Math 233 **Practice Exam 1 - Answers**

Instructions

- **Turn off all cell phones!** Put away all electronic devices such as iPods, iPads, laptops, etc.
- There are six (6) questions.
- Do all work in this exam booklet. You may continue work to the backs of pages and the blank page at the end, but if you do so indicate clearly where your work is for the grader.
- Calculators are **not** allowed, nor are formula sheets or any other external materials.
- Organize your work in an unambiguous order. Show all necessary steps.
- **Unless indicated otherwise, you must show work to obtain credit for your answers.**
- Be ready to show your UMass ID card when you hand in your exam booklet.

1. (20 points) For each question, please select the best response. Please clearly indicate your choice; ambiguous answers will not receive credit. In this problem, there is *no partial credit* awarded and it is *not necessary to show your work*.

(a) (4 points) Find the area of the triangle with vertices $(2,1,1)$, $(1,2,1)$, $(1,1,2)$. ****ANSWER**:** (iv). This is given by 1/2 the magnitude of $\vec{a} \times \vec{b}$, where $\vec{a}$ and $\vec{b}$ are vectors running along the sides of the triangle.

(i)	1	(ii)	2	(iii)	$\sqrt{3}$
(iv)	$\sqrt{3}/2$	(v)	$\sqrt{3}/4$	(vi)	$\sqrt{6}/2$

(b) (4 points) Find the **cosine** of the angle between the two planes $x + 2y = 0$ and $x + 2z = 3$. ****ANSWER**:** (iv) This is given by taking the two normal vectors $\vec{n}_1$, $\vec{n}_2$ to the planes and then computing $(\vec{n}_1 \cdot \vec{n}_2)/(|\vec{n}_1||\vec{n}_2|)$.

(i)	$2/3$	(ii)	$3/4$	(iii)	$\sqrt{3}/2$
(iv)	$1/5$	(v)	$1/\sqrt{2}$	(vi)	$1/2$

(c) (4 points) Find the **unit tangent vector** to the parametric curve $\vec{r}(t) = \langle \sin t, 2t, t^2 \rangle$ at $t = 0$. ****ANSWER**:** (i) First find $\vec{r}'(t)$, plug in $t = 0$, and then divide the result by its magnitude.

(i)	$\langle 1/\sqrt{5}, 2/\sqrt{5}, 0\rangle$	(ii)	$\langle 1/\sqrt{5}, -2/\sqrt{5}, 0\rangle$	(iii)	$\langle -2/\sqrt{5}, 1/\sqrt{5}, 0\rangle$
(iv)	$\langle 2/\sqrt{5}, 1/\sqrt{5}, 0\rangle$	(v)	$\langle -1/\sqrt{5}, -2/\sqrt{5}, 0\rangle$	(vi)	$\langle 2/\sqrt{5}, -1/\sqrt{5}, 0\rangle$

(d) (4 points) Describe the **level curves** of the function $f(x,y) = \sqrt{1 - x^2 - 2y^2}$. ****ANSWER**:** (ii) They are determined by setting $1 - x^2 - 2y^2$ to a constant, which means $x^2 + 2y^2$ is a constant. These are concentric ellipses.

(i)	concentric circles	(ii)	concentric ellipses (not circles)
(iii)	parabolas with the same vertex	(iv)	parabolas with different vertices
(v)	hyperbolas with the same vertex	(vi)	hyperbolas with different vertices

(e) (4 points) Find the linear approximation $L(x,y)$ of the function $f(x,y) = xye^x$ at $(x,y) = (1,1)$ and use it to **estimate** $f(1.1, 0.9)$. ****ANSWER**:** (i) Find f_x and f_y. Plug in the point $(1,1)$. Write the linear approximation equation at $(1,1)$ and plug in the point $(1.1, 0.9)$.

(i)	$1.1e$	(ii)	e	(iii)	0.5
(iv)	1.5	(v)	$2e$	(vi)	2

2. (20 points)

(a) (6 points) Let P be the plane through the points $(1,0,0)$, $(0,2,0)$, and $(0,0,3)$. Find an equation for P.

****ANSWER**:** Let $p = (1,0,0)$, $q = (0,2,0)$, $r = (0,0,3)$. We need to take two vectors lying in the plane, such as $\vec{v} = pq$, $\vec{w} = pr$, and then take the cross product $\vec{n} = \vec{v} \times \vec{w}$ to make a normal vector to P. We find $\vec{v} = \langle -1,2,0 \rangle$, $\vec{w} = \langle -1,0,3 \rangle$, and then $\vec{n} = \langle 6,3,2 \rangle$. Thus the equation has the form $6x + 3y + 2z = D$ for a constant D. Plugging in p we find $D = 6$. Therefore an equation for P is $6x + 3y + 2z = 6$.

