DISCLAIMER: This practice exam is intended to give you an idea about what a two-hour final exam is like. It is not possible for any practice exam to cover every topic, and the content, coverage and format of your actual exam could be different from this practice exam.

Part I: Multiple Choice Problems. You only need to give the answer; no justification needed.

1. Determine which of the following vector fields is conservative.

(a) $\langle -y, x, 0\rangle$ (b) $\langle xy, yz, zx\rangle$ (c) $\langle 2xy, x^2+2yz, y^2\rangle$

(d) $\langle y^2, z^2, x^2\rangle$ (e) $\langle x^2y, y^2z, z^2x\rangle$ (f) $\langle x, x, x\rangle$

ANS: c

SOLUTION: Note that we may disqualify any vector field with nonzero curl. In particular, we see that $\text{curl}\langle -y, x, 0\rangle = \langle 0,0,2\rangle$, $\text{curl}\langle xy, yz, zx\rangle = \langle -y, -z, -x\rangle$, $\text{curl}\langle y^2, z^2, x^2\rangle = \langle -2z, -2x, -2y\rangle$, $\text{curl}\langle x^2y, y^2z, z^2x\rangle = \langle -y^2, -z^2, -x^2\rangle$ and $\text{curl}\langle x, x, x\rangle = \langle 0, -1, 1\rangle$. On the other hand, one can check that $\text{curl}\langle 2xy, x^2+2yz, y^2\rangle = \vec{0}$ and find that $f(x,y,z) = x^2y + y^2z + C$ is a potential function.

2. Let $z = x^2y$ and let x, y be functions of t with $x(1) = 1$, $y(1) = 2$, $x(2) = 3$, $y(2) = 4$, $x'(1) = A$, $y'(1) = B$, $x'(2) = C$, $y'(2) = D$. Find $\frac{dz}{dt}$ when $t = 1$.

(a) 4A+D (b) 4A+B (c) 4C+D (d) A+2D (e) 4C+2D (f) A+4B

ANS: b

SOLUTION: Here we invoke the chain rule:

$$\frac{dz}{dt} = \frac{dz}{dx}\frac{dx}{dt} + \frac{dz}{dy}\frac{dy}{dt}$$

Now $z_x = 2xy$ and $z_y = x^2$, in particular at $t = 1$ we have $z_x(1) = 2(1)(2) = 4$ and $z_y(1) = (1)^2 = 1$. So now $\frac{dz}{dt}(1) = 4A + B$. The other given information is extraneous.

3. Find the minimum speed of the particle whose position function is $\vec{r}(t) = \langle t^2, 1-2t, t\rangle$.

(a) 0 (b) 1 (c) $\sqrt{2}$ (d) $\sqrt{3}$ (e) $\sqrt{5}$ (f) 6

ANS: e

SOLUTION: The speed (at time t) is given by the quantity $|\vec{r'}(t)|$. Since $\vec{r'}(t) = \langle 2t, -2, 1\rangle$, we have $|\vec{r'}(t)| = \sqrt{5+4t^2}$. This function's only critical point at $t = 0$ corresponds to a local (in fact, global) minimum with value $\sqrt{5}$.

4. Find the value of the gradient vector field of the function $z = x^2y^3$ at the point $(1,1)$.

(a) $\langle 2, -3\rangle$ (b) $\langle 2, 3\rangle$ (c) $\frac{1}{\sqrt{11}}\langle 3, -2\rangle$

(d) $\langle 2, -3, 1\rangle$ (e) $\frac{1}{\sqrt{11}}\langle 2, 3\rangle$ (f) $\frac{1}{\sqrt{14}}\langle 3, -2, 1\rangle$

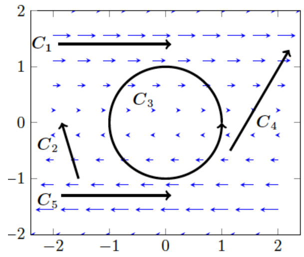
ANS: b

SOLUTION: $\nabla z = \langle 2xy^3, 3x^2y^2\rangle$ and so $\nabla z(1,1) = \langle 2(1)(1)^3, 3(1)^2(1)^2\rangle = \langle 2,3\rangle$.

---

5. The figure depicts a 2-dimensional vector field $\vec{F}$ as well as five oriented paths. Place in order, from smallest to largest, the five values.

$$A = \int_{C_1} \vec{F}\cdot d\vec{r} \qquad B = \int_{C_2} \vec{F}\cdot d\vec{r} \qquad C = \int_{C_3} \vec{F}\cdot d\vec{r} \qquad D = \int_{C_4} \vec{F}\cdot d\vec{r} \qquad E = \int_{C_5} \vec{F}\cdot d\vec{r}$$



smallest ☐ ☐ ☐ ☐ ☐ largest

ANS: E < C < B < D < A

SOLUTION: Take note of the antisymmetry across the x-axis in this vector field. Observe that $C = 0$ since the value integrating along the top half of the circle will be the negative of the value integrating along the bottom half. Next note that $E < 0$, since the entirety of the associated curve is in the negative tangent direction to $\vec{F}$. B and D are both positive since their curves (predominantly) form acute angles with $\vec{F}$ along their lengths, however B should be smaller as its curve's steeper slope implies a smaller tangential (to $\vec{F}$) component. Lastly, A is entirely in the tangent direction to $\vec{F}$ and the clear winner in terms of magnitude.

