined). We recall that Γ is said to be of *finite homological type* if (i) $\operatorname{vcd}\Gamma < \infty$

and (ii) for every $\mathbb{Z}\Gamma$-module M that is finitely generated as an abelian group, the homology group $H_i(\Gamma; M)$ is finitely generated for all i.

Definition 4.1. Let Γ be a torsion-free group of finite homological type, and let $H_*(\Gamma)$ denote the homology of Γ with (trivial) $\mathbb{Z}$-coefficients. The *Euler characteristic* $\chi(\Gamma)$ is

$$\chi(\Gamma) = \sum_i (-1)^i \operatorname{rank}_{\mathbb{Z}}(H_i(\Gamma)).$$

Proposition 4.2. [9, Theorem 6.3] *If Γ is torsion-free and of finite homological type and Γ' is a subgroup of finite index, then*

$$\chi(\Gamma') = [\Gamma \colon \Gamma'] \cdot \chi(\Gamma).$$

One can use Proposition 4.2 to extend the notion of Euler characteristic to groups with torsion. Namely, if Γ is an arbitrary group of finite homological type with a torsion-free subgroup Γ' of finite index, one sets

$$\chi(\Gamma) = \frac{\chi(\Gamma')}{[\Gamma \colon \Gamma']}. \tag{5}$$

By Proposition 4.2, this is independent of the choice of Γ'.

Proposition 4.3. *If Γ is a finite group, then $\chi(\Gamma) = 1/|\Gamma|$.*

Proof. Take Γ' to be the trivial subgroup in (5). □

Theorem 4.4. [16] *Let F be a number field with ring of integers $\mathcal{O}$, and let $\zeta_F(s)$ be the Dedekind zeta function of F. Then*

$$\chi(\mathrm{SL}_N(\mathcal{O})) = \prod_{k=2}^{N} \zeta_F(1-k).$$

In particular, since when F is imaginary quadratic $\zeta_F(m)$ vanishes for m any negative integer, we have $\chi(\mathrm{SL}_N(\mathcal{O})) = 0$ for $N \geq 2$.

Corollary 4.5. *Let F be an imaginary quadratic field with ring of integers $\mathcal{O}$. Then $\chi(\mathrm{GL}_N(\mathcal{O})) = 0$ for $N \geq 2$.*

Proof. This follows immediately from Theorem 4.4 and the definition of the Euler characteristic since $\mathrm{GL}_N(\mathcal{O})$ is of finite index in $\mathrm{SL}_N(\mathcal{O})$. □

Now we turn to a different concept, the *equivariant Euler characteristic* $\chi_\Gamma(X)$ of Γ. Here X is any cell complex with Γ action such that (i) X has finitely many cells mod Γ, and (ii) for each $\sigma \in X$, the stabilizer subgroup $\mathrm{Stab}_\Gamma(\sigma)$ is finite. One defines

$$\chi_\Gamma(X) = \sum_{\sigma \in S} (-1)^{\dim \sigma} \chi(\mathrm{Stab}_\Gamma(\sigma)),$$

where S is a set of representatives of cells of X mod Γ.

4.2. **The mass formula.** The well-rounded retract W defined in §2.2 is a proper, contractible Γ-complex and so its equivariant Euler characteristic is defined. We compute its equivariant Euler characteristic, phrased in terms of cells in Σ^* using Remark 2.7, to get a mass formula.

Theorem 4.6 (Mass Formula)**.** *We have*

$$\sum_{\sigma\in\Sigma_k^*}(-1)^k\frac{1}{|\operatorname{Stab}_\Gamma(\sigma)|}=0.$$

Proof. Let $\Gamma=\mathrm{GL}_N(\mathcal{O})$. By Proposition 4.3 we have $\chi(\operatorname{Stab}_\Gamma(\sigma))=1/|\operatorname{Stab}_\Gamma(\sigma)|$. Thus the result follows from Corollary 4.5 if we can show $\chi_\Gamma(W)=\chi(\Gamma)$, where W is the Γ-complex formed from the well-rounded retract, using Theorem 2.6 and Remark 2.7 to identify cells of W mod Γ with cells in Σ^*. But according to [9, Proposition 7.3 (e')], this equality is true if Γ has a torsion-free subgroup of finite index, which is a standard fact for Γ. ☐

Using the stabilizer information in Tables 1–12 one can easily verify that $\chi(\mathrm{GL}_n(\mathcal{O}))=0$ for each of our examples. For instance, if we add together the terms $1/|\operatorname{Stab}_\Gamma(\sigma)|$ for cells σ of the same dimension to a single term for $\mathrm{GL}_4(\mathcal{O}_{-4})$, we find (ordering the terms by increasing dimension in Σ^*)

$$-\frac{11}{3072}+\frac{127}{960}-\frac{4187}{2304}+\frac{28375}{2304}-\frac{868465}{18432}+\frac{126127}{1152}-\frac{81945}{512}\\ +\frac{340955}{2304}-\frac{48655}{576}+\frac{16075}{576}-\frac{21337}{4608}+\frac{101}{384}-\frac{17}{92160}=0.$$

The other groups can be checked similarly.

5. Explicit homology classes

By Theorem 3.7 we have $H_{N^2-1}(\mathrm{Vor}_{N,D}\otimes\mathbb{Q})\simeq H^0(\mathrm{GL}_N(\mathcal{O}),\mathbb{Q})$, which in turn is isomorphic to $\mathbb{Q}$. This suggests that there should be a canonical generator for this homology group, a fact already explored in [14, Section 5]. An obvious choice is the analogue of the chain presented there, namely

$$\xi:=\xi_{N,D}:=\sum_\sigma\frac{1}{|\operatorname{Stab}(\sigma)|}[\sigma],$$

where σ runs through the cells in $\Sigma_{N^2-1}(\mathrm{GL}_N(\mathcal{O}))$. In this section we verify that this is true for our examples. We should point out that all the cells in $\Sigma_{N^2-1}(\mathrm{GL}_N(\mathcal{O}))$ are orientable. The reason is that the group $\mathrm{GL}_n(\mathbb{C})$ is connected and so the determinant of its action on $\mathcal{H}^N(\mathbb{C})$ is positive. Since the stabilizers are included in $\mathrm{GL}_N(\mathcal{O})$ and the faces in $\Sigma_{N^2-1}(\mathrm{GL}_N(\mathcal{O}))$ are full-dimensional the orientation has to be preserved.

