Solution for Math 233, Practice Exam #2
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1. (20 points) For each question, please select the best response. Please clearly indicate your choice; ambiguous answers will not receive credit. In this problem, there is *no partial credit* awarded and it is *not necessary to show your work.*

(a) (4 points) The area of the region enclosed by the polar graph $r = 2\sin\theta$ is **Solution:** (v) By plotting a few points, we get the complete graph by letting θ go from 0 to π. So the area of this polar curve is $\int_0^\pi \int_0^{2\sin\theta} r dr\, d\theta = \int_0^\pi \frac{r^2}{2}\Big|_0^{2\sin\theta} d\theta = 2\int_0^\pi \sin^2\theta d\theta$
$= 2\int_0^\pi \frac{1-\cos 2\theta}{2}\, d\theta = \pi - \int_0^\pi \cos(2\theta) d\theta = \pi$

Alternatively: If you know that it's a circle then you can write down the answer immediately!

(i) $1/\sqrt{2}$ (ii) $\pi/\sqrt{2}$ (iii) 1
(iv) $\pi/2$ (v) π (vi) 2π

(b) (4 points) Evaluate the iterated integral $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy\, dx$

Solution: (iii) The domain of integration in polar coordinates is: $D = \{(r,\theta)|0 \le r \le 1, 0 \le \theta \le \frac{\pi}{2}\}$. In polar coordinates, $r = \sqrt{x^2+y^2}$, $dA = dydx = rdrd\theta$, so $\displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{x^2+y^2}\, dy\, dx = \int_0^{\pi/2}\int_0^1 r^2 dr d\theta = \int_0^{\pi/2} \left[\frac{r^3}{3}\right]_0^1 d\theta = \frac{1}{3}\int_0^{\pi/2} d\theta = \frac{\theta}{3}\Big|_0^{\pi/2} = \frac{\pi}{6}$

(i) $\frac{\pi}{2}$ (ii) $\frac{\sqrt{3}}{2}$ (iii) $\frac{\pi}{6}$
(iv) $\frac{5}{2}$ (v) $\frac{1}{2}$ (vi) $\frac{2}{\sqrt{\pi}}$

(c) (4 points) Find the critical points of the function:

$$f(x,y) = \frac{2}{3}x^3 + \frac{1}{3}y^3 - xy$$

Solution: (vi) Calculate $\nabla f(x,y) = \langle 2x^2 - y, y^2 - x\rangle = \langle 0,0\rangle$. Therefore, $2x^2 - y = 0 \Leftrightarrow y = 2x^2$, and setting to $y^2 - x = 0$ gives $4x^4 - x = x(4x^3-1) = 0 \Rightarrow x = 0$ or $x = 4^{-\frac{1}{3}}$. Substituting to $y = 2x^2$ gives a set of two critical points: $(0,0), (4^{-\frac{1}{3}}, 2\cdot 4^{-\frac{2}{3}})$.

(i) $(0,0),(1,0)$ (ii) $(1,2),(4^{\frac{1}{3}}, 2\cdot 4^{\frac{2}{3}})$ (iii) $(1,1),(2,2)$
(iv) $(0,1),(4^{-\frac{1}{3}}, 2\cdot 4^{-\frac{3}{2}})$ (v) $(1,1),(\sqrt{2},\sqrt[3]{2})$ (vi) $(0,0),(4^{-\frac{1}{3}}, 2\cdot 4^{-\frac{2}{3}})$

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*Continuation of **1.***

(d) (4 points) Set up the double integral $\iint_R f(x,y)\,dA$ over the shaded region R shown in Figure 1 in the order $dy\,dx$. (The region is bounded by $x = 1$, $y = 1 - x^2$, and $y = e^x$).

(i)	$\int_0^1\int_{x^2}^{\ln x} f(x,y)\,dy\,dx$	(ii)	$\int_0^1\int_{1-x^2}^{\ln x} f(x,y)\,dy\,dx$
(iii)	$\int_1^0\int_{1-x^2}^{e^x} f(x,y)\,dy\,dx$	(iv)	$\int_1^0\int_{e^x}^{1-x^2} f(x,y)\,dy\,dx$
(v)	$\boxed{\int_0^1\int_{1-x^2}^{e^x} f(x,y)\,dy\,dx}$	(vi)	$\int_0^1\int_{e^x}^{1-x^2} f(x,y)\,dy\,dx$

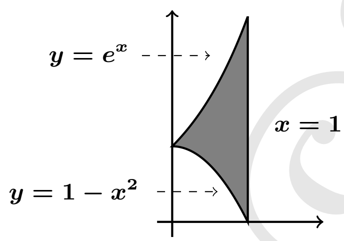

Figure 1: The region R from 1(d)

(e) (4 points) Let E be the solid region bounded by the paraboloid $z = 2 + x^2 + y^2$, the cylinder $x^2 + y^2 = 1$, and the xy-plane (Figure 2). In cylindrical coordinates, when written as an iterated integral the triple integral $\iiint_E e^z\,dV$ becomes

(i)	$\int_0^{2\pi}\int_0^1\int_0^{2+r^2} re^z\,dr\,d\theta\,dz$	(ii)	$\int_0^{2\pi}\int_0^1\int_0^{2+r^2} re^z\,d\theta\,dz\,dr$
(iii)	$\boxed{\int_0^{2\pi}\int_0^1\int_0^{2+r^2} re^z\,dz\,dr\,d\theta}$	(iv)	$\int_0^{2\pi}\int_0^2\int_0^{2+r^2} r^2e^z\,dr\,d\theta\,dz$
(v)	$\int_0^{2\pi}\int_0^2\int_0^{2+r^2} r^2e^z\,d\theta\,dz\,dr$	(vi)	$\int_0^{2\pi}\int_0^2\int_0^{2+r^2} r^2e^z\,dz\,dr\,d\theta$