---

6. Find a non-zero vector perpendicular to the plane that passes through the points $(1,0,0)$, $(5,4,0)$ and $(0,4,1)$.

(a) $\langle 1,-4,-5\rangle$ (b) $\langle 1,-4,5\rangle$ (c) $\langle -5,1,4\rangle$

(d) $\langle 4,-4,20\rangle$ (e) $\langle 4,1,5\rangle$ (f) $\langle 4,4,20\rangle$

ANS: d

SOLUTION: This is no different than asking for a normal vector to the specified plane. The difference vectors $\langle 5-1, 4-0, 0-0\rangle = \langle 4,4,0\rangle$ and $\langle 5-0, 4-4, 0-1\rangle = \langle 5,0,-1\rangle$ lie on the plane, so then $\langle 5,0,-1\rangle \times \langle 4,4,0\rangle = \langle 4,-4,20\rangle$ is a normal vector to the plane.

PART II: Written Problems. To earn full credit for the following problems you must show your work. You can leave answers in terms of fractions and square roots.

---

1. Find an equation for the plane consisting of all points that are equal distance from the two points $(1, 1, 0)$ and $(0, 1, 1)$

ANS: $\mathbf{z = x}$

SOLUTION: First, we can consider the line segment between these points given by $(t, 1, 1-t)$ for $0 \le t \le 1$. The midpoint at $t = \frac{1}{2}$ given by $(\frac{1}{2}, 1, \frac{1}{2})$ must be on the desired plane by definition. Furthermore, the direction vector $\langle 1, 0, -1\rangle$ of this line segment must be orthogonal to the plane, otherwise there would be points on the plane closer to $(1, 1, 0)$ than $(0, 1, 1)$ and vice versa. (Think projection vectors) This is all the data we need to have the equation of the plane: $(1)(x - \frac{1}{2}) + (0)(y - 1) + (-1)(z - \frac{1}{2}) = 0$, which simplifies to $x - z = 0$.

---

2. Find the maximum and minimum values of the function $f(x, y) = x^2 - y^2$ subject to the constraint $x^2 + y^2 \le 1$.

ANS: Max is 1, min is -1

SOLUTION: The extreme values of f on a closed region must occur either at a critical point within the region or on the boundary of the region. So, we first look at critical points: $\nabla f = \langle 2x, -2y\rangle$ which is the zero vector only at $(0, 0)$. This point is in the region since $(0) + (0) \le 1$, and we have $f(0, 0) = (0)^2 - (0)^2 = 0$. Now we look at the boundary $x^2 + y^2 = 1$. We can optimize f with input restricted to this circle by noting the optimizing inputs must satisfy $\langle 2x, -2y\rangle = \lambda\langle 2x, 2y\rangle$ for some constant λ. The first component of this equality tells us either $x = 0$ or $\lambda = 1$. In the former case, we have $0 + y^2 = 1 \implies y = \pm 1$ giving the candidate points $(0, 1)$ and $(0, -1)$; now $f(0, \pm 1) = 0 - (\pm 1)^2 = -1$. On the other hand, if $\lambda = 1$, then the second component of the equality above forces $y = 0$. So we get two more candidates at $(1, 0)$ and $(-1, 0)$, and $f(\pm 1, 0) = (\pm 1)^2 - 0 = 1$. Comparing all the candidates, we find that the minimum is -1 at $(0, \pm 1)$ and maximum is 1 at $(\pm 1, 0)$.

---

3. Consider the 2-dimensional vector field $\vec{F}(x, y) = e^y \sin x\vec{i} + e^y \cos x\vec{j}$.
   (a) Is this vector field conservative?
   (b) Compute the work done by $\vec{F}$ along the line segment from $(0, 0)$ to $(\pi, 1)$.

(a) Is this vector field conservative? **ANS: No**

SOLUTION: Let us see if we can get a potential function. $\int e^y \sin(x)\, dx = -e^y \cos(x) + f(y)$ whereas $\int e^y \cos(x)\, dy = e^y \cos(x) + g(x)$. Since $-e^y \cos(x)$ is a term in the first antiderivative but not the second and it depends on *both* x and y, it cannot be part of $f(y)$ and so no potential function for $\vec{F}$ can exist.