Theorem 5.1. *When $N=3$ and $D\geq -24$ or $N=4$ and $D=-3,-4$, the chain ξ is a cycle and thus generates $H_{N^2-1}(\mathrm{Vor}_{N,D}\otimes\mathbb{Q})$.*

Proof. The proof is an explicit computation with differential matrix A representing the map $V_{N^2-1}(\Gamma)\to V_{N^2-2}(\Gamma)$ (cf. §3.2). Note that the signs of the entries of A depend on a choice of orientation for each of the cells in Σ_{N^2-1} and Σ_{N^2-2}. In each non-zero row of A, there are exactly two non-zero entries. Each non-zero entry $A_{i,j}$ has absolute value $|\operatorname{Stab}(\sigma_j)|/|\operatorname{Stab}(\tau_i)|$, where $\sigma_j\in\Sigma_{N^2-1}(\mathrm{GL}_N(\mathcal{O}))$ and $\tau_i\in\Sigma_{N^2-2}(\mathrm{GL}_N(\mathcal{O}))$. One then checks that there is a choice of orientations such that the non-zero entries in a given row have opposite signs. ☐

For example, consider $N=4$ and $D=-4$. The differential matrix is

$$d_{15}=\begin{bmatrix}0&0\\1920&-256\end{bmatrix},$$

with kernel generated by $(2,15) = 92160(1/46080, 1/6144)$. The orders of the two stabilizer groups for the cells in $\Sigma_{15}(\mathrm{GL}_N(\mathcal{O}_{-4}))$ are 46080 and 6144, respectively, and thus $\xi_{4,-4}$ is a cycle.

It seems likely that Theorem 5.1 holds for all $\mathrm{GL}_N(\mathcal{O}_D)$, although we do not know a proof.

6. Tables

We conclude by presenting the results of our computations. Here is a guide to the notation in the tables.

- The first three columns concern the cell decomposition of X_N^* mod Γ:
  - n is dimension of the cells in the (partially) compactified symmetric space X_N^*.
  - $|\Sigma_n^*|$ is the number of Γ-orbits in the cells that meet $X_N \subset X_N^*$.
  - $|\,\mathrm{Stab}\,|$ gives the sizes of the stabilizer subgroups, in factored form. The notation $A(k)$ means that, of the $|\Sigma_n^*|$ cells of dimension n, k of them have a stabilizer subgroup of order A.
- The next four columns concern the differentials d_n of the Voronoi complex $\mathrm{Vor}_{N,D}$:
  - $|\Sigma_n|$ is the number of orientable Γ-orbits in Σ_n^*.
  - Ω is the number of nonzero entries in the differential $d_n\colon V_n(\Gamma) \to V_{n-1}(\Gamma)$.
  - rank is the rank of d_n.
  - elem. div. gives the elementary divisors of d_n. As in the stabilizer column, the notation $d(k)$ means that the elementary divisor d occurs with multiplicity k. If the rank of d_n vanishes, then this column is empty.
- Finally, the last column gives the homology of the Voronoi complex. One can easily check that $H_n \simeq \mathbb{Z}^r \oplus \bigoplus(\mathbb{Z}/d\mathbb{Z})^k$, where $r = |\Sigma_n| - \mathrm{rank}(d_n) - \mathrm{rank}(d_{n+1})$ and the sum is taken over the elementary divisors $d(k)$ from row $n+1$. To save space, we abbreviate $\mathbb{Z}/d\mathbb{Z}$ by $\mathbb{Z}_d$. By Theorem 3.7 we have $H_n(\mathrm{Vor}_{N,D}) \simeq H^{N^2-1-n}(\mathrm{GL}_N(\mathcal{O}))$ modulo the torsion primes in $\mathrm{GL}_N(\mathcal{O})$. These primes are visible in the third column of each table.

The tables suggest that the rank of $H_n(\mathrm{GL}_3(\mathcal{O})$ is nonzero if and only if $n = 0$, 4 or 5, and similarly that the rank of $H_n(\mathrm{GL}_4(\mathcal{O}))$ is nonzero if and only if $n = 0$, 3, 5, 6, 8 or 9.

Table 1. Invariants for the cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-3})$.

n	$\lvert\Sigma_n^*\rvert$	$\lvert\,\mathrm{Stab}\,\rvert$	$\lvert\Sigma_n\rvert$	Ω	rank	elem. div.	H_n
2	1	$2^4 3^4(1)$	0	0	0		0
3	2	$2^4 3^2(1)$, $2^3 3^3(1)$	0	0	0		0
4	3	$2^2 3^2(1)$, $2^4 3^3(1)$, $2^4 3^1(1)$	1	0	0		$\mathbb{Z}$
5	4	$2^4 3^2(1)$, $2^2 3^2(1)$, $2^1 3^2(1)$, $2^2 3^3(1)$	2	0	0		$\mathbb{Z}$
6	3	$2^2 3^2(1)$, $2^1 3^2(1)$, $2^2 3^3(1)$	1	2	1	1(1)	0
7	2	$2^4 3^2(1)$, $2^2 3^2(1)$	1	0	0		$\mathbb{Z}_9$
8	2	$2^4 3^4(2)$	2	2	1	9(1)	$\mathbb{Z}$