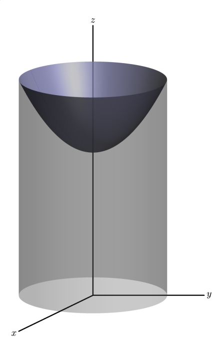

Figure 2: The region E from 1(e)

2. (15 points) Let $f(x, y) = x + y$ and let E be the ellipse

$$x^2 + \frac{y^2}{8} = 1.$$

Find the minimum and maximum value of f on E.
Solution: We want to maximize/minimize $f(x, y) = x + y$ subject to the contraint $g(x, y) = 1$ where $g(x, y) = x^2 + y^2/8$. By Lagrange multiplier, we have $\nabla f = \lambda \nabla g$ for some λ. In terms of coordinates, that means $\langle 1, 1\rangle = \lambda\langle 2x, y/4\rangle$.

Note that this implies in particular $\lambda \neq 0$, whence $y = 8x$. Substitute this back to the constraint $g(x, y) = 1$, we get $9x^2 = 1$, whence $x = \pm 1/3$.

By $y = 2x$ we find that $(x, y) = (1/3, 8/3)$ or $(-1/3, -8/3)$.
Therefore the maximum and minimum values of f, are respectively,
$\boxed{3 \text{ and } -3}$.

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3. (15 points) Let R be the triangular region in the xy-plane with vertices $(0, 0)$, $(1, 1)$, and $(1, 0)$. Find the volume over R and under the paraboloid $z = 2 - x^2 - y^2$.
Solution: The volume of this solid is given by

$$\int_0^1 \int_0^x (2 - x^2 - y^2)dydx$$
$$= \int_0^1 (2y - x^2y - y^3/3)\Big|_0^x dx$$
$$= \int_0^1 (2x - 4x^3/3)dx$$
$$= (x^2 - x^4/3)\Big|_0^1$$
$$= \boxed{2/3}$$

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4. (15 points) Find the surface area of the part of the graph of $z = 3 + 2y + x^4/4$ that lies over the region R in the xy-plane bounded by $y = x^5$, $x = 1$, and the x-axis.
Solution: The surface area is given by

$$\int_0^1 \int_0^{x^5} \sqrt{1 + (z_x)^2 + (z_y)^2}\ dy\ dx = \int_0^1 \int_0^{x^5} \sqrt{1 + (x^3)^2 + (2)^2}\ dy\ dx$$
$$\int_0^1 \int_0^{x^5} \sqrt{5 + x^6}\ dy\ dx = \int_0^1 x^5\sqrt{5 + x^6}\ dx =$$
$$= \int_5^6 \frac{1}{6}\sqrt{u}\ du = \boxed{\frac{6^{3/2} - 5^{3/2}}{9}}$$

5. (15 points) Let E be the solid region bounded by the unit sphere $x^2+y^2+z^2=1$ and inside the cone $z=\sqrt{x^2+y^2}$. Evaluate $\iiint_E z\,dV$.

Solution: First we compute the intersection of the sphere and the cone:

$$1-(x^2+y^2)=z^2=x^2+y^2$$

so the intersection is $x^2+y^2=1/2, z=\sqrt{1/2}$.

The spherical top suggests to compute this triple integral using *spherical coordinates*.

Since the solid is symmetric around the z-axis, θ goes from 0 to 2π.

To determine the range of ϕ, consider the xz-cross section of the solid, where ϕ goes from 0 to $\pi/4$. Putting everything together, the triple integral in spherical coordinates is $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^1(\rho\cos\phi)(\rho^2\sin\phi)\ d\rho\ d\phi\ d\theta$

$\displaystyle=\int_0^{2\pi}\int_0^{\pi/4}\frac{1}{4}\cos\phi\ \sin\phi\ d\phi\ d\theta$

$\displaystyle=\int_0^{2\pi}\frac{1}{16}\ d\theta=\boxed{\frac{\pi}{8}}$

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6. (20 points) Find the x and y coordinates of all critical points of the function

$$f(x,y)=2x^3-6x^2+xy^2+y^2$$

and use the Second Derivatives Test to classify them as local minima, local maxima or saddle points.

Solution: Setting $\nabla f(x,y)=\langle 0,0\rangle$ gives:

$\nabla f(x,y)=\langle 6x^2-12x+y^2, 2xy+2y\rangle=\langle 0,0\rangle\Rightarrow$

$2y(x+1)=0\Rightarrow y=0$ or $x=-1$.

For $y=0$, from $6x^2-12x+y^2=0$, $6x^2-12x=6x(x-2)=0\Rightarrow x=0$ or $x=2$.

For $x=-1$, from $6x^2-12x+y^2=0$, $6+12+y^2=0$ which is impossible.

Therefore the critical points are $(0,0),(2,0)$.

Using the Second Derivatives Test to check the points:

$D=\begin{vmatrix} f_{xx} & f_{xy}\\ f_{yx} & f_{yy}\end{vmatrix}=\begin{vmatrix} 12x-12 & 2y\\ 2y & 2x+2\end{vmatrix}=24x^2-4y^2-24.$

$D(0,0)=-24<0$, so $\boxed{(0,0)}$ is a $\underline{\text{saddle point}}$.

$D(2,0)=96-24=72>0$ and $f_{xx}(2,0)=12>0$, so $\boxed{(2,0)}$ is a $\underline{\text{local minimum}}$.