STRATIFIED MORSE THEORY

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The sea was clearly visible for a one-mile radius around the Nautilus. What a sight! What pen could describe it? Who could portray the effects of the light through those translucent sheets of water, the subtlety of its progressive shadings into the ocean's upper and lower strata?

—Jules Verne, *Twenty Thousand Leagues Under the Sea*
(trans. F.P. Walter)

INTRODUCTION

Classical Morse Theory establishes a connection between the topology of a smooth manifold X and the critical points of an open dense family of functions $f : X \to \mathbb{R}$. Stratified Morse Theory generalizes Classical Morse Theory to spaces with singularities—in particular to subspaces of smooth manifolds that admit a Whitney stratification. (Many singular spaces, including all real and complex analytic varieties, fall into this broad category.)

Specifically, Stratified Morse Theory generalizes three theorems at the heart of Classical Morse Theory.

The first two classical theorems, Theorems A and B, say that the topology of the space $X_{\leq v} := \{x \in X : f(x) \leq v\}$ changes only when v passes through a critical value, and that when it does the topological change can be described as attaching a "handle" (see p. 3). These theorems generalize elegantly to Whitney stratified spaces (see p. 4)—although their proofs are surprisingly technical.

The third classical theorem, Theorem C, provides a homological characterization of the index of a critical point; it says that if v is a critical value of index λ, then:

$$\mathrm{H}_k(X_{\leq v+\varepsilon}, X_{\leq v-\varepsilon}) = \begin{cases} \mathbb{Z} & \text{if } k = \lambda \\ 0 & \text{otherwise} \end{cases}$$

for sufficiently small $\varepsilon > 0$.

Does Theorem C generalize to Whitney stratified spaces? Not directly, as simple examples show (see p. 5). But if we replace ordinary homology with intersection homology and if we consider only complex analytic varieties of pure dimension, then we obtain Theorem C' on p. 5. Its proof is our main goal in this essay.

The need for intersection homology in the statement of Theorem C' is perhaps unsurprising. After all, Theorem C plays a central role in the Morse-theoretic proof of classical Poincaré duality, and intersection homology is a homeomorphism invariant that recovers Poincaré duality for complex analytic varieties of pure dimension (see p. 16).

But intersection homology recovers Poincaré duality for many more spaces than just complex analytic ones (for example Witt spaces). Does Theorem C generalize to these spaces as well? This is unknown, but our proof of Theorem C' makes it seem unlikely.

Indeed our proof of Theorem C' depends heavily on the assumption that X is complex analytic. To begin with, it relies on an analysis of the local structure of complex analytic varieties using the so-called Milnor fibration. Next it proceeds by induction on dimension by passing to the complex link in the normal slice of the critical point. This inductive step uses the Lefschetz hyperplane theorem for complex analytic varieties.

Because our proof of Theorem C' depends so heavily on the assumption that X is complex analytic, we simplify our exposition by restricting to complex analytic varieties from the outset, even though many of the intermediate results—in particular the generalizations of Theorems A and B—hold for any Whitney stratified space.

We conclude the essay by considering the special case of complex algebraic curves.

The results in this essay are due to Goresky and MacPherson [GM83c]. The principal references used were the book [GM88] and the article [GM83b].

CONTENTS

1. THEOREMS A, B, AND C OF CLASSICAL MORSE THEORY

Let X be a *smooth* complex analytic variety of complex dimension n. Assume that X is analytically embedded in some ambient complex analytic manifold M. In this way X could be virtually any sort of 'complex variety' you could imagine, including an affine or projective algebraic variety.

Classical Morse Theory [Mil63] establishes a connection between the topology of X—or more generally any smooth manifold—and the critical points of an open dense family of proper[1] functions $X \to \mathbb{R}$ called Morse functions. In particular, it shows that local information near the critical points of a proper Morse function $f : X \to \mathbb{R}$ can be used to construct X through a sequence of surgeries. This in turn reveals a relationship between the critical points of f and the homology of X. Theorems A, B, and C below codify these results.

First recall that a point p is called a *critical point* for a smooth function $f : X \to \mathbb{R}$ if the induced map $f_* : T_pX \to \mathbb{R}$ is zero. The real number $f(p)$ is then called a *critical value*. We call f a *Morse function* if (i) its critical values are distinct and (ii) its critical points are nondegenerate—ie its Hessian matrix of second derivatives at p has nonvanishing determinant. The number of negative eigenvalues of this Hessian matrix is called the *index* of f at p.

Let $X_{\leq v}$ denote the subspace $\{x \in X : f(x) \leq v\}$ and let $(D^a, \partial D^a)$ denote the (closed) a-dimensional disk and its boundary sphere.

[1] We call f *proper* if $f^{-1}(K)$ is compact whenever K is.

The key theorems of Classical Morse Theory can now be stated as follows.