TABLE 2. Invariants for the cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-4})$.

n	$\lvert\Sigma_n^*\rvert$	$\lvert\,\mathrm{Stab}\,\rvert$	$\lvert\Sigma_n\rvert$	Ω	rank	elem. div.	H_n
2	2	$2^7 3^1(2)$	0	0	0		0
3	3	$2^3 3^1(1)$, $2^5 3^1(2)$	0	0	0		0
4	4	$2^4(1)$, $2^3(1)$, $2^5(1)$, $2^7 3^1(1)$	1	0	0		$\mathbb{Z}$
5	5	$2^2 3^1(2)$, $2^3(1)$, $2^5 3^1(1)$, $2^5(1)$	4	0	0		$\mathbb{Z} \oplus \mathbb{Z}_2$
6	3	$2^2 3^1(2)$, $2^4(1)$	3	10	3	1(2), 2(1)	0
7	1	$2^4(1)$	0	0	0		0
8	1	$2^7 3^1(1)$	1	0	0		$\mathbb{Z}$

TABLE 3. Invariants for the cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-7})$.

n	$\lvert\Sigma_n^*\rvert$	$\lvert\,\mathrm{Stab}\,\rvert$	$\lvert\Sigma_n\rvert$	Ω	rank	elem. div.	H_n
2	3	$2^4 3^1(3)$	0	0	0		0
3	6	$2^2 3^1(2)$, $2^3 3^1(1)$, $2^2(1)$, $2^4 3^1(1)$, $2^4(1)$	0	0	0		0
4	9	$2^2 3^1(1)$, $2^1(1)$, $2^3 3^1(1)$, $2^2(2)$, $2^4(1)$, $2^3(3)$	3	0	0		$\mathbb{Z}$
5	11	$2^2 3^1(1)$, $2^1(2)$, $2^2(1)$, $2^4 3^1(3)$, $2^1 3^1(3)$, $2^3(1)$	10	8	2	1(2)	$\mathbb{Z}^2 \oplus \mathbb{Z}_7$
6	8	$2^2(2)$, $2^1 3^1(2)$, $2^3(3)$, $2^1 3^1 7^1(1)$	6	19	6	1(5), 7(1)	0
7	2	$2^1 7^1(1)$, $2^2(1)$	1	0	0		$\mathbb{Z}_3$
8	2	$2^4 3^1 7^1(1)$, $2^1 3^1 7^1(1)$	2	2	1	3(1)	$\mathbb{Z}$

TABLE 4. Invariants for the cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-8})$.

n	$\lvert\Sigma_n^*\rvert$	$\lvert\,\mathrm{Stab}\,\rvert$	$\lvert\Sigma_n\rvert$	Ω	rank	elem. div.	H_n
2	5	$2^2 3^1(2)$, $2^4 3^1(2)$, $2^4(1)$	0	0	0		0
3	16	$2^1(2)$, $2^2(5)$, $2^3 3^1(1)$, $2^2 3^1(5)$, $2^4 3^1(1)$, $2^5(2)$	2	0	0		0
4	26	$2^1(14)$, $2^2(9)$, $2^3(1)$, $2^5 3^1(1)$, $2^4(1)$	16	12	2	1(2)	$\mathbb{Z} \oplus \mathbb{Z}_2$
5	37	$2^1(25)$, $2^2(4)$, $2^1 3^1(4)$, $2^3(2)$, $2^4 3^1(2)$	36	104	13	1(12), 2(1)	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$
6	28	$2^1(18)$, $2^2(2)$, $2^1 3^1(4)$, $2^3(2)$, $2^2 3^1(2)$	26	166	21	1(20), 2(1)	0
7	7	$2^1(6)$, $2^2(1)$	6	45	5	1(5)	0
8	2	$2^1 3^1(1)$, $2^5(1)$	2	6	1	1(1)	$\mathbb{Z}$

TABLE 5. Invariants for the cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-11})$.

n	$\lvert\Sigma_n^*\rvert$	$\lvert\mathrm{Stab}\rvert$	$\lvert\Sigma_n\rvert$	Ω	rank	elem. div.	H_n
2	8	$2^2 3^1(2)$, $2^3 3^1(1)$, $2^2(2)$, $2^4 3^1(1)$, $2^4(2)$	1	0	0		0
3	34	$2^1(13)$, $2^2(11)$, $2^3(2)$, $2^4 3^1(1)$, $2^2 3^1(4)$, $2^3 3^1(3)$	15	6	1	1(1)	0
4	91	$2^1(67)$, $2^2(15)$, $2^1 3^1(1)$, $2^3(5)$, $2^2 3^1(1)$, $2^4(1)$, $2^4 3^1(1)$	75	193	14	1(14)	$\mathbb{Z}$
5	150	$2^1(124)$, $2^2(13)$, $2^1 3^1(7)$, $2^3(2)$, $2^2 3^1(3)$, $2^4 3^1(1)$	147	700	60	1(60)	$\mathbb{Z}^2 \oplus \mathbb{Z}_4$
6	125	$2^1(110)$, $2^2(6)$, $2^1 3^1(6)$, $2^3(3)$	122	859	85	1(84), 4(1)	0
7	51	$2^1(44)$, $2^2(2)$, $2^1 3^1(3)$, $2^2 3^1(1)$, $2^4(1)$	48	404	37	1(37)	$\mathbb{Z}_3^2$
8	12	$2^1(4)$, $2^1 3^1(6)$, $2^4(2)$	12	88	11	1(9), 3(2)	$\mathbb{Z}$

TABLE 6. Invariants for the cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-15})$.

