HENRY JACOB MATHEMATICS COMPETITION

SAMPLE PROBLEMS FROM PREVIOUS COMPETITIONS

No calculators are allowed!

(1) (a) If $f(x) = x^2 e^x$, find a formula for the nth derivative of f.
(b) Prove that your formula is correct.

(2) What is the smallest possible area of a triangle whose sides are formed by the positive x and y axes and a line through the point $(1, 2)$?

(3) Prove that for any positive integer n, the expression $1110^n + 1102^n - 200^n - 10^n$ is divisible by 2002.

(4) The $(5, 12, 13)$ right triangle has area and perimeter both equal to 30. Find all right triangles T with integral sides such that the area of T equals the perimeter of T.

(5) Find
$$\int_{-1}^{1} \frac{e^{1/x}\, dx}{x^2(1 + e^{1/x})}.$$

(6) Let $R(t)$ be the area of the region bounded by the y-axis, a positive continuous function $f(x)$, a negative continuous function $g(x)$, and the line $x = t^4$. Compute $R'(2)$.

(7) A function is *odd* if $f(-x) = -f(x)$. Prove that $f(x) = \ln(x + \sqrt{x^2 + 1})$ is odd.

(8) Show that there are infinitely many squares that are the sum of a square and a prime.

(9) Let $g(x)$ be the greatest integer less than or equal to x. For example $g(\pi) = 3$ and $g(-\pi) = -4$. Sketch the graphs of the following:
(a) $y = g(x)$
(b) $g(y) = g(x)$
(c) $g(y) = |g(x)|$

(10) Find a polynomial P with $P(1) = 1$, $P(2) = 2$, $P(3) = 2003$.

(11) An "hourglass" is formed by rotating the graph of $y = e^x$ about the line $y = x$. Find the largest tangent sphere that can be passed through the neck of the hourglass.

(12) (a) Show that $x/y + y/z + z/x = 1$ has no solution in positive integers.
(b) Find a solution to $x/y + y/z + z/x = 5$ in positive integers.