(b) (6 points) Let L be the line through the origin in the direction of $\vec{r} = \langle 2,-2,-3 \rangle$. Find parametric equations for L.

****ANSWER**:** The direction vector is $\langle 2,-2,-3 \rangle$ and a point on the line L is $(0,0,0)$. Thus parametric equations are $x = 2t$, $y = -2t$, and $z = -3t$ where t ranges over all real numbers.

(c) (8 points) Does L intersect P? If yes, find the point of intersection. If not, find the distance between L and P.

****ANSWER**:** The dot product $\vec{n} \cdot \vec{r}$ is zero. This means the direction vector of L is perpendicular to the normal vector of P, which implies either L is contained in P or is parallel to P. The origin is on L but not on P, so L must be parallel to P. Thus we have to find the distance. We can compute it as $d = |\vec{a} \cdot \hat{n}|$, where $\vec{a}$ is a vector running from L to P and $\hat{n}$ is a unit vector in the direction of the normal vector $\vec{n}$ (this is the direction along which the distance is measured). We can take $\vec{a}$ to run from the origin to p, i.e. $\vec{a} = \langle 1,0,0 \rangle$. The length of $\vec{n} = \langle 6,3,2 \rangle$ is $\sqrt{36+9+4} = 7$, so $\hat{n} = \langle 6/7, 3/7, 2/7 \rangle$ and $d = |\vec{a} \cdot \hat{n}| = 6/7$.

3. (15 points) Suppose a particle moves with position function $\vec{\boldsymbol{r}}(t) = \langle t^2, t\sqrt{2}, (\ln t)/2\rangle$, where $t > 0$.

(a) (8 points) Find the velocity and acceleration of the particle.

****ANSWER**:**

$$\vec{\boldsymbol{v}}(t) = \vec{\boldsymbol{r}}'(t) = \left\langle 2t, \sqrt{2}, \frac{1}{2t} \right\rangle \text{ is the velocity of the particle, and}$$

$$\vec{\boldsymbol{a}}(t) = \vec{\boldsymbol{v}}'(t) = \vec{\boldsymbol{r}}''(t) = \left\langle 2, 0, -\frac{1}{2t^2} \right\rangle \text{ is the acceleration of the particle.}$$

(b) (12 points) Find the distance traveled by the particle from $t = 1$ to $t = e$.

****ANSWER**:** Arclength for a curve $r(t)$ between $t = a$ and $t = b$ is

$$Arc(a, b) = \int_a^b \left|\vec{\boldsymbol{r}}'(t)\right| dt = \int_a^b |\vec{\boldsymbol{v}}(t)|\, dt.$$

Plugging in from (a) we have

$$Arc(1, e) = \int_1^e \left|\left\langle 2t, \sqrt{2}, \frac{1}{2t}\right\rangle\right| dt = \int_1^e \sqrt{(2t)^2 + (\sqrt{2})^2 + \left(\frac{1}{2t}\right)^2} dt$$

$$= \int_1^e \sqrt{4t^2 + 2 + \frac{1}{4t^2}} dt = \int_1^e \sqrt{\frac{1}{4t^2}\left(16t^4 + 8t^2 + 1\right)} dt = \int_1^e \frac{1}{2t}\sqrt{(4t^2 + 1)^2} dt$$

$$= \int_1^e \frac{1}{2t}(4t^2+1) dt = \int_1^e \left(2t + \frac{1}{2t}\right) dt = \left(t^2 + \frac{1}{2}\ln t\right)\Big|_1^e = e^2 + \frac{1}{2}\ln e - 1 - \frac{1}{2}\ln 1$$

$$= e^2 + \frac{1}{2}\cdot 1 - 1 - \frac{1}{2}\cdot 0 = e^2 - \frac{1}{2}.$$

4. (15 points) Let $f(x,y) = x^2y + ye^{xy}$.

(a) (5 points) Find the linearization $L(x,y)$ of f at the point (0.5) and use it to approximate the value of f at the point $(0.1, 4.9)$.

****ANSWER**:** $f_x(x,y) = 2xy + y^2e^{xy}$, $f_y(x,y) = x^2 + e^{xy} + xye^{xy}$

$f_x(0,5) = 25$, $f_y(0,5) = 1$

Let $L(x,y)$ be the linear approximation at $(0,5)$.