n	$\lvert\Sigma_n^*\rvert$	$\lvert\mathrm{Stab}\rvert$	$\lvert\Sigma_n\rvert$	Ω	rank	elem. div.	H_n
2	34	$2^1(4)$, $2^2(8)$, $2^1 3^1(1)$, $2^3(8)$, $2^4 3^1(3)$, $2^2 3^1(3)$, $2^4(7)$	10	0	0		$\mathbb{Z}$
3	217	$2^1(102)$, $2^2(77)$, $2^1 3^1(2)$, $2^3(23)$, $2^2 3^1(5)$, $2^4(2)$, $2^3 3^1(3)$, $2^4 3^1(3)$	128	175	9	1(9)	0
4	689	$2^1(546)$, $2^2(114)$, $2^1 3^1(1)$, $2^3(20)$, $2^2 3^1(2)$, $2^4(5)$, $2^3 3^1(1)$	604	2112	119	1(119)	$\mathbb{Z}^3 \oplus \mathbb{Z}_2^4$
5	1224	$2^1(1109)$, $2^2(84)$, $2^1 3^1(7)$, $2^3(13)$, $2^2 3^1(5)$, $2^4(3)$, $2^4 3^1(3)$	1185	6373	482	1(478), 2(4)	$\mathbb{Z}^5$
6	1139	$2^1(1081)$, $2^2(47)$, $2^1 3^1(7)$, $2^3(4)$	1102	7771	698	1(698)	0
7	522	$2^1(489)$, $2^2(30)$, $2^1 3^1(1)$, $2^3(1)$, $2^2 3^1(1)$	493	4162	404	1(404)	$\mathbb{Z}_3^2 \oplus \mathbb{Z}_6^2 \oplus \mathbb{Z}_{12}$
8	90	$2^1(78)$, $2^2(3)$, $2^1 3^1(5)$, $2^3(2)$, $2^2 3^1(2)$	90	972	89	1(84), 3(2), 6(2), 12(1)	$\mathbb{Z}$

TABLE 7. Invariants for the cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-19})$.

n	$\lvert\Sigma_n^*\rvert$	$\lvert\,\mathrm{Stab}\,\rvert$	$\lvert\Sigma_n\rvert$	Ω	rank	elem. div.	H_n
2	43	$2^1(23)$, $2^2(10)$, $2^43^1(1)$, $2^13^1(3)$, $2^3(1)$, $2^23^1(2)$, $2^4(2)$, $2^33^1(1)$	29	0	0		0
3	359	$2^1(304)$, $2^2(38)$, $2^13^1(3)$, $2^3(7)$, $2^23^1(4)$, $2^33^1(2)$, $2^43^1(1)$	314	664	29	1(29)	0
4	1293	$2^1(1234)$, $2^2(52)$, $2^3(4)$, $2^4(1)$, $2^33^1(1)$, $2^43^1(1)$	1255	5410	285	1(285)	$\mathbb{Z}^2$
5	2347	$2^1(2287)$, $2^2(37)$, $2^13^1(13)$, $2^3(7)$, $2^23^1(2)$, $2^43^1(1)$	2339	13596	968	1(968)	$\mathbb{Z}^3 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8$
6	2169	$2^1(2137)$, $2^2(15)$, $2^13^1(10)$, $2^3(7)$	2164	15404	1368	1(1366), 2(1), 8(1)	0
7	958	$2^1(950)$, $2^2(6)$, $2^13^1(1)$, $2^4(1)$	952	8181	796	1(796)	$\mathbb{Z}_3^4$
8	157	$2^1(149)$, $2^13^1(6)$, $2^4(2)$	157	1884	156	1(152), 3(4)	$\mathbb{Z}$

TABLE 8. Invariants for the cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-20})$.

n	$\lvert\Sigma_n^*\rvert$	$\lvert\,\mathrm{Stab}\,\rvert$	$\lvert\Sigma_n\rvert$	Ω	rank	elem. div.	H_n
2	69	$2^1(21)$, $2^2(26)$, $2^13^1(4)$, $2^3(6)$, $2^23^1(3)$, $2^4(7)$, $2^43^1(2)$	31	0	0		$\mathbb{Z}$
3	538	$2^1(398)$, $2^2(98)$, $2^13^1(4)$, $2^3(22)$, $2^23^1(7)$, $2^4(3)$, $2^33^1(4)$, $2^43^1(2)$	425	772	30	1(30)	$\mathbb{Z}_2$
4	1895	$2^1(1721)$, $2^2(153)$, $2^3(15)$, $2^23^1(1)$, $2^4(4)$, $2^43^1(1)$	1804	7464	395	1(394), 2(1)	$\mathbb{Z}^4 \oplus \mathbb{Z}_2^4$
5	3382	$2^1(3223)$, $2^2(117)$, $2^13^1(15)$, $2^3(15)$, $2^23^1(7)$, $2^4(4)$, $2^43^1(1)$	3345	19167	1405	1(1401), 2(4)	$\mathbb{Z}^6 \oplus \mathbb{Z}_3$
6	3061	$2^1(2976)$, $2^2(61)$, $2^13^1(15)$, $2^3(8)$, $2^23^1(1)$	3017	21502	1934	1(1933), 3(1)	0
7	1330	$2^1(1293)$, $2^2(37)$	1294	11127	1083	1(1083)	$\mathbb{Z}_3^2 \oplus \mathbb{Z}_6^2 \oplus \mathbb{Z}_{12}$
8	212	$2^1(202)$, $2^2(2)$, $2^13^1(5)$, $2^3(2)$, $2^23^1(1)$	212	2532	211	1(206), 3(2), 6(2), 12(1)	$\mathbb{Z}$

Table 9. Invariants for the cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-23})$.