(b) Compute the work done... **ANS: $\mathbf{(e + 1)(\pi^2 - 1)(\pi^2 + 1)^{-1}}$ Joules**

SOLUTION: We parametrize the line segment by $\vec{r}(t) = \langle \pi t, t\rangle$ for $0 \le t \le 1$. Then $\vec{r'}(t) = \langle \pi, 1\rangle$ and we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_0^1 \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt = \int_0^1 \langle e^t \sin(\pi t), e^t \cos(\pi t)\rangle \cdot \langle \pi, 1\rangle\, dt$$
$$= \int_0^1 e^t(\pi \sin(\pi t) + \cos(\pi t))\, dt.$$

This is actually a very nonobvious integral, but here follows the usual argument. Integration by parts tells us $\int v\ du = uv - \int u\ dv$. Set $du = e^t\ dt$ and $v = \pi\sin(\pi t) + \cos(\pi t)$. Then $u = e^t$ and $dv = \pi(\pi\cos(\pi t) - \sin(\pi t))\ dt$, so that the integral above becomes

$$\left[e^t(\pi\sin(\pi t) + \cos(\pi t))\right]_0^1 - \pi\int_0^1 e^t(\pi\cos(\pi t) - \sin(\pi t))\ dt$$

and the bracketed portion on the left evaluates as $-(e+1)$. For the remaining integral, we apply the same integration by parts trick to get

$$-(e+1) - \pi\left(\left[e^t(\pi\cos(\pi t) - \sin(\pi t)\right]_0^1 + \pi\int_0^1 e^t(\pi\sin(\pi t) + \cos(\pi t))\ dt\right)$$

The bracketed part here evaluates as $-\pi(e+1)$. Since this expression is equal to the original integral, we can rearrange algebraically to get

$$(1+\pi^2)\int_0^1 e^t(\pi\sin(\pi t) + \cos(\pi t))\ dt = \pi^2(e+1) - (e+1)$$
$$\int_0^1 e^t(\pi\sin(\pi t) + \cos(\pi t))\ dt = (e+1)\left(\frac{\pi^2-1}{\pi^2+1}\right)$$

---

4. Determine the flux of the vector field $\vec{F} = ye^{z^2}\vec{i} + y^2\vec{j} + e^{xy}\vec{k}$ across the cylinder S defined by $x^2+y^2 = 9$ and $0 \le z \le 4$, with positive orientation.

ANS: 0

SOLUTION: Our surface here is closed and positively oriented, so the standard form of divergence theorem applies. (Below, E is the *solid* cylinder associated to S.)

$$\iint_S \vec{F}\cdot dS = \iiint_E \operatorname{div}(\vec{F})\ dV$$
$$= \iiint_E 2y\ dV$$

The natural way to complete this computation is to use cylindrical coordinates:

$$\iiint_E 2y\ dV = 2\int_0^{2\pi}\int_0^3\int_0^4 r\sin(\theta)\ r dz dr d\theta$$
$$= 2\left(\int_0^{2\pi}\sin(\theta)\ d\theta\right)\left(\int_0^3 r^2\ dr\right)\left(\int_0^4\ dz\right) = 0$$

since the integral with respect to θ is 0.

---

5. Evaluate $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F} = \langle x^2z, xy^2, z^2\rangle$ and C is the curve of intersection of the plane $x+y+z=1$ with the cylinder $x^2+y^2=9$, oriented counterclockwise as viewed from above.

ANS: $\frac{81\pi}{2}$

SOLUTION: Observe that C is also a boundary curve for the part of the plane $z = 1-x-y$ contained within the cylinder $x^2+y^2 = 9$. Furthermore, $\operatorname{curl}(\vec{F}) = \langle 0, x^2, y^2\rangle$ is a simpler vector field to work with. This suggests we should apply Stokes' theorem. We can parametrize the surface of the part of the plane precisely as $\vec{s}(x, y) = \langle x, y, 1 - x - y\rangle$ and then calculate

$$\begin{aligned} \vec{s}_x &= \langle 1, 0, -1\rangle \\ \vec{s}_y &= \langle 0, 1, -1\rangle \\ \vec{s}_x \times \vec{s}_y &= \langle 1, 1, 1\rangle \end{aligned}$$

and verify visually that the direction of this normal to the surface is in agreement (by right hand rule) with the counterclockwise orientation of the boundary curve. So:

$$\begin{aligned} \int_C \vec{F} \cdot d\vec{r} &= \iint_S \text{curl}(\vec{F}) \cdot d\vec{s} \\ &= \iint_R \langle 0, x^2, y^2\rangle \cdot \langle 1, 1, 1\rangle \, dA \\ &= \iint_R x^2 + y^2 \, dA \end{aligned}$$

To complete the calculation, observe that the region of integration in parameter space is just a disk of radius 3. So it is natural to use polar coordinates here.

$$\begin{aligned} \iint_R x^2 + y^2 \, dA &= \int_0^{2\pi} \int_0^3 (r^2) \, r dr d\theta \\ &= 2\pi \int_0^3 r^3 \, dr \\ &= 2\pi \left[\frac{1}{4} r^4\right]_0^3 = \frac{81\pi}{2} \end{aligned}$$

In theory you could avoid Stokes' theorem and calculate the line integral along the intersection curve C directly using θ as a parameter, but you'll be giving yourself a lot more calculations.