MATH 411 EXAM I

This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and then *any four* of the remaining problems. Unless indicated, you must show your work to receive credit for a solution. Make sure you answer every part of every problem you do.

When submitting your exam, please indicate which problems you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly four problems from problems 2 and higher (problem 1 is automatically selected and you need not indicate it); any unselected problems will not be graded, and if you select more than four only the first four (in numerical order) will be graded.

(1) Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem.
   (a) (4 pts) The binary operation of division on the nonzero real numbers $\mathbb{R}^*$ is associative.
   (b) (4 pts) For any positive integer n, there exists a group of order n.
   (c) (4 pts) For any group G, one can find a proper subgroup H of G that is not trivial (i.e. $H \neq G$ and $H \neq \{e\}$).
   (d) (4 pts) The direct product of two cyclic groups is always cyclic.
   (e) (4 pts) If G is a cyclic group, then G is abelian.d

(2) (20 pts) Let $\mathbb{Z}$ be given the binary operation $a * b = a + b - 1$.
   (a) (15 pts) Show that $(\mathbb{Z}, *)$ forms a group.
   (b) (5 pts) Is this group cyclic? Why or why not?

(3) (20 pts) Let G be the set of triples (a, b, c) where a, b, c are taken from $\mathbb{Z}$. Give G the binary operation

$$(a_1, b_1, c_1) * (a_2, b_2, c_2) = (a_1 + a_2, b_1 + b_2, c_1 + c_2 + a_1 b_2),$$

   where the indicated additions and multiplication take place in $\mathbb{Z}$. The operation $*$ is known to be associative (you don't need to prove that).
   (a) (5 pts) Compute $(1, 2, 3) * (-1, -2, -1)$ and $(0, 0, 0) * (3, 5, 7)$.
   (b) (15 pts) Show that $(G, *)$ is a group.

(4) Let G be a cyclic group of order 60 with a generator g.
   (a) (5 pts) What are the orders of the subgroups of G?
   (b) (5 pts) For each subgroup in part (4a), give a generator for it in terms of g.
   (c) (5 pts) How many generators does G have?
   (d) (5 pts) Give a list of all generators of G in terms of g.

(5) (20 pts)
   (a) (15 pts) Prove that if every element g of a group G satisfies $g^2 = e$, then G is abelian.
   (b) (5 pts) Give a counterexample to the converse of the previous statement. That is, give an example of a group G that is abelian but not every element $g \in G$ satisfies $g^2 = e$.

(6) (20 pts) Let G be a group and let $x \in G$. Show that $o(x) = o(x^{-1})$.

(7) (20 pts) Let A be a subgroup of G and B be a subgroup of H. Show that $A \times B$ is a subgroup of $G \times H$.

Date: Wednesday, 9 Oct 2024.