n	$\lvert\Sigma_n^*\rvert$	$\lvert\mathrm{Stab}\rvert$	$\lvert\Sigma_n\rvert$	Ω	rank	elem. div.	H_n
2	204	$2^1(89)$, $2^2(65)$, $2^13^1(4)$, $2^3(27)$, $2^23^1(6)$, $2^4(10)$, $2^43^1(3)$	126	0	0		$\mathbb{Z}^4$
3	1777	$2^1(1402)$, $2^2(295)$, $2^13^1(2)$, $2^3(56)$, $2^23^1(11)$, $2^4(3)$, $2^33^1(5)$, $2^43^1(3)$	1477	3272	122	1(122)	$\mathbb{Z}_2$
4	6589	$2^1(6112)$, $2^2(434)$, $2^3(35)$, $2^23^1(2)$, $2^4(5)$, $2^33^1(1)$	6285	26837	1355	1(1354), 2(1)	$\mathbb{Z}^5 \oplus \mathbb{Z}_2^{10} \oplus \mathbb{Z}_6^3$
5	12214	$2^1(11866)$, $2^2(291)$, $2^13^1(19)$, $2^3(23)$, $2^23^1(6)$, $2^4(6)$, $2^43^1(3)$	12119	69891	4925	1(4912), 2(10), 6(3)	$\mathbb{Z}^{10} \oplus \mathbb{Z}_2^5$
6	11627	$2^1(11461)$, $2^2(138)$, $2^13^1(16)$, $2^3(10)$, $2^23^1(2)$	11568	81720	7184	1(7179), 2(5)	0
7	5303	$2^1(5250)$, $2^2(48)$, $2^13^1(2)$, $2^23^1(2)$, $2^4(1)$	5253	44741	4384	1(4384)	$\mathbb{Z}_{12}^2 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_6^3$
8	870	$2^1(853)$, $2^2(3)$, $2^13^1(10)$, $2^3(2)$, $2^4(2)$	870	10464	869	1(862), 12(2), 3(2), 6(3)	$\mathbb{Z}$

Table 10. Invariants for the cell complex, differentials, and homology for $\mathrm{GL}_3(\mathcal{O}_{-24})$.

n	$\lvert\Sigma_n^*\rvert$	$\lvert\mathrm{Stab}\rvert$	$\lvert\Sigma_n\rvert$	Ω	rank	elem. div.	H_n
2	158	$2^1(90)$, $2^2(41)$, $2^13^1(3)$, $2^3(13)$, $2^23^1(1)$, $2^4(8)$, $2^43^1(2)$	104	0	0		$\mathbb{Z}$
3	1396	$2^1(1214)$, $2^2(142)$, $2^13^1(4)$, $2^3(24)$, $2^23^1(6)$, $2^4(1)$, $2^33^1(3)$, $2^43^1(2)$	1247	2967	103	1(103)	$\mathbb{Z}_2^7$
4	5090	$2^1(4859)$, $2^2(199)$, $2^13^1(1)$, $2^3(23)$, $2^23^1(1)$, $2^4(5)$, $2^33^1(1)$, $2^43^1(1)$	4957	22280	1144	1(1137), 2(7)	$\mathbb{Z}^5 \oplus \mathbb{Z}_2^5$
5	9091	$2^1(8889)$, $2^2(161)$, $2^13^1(12)$, $2^3(20)$, $2^23^1(5)$, $2^4(3)$, $2^43^1(1)$	9043	53385	3808	1(3803), 2(5)	$\mathbb{Z}^7 \oplus \mathbb{Z}_2^8 \oplus \mathbb{Z}_4^2$
6	8319	$2^1(8187)$, $2^2(102)$, $2^13^1(13)$, $2^3(15)$, $2^23^1(2)$	8263	58948	5228	1(5218), 2(8), 4(2)	$\mathbb{Z}_2$
7	3662	$2^1(3617)$, $2^2(42)$, $2^3(2)$, $2^4(1)$	3630	31020	3035	1(3034), 2(1)	$\mathbb{Z}_{12}^2 \oplus \mathbb{Z}_2^3 \oplus \mathbb{Z}_6$
8	596	$2^1(578)$, $2^2(8)$, $2^13^1(2)$, $2^3(4)$, $2^23^1(2)$, $2^4(2)$	596	7188	595	1(589), 12(2), 2(3), 6(1)	$\mathbb{Z}$

TABLE 11. Invariants for the cell complex, differentials, and homology for $GL_4(\mathcal{O}_{-3})$.