$L(x,y) = f(0,5) + f_x(0,5)(x-0) + f_y(0,5)(y-5)$

$L(x,y) = 5 + 25x + (y-5)$

Calculating at $(0.1, 4.9)$, $L(0.1, 4.9) = 5 + 25(0.1) + (4.9 - 5) = 7.4$.

(b) (5 points) Suppose that $x(r,\theta) = r\cos\theta$ and $y(r,\theta) = r\sin\theta$. Calculate f_θ at $r = 5$ and $\theta = \frac{\pi}{2}$.

****ANSWER**:** The Chain rule gives

$$\frac{\partial f}{\partial \theta} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \theta}$$

$= (2xy + y^2e^{xy})(-r\sin\theta) + (x^2 + e^{xy} + xye^{xy})(r\cos\theta)$

When $r = 5$ and $\theta = \frac{\pi}{2}$, then $x = 0$ and $y = 5$.

Thus, $\frac{\partial f}{\partial \theta} = (0 + 25e^0)(-5) + 0 = -125$

(c) (5 points) Consider the equation $x^2 + y^2/9 + z^2/4 = 1$. Calculate z_x and z_y at an arbitrary point (x,y,z) on the surface. (wherever possible)

****ANSWER**:** Using implicit differentiation:

$2x + \frac{z}{2}\frac{\partial z}{\partial x} = 0$, so $\frac{\partial z}{\partial x} = -\frac{4x}{z}$

$\frac{2y}{9} + \frac{z}{2}\frac{\partial z}{\partial x} = 0$, so $\frac{\partial z}{\partial y} = -\frac{4y}{9z}$

Both partial derivatives make sense when $z \neq 0$.

5. (15 points) Use the Chain Rule to find $\dfrac{\partial z}{\partial v}$ when $u = 1$ and $v = 1$, where
$z = x^3y^2 + y^3x$; $\quad x = u^2 + v$ and $\quad y = 2u - v$.

****ANSWER**:** Substituting $u = v = 1$, gives $x = 1^2+1 = 2$, $y = 2\cdot 1-1 = 1$, $\dfrac{\partial x}{\partial v} = 1$ and $\dfrac{\partial y}{\partial v} = -1$.
Using the Chain Rule we have $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v} =$
$(3x^2y^2 + y^3)(1) + (2x^3y + 3y^2x)(-1) = 3x^2y^2 + y^3 - 2x^3y - 3y^2x$.
So for $u = 1$ and $v = 1$, we get:
$\dfrac{\partial z}{\partial v}(1,1) = 3(4) + 1 - 2(8) - 3(2) = -9$.

6. (15 points) A projectile is fired from a point 5 m above the ground at an angle of 30 degrees and an initial speed of 100 m/s.

(a) (4 points) Write a vector for the initial velocity $\vec{v}(0)$ and the initial position $\vec{r}(0)$.

****ANSWER**:** Since the force due to gravity acts downward, we have $\vec{F} = m\vec{a} = -mg\vec{j}$. Therefore, $\vec{a} = -g\vec{j}$.
Initial velocity is: $\vec{v}(0) = 100(\cos(30°)\vec{i} + \sin(30°)\vec{j}) = 50\sqrt{3}\,\vec{i} + 50\vec{j}$ (in units of m/s).
Initial position is: $\vec{r}(0) = 5\vec{j}$ (in units of m).

(b) (7 points) At what time does the projectile hit the ground?

****ANSWER**:** The velocity and position functions are:
$\vec{v} = \vec{r}'(t) = -gt\vec{j} + \vec{v}(0)$
$\vec{r}(t) = -\frac{1}{2}gt^2\vec{j} + t\vec{v}(0) + \vec{r}(0) = 50\sqrt{3}\,t\,\vec{i} + (5 + 50t - \frac{1}{2}gt^2)\,\vec{j}$. The projectile hits the ground when $y = 0$, so $5 + 50t - \frac{1}{2}gt^2 = 0$. Using the quadratic formula, we find $t = \dfrac{100 + \sqrt{100^2 + 40g}}{2g}$.

(c) (4 points) How far did the projectile travel, horizontally, before it hit the ground?

****ANSWER**:** Substituting t from the previous part to the value of the x-coordinate of $\vec{r}(t)$, $x(t) = 50\sqrt{3}\,t$ which gives the range d, we find $d = 50\sqrt{3}\Big(\dfrac{100 + \sqrt{100^2 + 40g}}{2g}\Big)$.

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