n	$\lvert\Sigma_n^*\rvert$	$\lvert$ Stab $\rvert$	$\lvert\Sigma_n\rvert$	Ω	rank	elem. div.	H_n
3	2	$2^4 3^5(1)$, $2^7 3^5(1)$	0	0	0		0
4	5	$2^4 3^2(1)$, $2^4 3^2 5^1(1)$, $2^5 3^3(1)$, $2^5 3^4(1)$, $2^3 3^4(1)$	0	0	0		0
5	12	$2^2 3^2(2)$, $2^3 3^2(1)$, $2^2 3^3(1)$, $2^3 3^1(1)$, $2^5 3^4(1)$, $2^3 3^3(1)$, $2^4 3^1(2)$, $2^5 3^2(2)$, $2^6 3^4(1)$	0	0	0		0
6	34	$2^5 3^4(1)$, $2^1 3^1(1)$, $2^4 3^1(2)$, $2^2 3^1(8)$, $2^5 3^3(1)$, $2^1 3^2(2)$, $2^3 3^4(1)$, $2^3 3^1(4)$, $2^5 3^2(1)$, $2^3 3^3(1)$, $2^4 3^2(2)$, $2^2 3^3(4)$, $2^2 3^2(6)$	8	0	0		$\mathbb{Z}$
7	82	$2^1 3^1(21)$, $2^5 3^1(1)$, $2^2 3^1(23)$, $2^1 3^2(7)$, $2^2 3^3(3)$, $2^3 3^1(5)$, $2^3 3^4(1)$, $2^2 3^2(8)$, $2^3 3^3(2)$, $2^7 3^4(1)$, $2^4 3^1(4)$, $2^1 3^3(2)$, $2^4 3^2(1)$, $2^3 3^2(3)$	50	58	7	1(7)	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$
8	166	$2^3 3^3(2)$, $2^1 3^1(88)$, $2^4 3^3(1)$, $2^2 3^1(36)$, $2^1 3^2(13)$, $2^5 3^3(1)$, $2^3 3^1(5)$, $2^2 3^2(7)$, $2^4 3^1(2)$, $2^1 3^3(2)$, $2^3 3^2(5)$, $2^2 3^3(3)$, $2^2 3^5(1)$	129	604	41	1(40), 2(1)	$\mathbb{Z}_9$
9	277	$2^1 3^1(191)$, $2^5 3^2(1)$, $2^2 3^1(34)$, $2^1 3^2(17)$, $2^5 3^5(2)$, $2^3 3^1(6)$, $2^1 3^1 5^1(1)$, $2^2 3^2(9)$, $2^1 3^3(2)$, $2^3 3^2(3)$, $2^5 3^1(1)$, $2^2 3^3(7)$, $2^7 3^3(1)$, $2^2 3^5(1)$, $2^4 3^2 5^1(1)$	228	1616	88	1(87), 9(1)	$\mathbb{Z} \oplus \mathbb{Z}_{24}$
10	324	$2^1 3^1(246)$, $2^2 3^1(35)$, $2^1 3^2(16)$, $2^3 3^1(7)$, $2^2 3^2(9)$, $2^1 3^3(2)$, $2^4 3^3(1)$, $2^2 3^3(5)$, $2^3 3^3(2)$, $2^5 3^2(1)$	286	2531	139	1(138), 24(1)	$\mathbb{Z} \oplus \mathbb{Z}_2^4$
11	259	$2^1 3^1(200)$, $2^2 3^1(24)$, $2^1 3^2(11)$, $2^3 3^1(6)$, $2^2 3^2(9)$, $2^2 3^4(1)$, $2^4 3^1(1)$, $2^1 3^3(1)$, $2^2 3^3(2)$, $2^4 3^2(2)$, $2^4 3^3(2)$	237	2283	146	1(142), 2(4)	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_6$
12	142	$2^1 3^1(91)$, $2^2 3^1(20)$, $2^1 3^2(9)$, $2^3 3^1(5)$, $2^2 3^2(11)$, $2^1 3^3(4)$, $2^2 3^3(1)$, $2^4 3^2(1)$	122	1252	91	1(88), 2(2), 6(1)	$\mathbb{Z} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_{12}$
13	48	$2^1 3^3(1)$, $2^3 3^3(1)$, $2^1 3^1(22)$, $2^2 3^1(6)$, $2^1 3^2(7)$, $2^3 3^1(2)$, $2^2 3^2(8)$, $2^4 3^1(1)$	36	369	30	1(28), 3(1), 12(1)	$\mathbb{Z}_{15}$
14	15	$2^1 3^2(2)$, $2^2 3^2(2)$, $2^1 3^3(1)$, $2^2 3^3(3)$, $2^1 3^1(1)$, $2^4 3^2 5^1(1)$, $2^3 3^1(1)$, $2^2 3^1(2)$, $2^1 3^1 5^1(1)$, $2^6 3^2(1)$	10	51	6	1(5), 15(1)	$\mathbb{Z}_{12} \oplus \mathbb{Z}_{288}$
15	5	$2^6 3^3(1)$, $2^7 3^5 5^1(1)$, $2^4 3^5(1)$, $2^2 3^3(2)$	5	16	4	1(2), 12(1), 288(1)	$\mathbb{Z}$

TABLE 12. Invariants for the cell complex, differentials, and homology for $\mathrm{GL}_4(\mathcal{O}_{-4})$.

n	$\lvert\Sigma_n^*\rvert$	$\lvert\mathrm{Stab}\rvert$	$\lvert\Sigma_n\rvert$	Ω	rank	elem. div.	H_n
3	4	$2^{11}3^1(2)$, $2^9 3^1(1)$, $2^7 3^1(1)$	0	0	0		0
4	10	$2^3 3^1(1)$, $2^8(1)$, $2^4 3^1(1)$, $2^5 3^1 5^1(1)$, $2^7(1)$, $2^5 3^1(2)$, $2^8 3^1(1)$, $2^7 3^1(1)$, $2^5(1)$	0	0	0		0
5	33	$2^{10}3^1(1)$, $2^2(1)$, $2^7 3^2(1)$, $2^6 3^1(1)$, $2^3(6)$, $2^4 3^1(2)$, $2^4(7)$, $2^7(1)$, $2^5 3^1(2)$, $2^3 3^1(2)$, $2^{10}(1)$, $2^6(2)$, $2^5(6)$	5	0	0		0
6	98	$2^7(1)$, $2^7 3^1(2)$, $2^2(26)$, $2^3(37)$, $2^2 3^1(1)$, $2^4(10)$, $2^8 3^2(1)$, $2^3 3^1(1)$, $2^5(9)$, $2^4 3^1(4)$, $2^6(1)$, $2^6 3^1(1)$, $2^5 3^1(4)$	48	35	5	1(5)	$\mathbb{Z} \oplus \mathbb{Z}_2^4$
7	258	$2^2(147)$, $2^3(69)$, $2^2 3^1(1)$, $2^4(18)$, $2^3 3^1(2)$, $2^5(10)$, $2^4 3^1(3)$, $2^6(2)$, $2^5 3^1(2)$, $2^7(3)$, $2^{11}3^2(1)$	189	682	42	1(38), 2(4)	$\mathbb{Z}^2 \oplus \mathbb{Z}_2^3$
8	501	$2^2(397)$, $2^3(67)$, $2^2 3^1(4)$, $2^4(18)$, $2^3 3^1(4)$, $2^5(5)$, $2^4 3^1(1)$, $2^6(3)$, $2^7(1)$, $2^5 3^2(1)$	435	2972	145	1(142), 2(3)	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$
9	704	$2^2(603)$, $2^3(58)$, $2^2 3^1(6)$, $2^4(15)$, $2^2 5^1(1)$, $2^3 3^1(7)$, $2^5(6)$, $2^4 3^1(2)$, $2^6(1)$, $2^3 3^2(1)$, $2^7(1)$, $2^5 3^1 5^1(1)$, $2^8 3^2(1)$, $2^9 3^1(1)$	639	5928	290	1(287), 2(2), 4(1)	$\mathbb{Z} \oplus \mathbb{Z}_2^4 \oplus \mathbb{Z}_4$
10	628	$2^2(571)$, $2^3(31)$, $2^2 3^1(4)$, $2^4(13)$, $2^3 3^1(3)$, $2^5(1)$, $2^6(2)$, $2^3 3^2(1)$, $2^7(1)$, $2^8(1)$	597	6701	348	1(343), 2(4), 4(1)	$\mathbb{Z} \oplus \mathbb{Z}_2^7 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{24}$
11	369	$2^2(320)$, $2^3(25)$, $2^2 3^1(4)$, $2^4(12)$, $2^3 3^1(4)$, $2^5(3)$, $2^6 3^2(1)$	346	4544	248	1(237), 2(7), 4(2), 8(1), 24(1)	0
12	130	$2^2(103)$, $2^3(9)$, $2^2 3^1(8)$, $2^4(3)$, $2^3 3^1(1)$, $2^5(2)$, $2^2 3^2(1)$, $2^4 3^1(1)$, $2^6(1)$, $2^5 3^1(1)$	120	1787	98	1(98)	$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_8$
13	31	$2^2(13)$, $2^6 3^1(1)$, $2^3(7)$, $2^2 3^1(4)$, $2^4(1)$, $2^3 3^1(2)$, $2^6(1)$, $2^5 3^2(1)$, $2^9(1)$	22	337	21	1(19), 2(1), 8(1)	$\mathbb{Z}_5$
14	7	$2^3 3^1(2)$, $2^4(2)$, $2^5 3^1 5^1(1)$, $2^2 5^1(1)$, $2^7 3^1(1)$	2	3	1	5(1)	$\mathbb{Z}_{128}$
15	2	$2^{10}3^2 5^1(1)$, $2^{11}3^1(1)$	2	2	1	128(1)	$\mathbb{Z}$

References

[1] A. Ash, *Deformation retracts with lowest possible dimension of arithmetic quotients of self-adjoint homogeneous cones*, Math. Ann. **225** (1977), no. 1, 69–76.

[2] A. Ash, *Small-dimensional classifying spaces for arithmetic subgroups of general linear groups*, Duke Math. J. **51** (1984), no. 2, 459–468, URL: http://dx.doi.org/10.1215/S0012-7094-84-05123-8, doi:10.1215/S0012-7094-84-05123-8.

[3] A. Ash, P. E. Gunnells, and M. McConnell, *Torsion in the cohomology of congruence subgroups of* $\mathrm{SL}(4,\mathbb{Z})$ *and Galois representations*, J. Algebra **325** (2011), 404–415, URL: http://dx.doi.org/10.1016/j.jalgebra.2010.07.024, doi:10.1016/j.jalgebra.2010.07.024.

[4] A. Ash, D. Mumford, M. Rapoport, and Y.-S. Tai, *Smooth compactifications of locally symmetric varieties*, second ed., Cambridge Mathematical Library, Cambridge University Press, Cambridge, 2010, With the collaboration of Peter Scholze, URL: http://dx.doi.org/10.1017/CBO9780511674693, doi:10.1017/CBO9780511674693.

[5] E. Berkove, *The mod-2 cohomology of the Bianchi groups*, Trans. Amer. Math. Soc. **352** (2000), no. 10, 4585–4602, URL: http://dx.doi.org/10.1090/S0002-9947-00-02505-8, doi:10.1090/S0002-9947-00-02505-8.

[6] A. Borel and J.-P. Serre, *Corners and arithmetic groups*, Comment. Math. Helv. **48** (1973), 436–491, Avec un appendice: Arrondissement des variétés à coins, par A. Douady et L. Hérault.

[7] A. Borel, *Some metric properties of arithmetic quotients of symmetric spaces and an extension theorem*, J. Differential Geometry **6** (1972), 543–560, Collection of articles dedicated to S. S. Chern and D. C. Spencer on their sixtieth birthdays.

[8] O. Braun and R. Coulangeon, *Perfect lattices over imaginary quadratic number fields*, preprint at `arxiv:1304.0559`, April 2013.

[9] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, vol. 87, Springer-Verlag, New York, 1994, Corrected reprint of the 1982 original.

[10] J. E. Cremona, *Hyperbolic tessellations, modular symbols, and elliptic curves over complex quadratic fields*, Compositio Math. **51** (1984), no. 3, 275–324, URL: http://www.numdam.org/item?id=CM_1984__51_3_275_0.

[11] M. Dutour Sikirić, H. Gangl, P. E. Gunnells, J. Hanke, A. Schürmann, and D. Yasaki, *On* K_4 *of the Gaussian and Eisenstein integers*, in preparation.

[12] M. Dutour Sikirić, H. Gangl, P. E. Gunnells, J. Hanke, A. Schürmann, and D. Yasaki, *The Voronoi cone method for homology computation*, in preparation.

[13] P. Elbaz-Vincent, H. Gangl, and C. Soulé, *Quelques calculs de la cohomologie de* $\mathrm{GL}_N(\mathbb{Z})$ *et de la* K*-théorie de* $\mathbb{Z}$, C. R. Math. Acad. Sci. Paris **335** (2002), no. 4, 321–324, URL: http://dx.doi.org/10.1016/S1631-073X(02)02481-0, doi:10.1016/S1631-073X(02)02481-0.

[14] P. Elbaz-Vincent, H. Gangl, and C. Soulé, *Perfect forms,* K*-theory and the cohomology of modular groups*, preprint at `arxiv:1001.0789`, 2010.

[15] P. E. Gunnells and D. Yasaki, *Modular forms and elliptic curves over the cubic field of discriminant* -23, preprint at `arXiv:1201.4132`, 2012.

[16] G. Harder, *A Gauss-Bonnet formula for discrete arithmetically defined groups*, Ann. Sci. École Norm. Sup. (4) **4** (1971), 409–455.

[17] A. Hatcher, *Algebraic topology*, Cambridge University Press, Cambridge, 2002.

[18] M. Koecher, *Beiträge zu einer Reduktionstheorie in Positivitätsbereichen. I*, Math. Ann. **141** (1960), 384–432.

[19] M. McConnell, *Classical projective geometry and arithmetic groups*, Math. Ann. **290** (1991), no. 3, 441–462, URL: http://dx.doi.org/10.1007/BF01459253, doi:10.1007/BF01459253.

[20] K. Okuda and S. Yano, *A generalization of Voronoï's theorem to algebraic lattices*, J. Théor. Nombres Bordeaux **22** (2010), no. 3, 727–740, URL: http://jtnb.cedram.org/item?id=JTNB_2010__22_3_727_0.

[21] A. D. Rahm, *The homological torsion of* PSL_2 *of the imaginary quadratic integers*, Trans. Amer. Math. Soc. **365** (2013), no. 3, 1603–1635, URL: http://dx.doi.org/10.1090/S0002-9947-2012-05690-X, doi:10.1090/S0002-9947-2012-05690-X.

[22] A. D. Rahm and M. Fuchs, *The integral homology of* PSL_2 *of imaginary quadratic integers with nontrivial class group*, J. Pure Appl. Algebra **215** (2011), no. 6, 1443–1472, URL: http://dx.doi.org/10.1016/j.jpaa.2010.09.005, doi:10.1016/j.jpaa.2010.09.005.

[23] J. Schwermer and K. Vogtmann, *The integral homology of* SL_2 *and* PSL_2 *of Euclidean imaginary quadratic integers*, Comment. Math. Helv. **58** (1983), no. 4, 573–598, URL: http://dx.doi.org/10.1007/BF02564653, doi:10.1007/BF02564653.

[24] M. H. Şengün, *On the integral cohomology of Bianchi groups*, Exp. Math. **20** (2011), no. 4, 487–505, URL: http://dx.doi.org/10.1080/10586458.2011.594671, doi:10.1080/10586458.2011.594671.

[25] J.-P. Serre, *Groupes d'homotopie et classes de groupes abéliens*, Ann. of Math. (2) **58** (1953), 258–294.

[26] C. Soulé, *Addendum to the article: "On the torsion in* $K_*(\mathbf{Z})$*" (Duke Math. J.* **45** *(1978), no. 1, 101–129) by R. Lee and R. H. Szczarba*, Duke Math. J. **45** (1978), no. 1, 131–132.

[27] C. Soulé, *On the* 3*-torsion in* $K_4(\mathbf{Z})$, Topology **39** (2000), no. 2, 259–265, URL: http://dx.doi.org/10.1016/S0040-9383(99)00006-3, doi:10.1016/S0040-9383(99)00006-3.

[28] R. E. Staffeldt, *Reduction theory and* K_3 *of the Gaussian integers*, Duke Math. J. **46** (1979), no. 4, 773–798, URL: http://projecteuclid.org/getRecord?id=euclid.dmj/1077313722.

[29] W. Stein, *Modular forms, a computational approach*, Graduate Studies in Mathematics, vol. 79, American Mathematical Society, Providence, RI, 2007, With an appendix by Paul E. Gunnells.

[30] K. Vogtmann, *Rational homology of Bianchi groups*, Math. Ann. **272** (1985), no. 3, 399–419, URL: http://dx.doi.org/10.1007/BF01455567, doi:10.1007/BF01455567.

[31] G. Voronoi, *Nouvelles applications des paramètres continues à la théorie des formes quadratiques 1: Sur quelques propriétés des formes quadratiques positives parfaites*, J. Reine Angew. Math **133** (1908), no. 1, 97–178.

[32] D. Yasaki, *Hyperbolic tessellations associated to Bianchi groups*, Algorithmic number theory, Lecture Notes in Comput. Sci., vol. 6197, Springer, Berlin, 2010, pp. 385–396.

H. Gangl, Department of Mathematical Sciences, South Road, Durham DH1 3LE, United Kingdom

E-mail address: herbert.gangl@durham.ac.uk

URL: http://maths.dur.ac.uk/~dma0hg/

P. E. Gunnells, Department of Mathematics and Statistics, LGRT 1115L, University of Massachusetts, Amherst, MA 01003, USA

E-mail address: gunnells@math.umass.edu

URL: https://www.math.umass.edu/~gunnells/

J. Hanke, P.O. Box 1271, Princeton, NJ 08542, USA

E-mail address: jonhanke@gmail.com

URL: http://www.wordpress.jonhanke.com

A. Schürmann, Universität Rostock, Institute of Mathematics, 18051 Rostock, Germany

E-mail address: achill.schuermann@uni-rostock.de

URL: http://www.geometrie.uni-rostock.de/

M. D. Sikirić, Rudjer Bosković Institute, Bijenicka 54, 10000 Zagreb, Croatia

E-mail address: mathieu.dutour@gmail.com

URL: http://drobilica.irb.hr/~mathieu/

D. Yasaki, Department of Mathematics and Statistics, University of North Carolina at Greensboro, Greensboro, NC 27412, USA

E-mail address: d_yasaki@uncg.edu

URL: http://www.uncg.edu/~d_yasaki/