Theorem A. *As v varies within the open interval between two adjacent critical values, the topological type of $X_{\leq v}$ does not change.*

But as v crosses a critical value $f(p)$, the topological change of $X_{\leq v}$ can be described as follows.

Theorem B. *The space $X_{\leq f(p)+\varepsilon}$ is homeomorphic to the space obtained from $X_{\leq f(p)-\varepsilon}$ by attaching a "handle" $D^\lambda \times D^{2n-\lambda}$ along $\partial D^\lambda \times D^{2n-\lambda}$, where λ is the index of f at the critical point p and $\varepsilon > 0$ is sufficiently small.*

We can restate Theorem B by saying that the *Morse Data* for f at p is:

$$(D^\lambda \times D^{2n-\lambda}, \partial D^\lambda \times D^{2n-\lambda}) .$$

Theorems A and B together tell us how to construct X through a sequence of surgeries, one for each critical value.

A consequence of Theorem B is the following beautiful result.

Theorem C. *If p is a critical point of index λ, then:*

$$H_k(X_{\leq f(p)+\varepsilon}, X_{\leq f(p)-\varepsilon}) = \begin{cases} \mathbb{Z} & \textit{if } k = \lambda \\ 0 & \textit{otherwise} \end{cases}$$

for sufficiently small $\varepsilon > 0$.

Theorem C gives a purely homological characterization of the index λ and leads to the Morse inequalities—inequalities which constrain the number of critical points of f by the Betti numbers of X (and vice versa).

2. Theorems A', B', and C' of Stratified Morse Theory

Now let X be a *singular* complex analytic variety (again analytically embedded in some ambient smooth complex manifold M). Can the Morse theory described above tell us anything about the topology of X? Not directly; since X is not smooth, we have no notion of 'smooth function' on X, let alone critical points and indices. But the theory described above can nevertheless be modified to apply to X as follows.

First of all, a theorem (see for example [Hir73]) says that X admits a *Whitney stratification*. Loosely speaking, this means that X can be decomposed into smooth submanifolds, called *strata*, such that the topology of X near each stratum is locally constant.

To define a Whitney stratification formally, we first introduce the more general notion of an $\mathscr{I}$-decomposition.

Definition 2.1. Let $(\mathscr{I}, <)$ be a partially ordered set. An *$\mathscr{I}$-decomposition* of a topological space Z is a locally finite collection of disjoint, locally closed subsets called *pieces* $S_\alpha \subset Z$ (one for each $\alpha \in \mathscr{I}$) such that:

(1) $Z = \bigcup_{\alpha \in \mathscr{I}} S_\alpha$ and
(2) $S_\alpha \cap \overline{S_\beta} \neq \emptyset \iff S_\alpha \subset \overline{S_\beta} \iff \alpha = \beta \vee \alpha < \beta$
(in which case we write $S_\alpha < S_\beta$).

Definition 2.2. Let Z be a closed subset of a smooth manifold M. A *Whitney stratification* of Z is an $\mathscr{I}$-decomposition $Z = \bigcup_{\alpha \in \mathscr{I}} S_\alpha$ for some partially-ordered set $\mathscr{I}$ such that:

(1) Each piece S_α is a locally closed smooth submanifold of M.

(2) Let $S_\alpha < S_\beta$ and $y \in S_\alpha$. Suppose that two sequences $x_i \in S_\beta$ and $y_i \in S_\alpha$ converge to y, that the secant lines $\overline{x_i y_i}$ converge to some limiting line ℓ, and that the tangent planes $T_{x_i} S_\beta$ converge to some limiting plane τ. Then $\ell \subset \tau$.

We call the pieces of a Whitney stratification its *strata*, and we assume that *each stratum is connected.*

Condition (2) of Definition 2.2 is referred to as "Whitney's Condition B". It implies a weaker condition called "Whitney's Condition A" (not stated). Although these important conditions are needed to prove most of the results on which our arguments depend, our arguments themselves never directly appeal to these conditions. Therefore, the reader should perhaps simply bear in mind the most important and easily grasped consequence of Whitney's conditions: that the topology of a Whitney stratified space is locally constant over each stratum; that is, each stratum has a neighborhood (in Z) that is a fiber bundle with base space the stratum. (The fiber is a space we define later called the *normal slice*.) For a thorough treatment of these conditions see [Mat70].

We are making progress in extending Morse theory to X, for if the point p lies within the stratum S, we can now speak of the tangent space T_pS. But if X is singular along S, there could be many more planes "tangent" to X at p; we call each such plane a generalized tangent space. More precisely, a *generalized tangent space* at p is any plane Q of the form:

$$Q = \lim_{p_i \to p} T_{p_i} S'$$

where p_i is a sequence of points in a stratum S' converging to p.

With these concepts in place, Lazzeri [Laz73] introduced the following generalization of Morse function.

Definition 2.3. A function $f : X \to \mathbb{R}$ is called a *Morse function* if:

(1) f is the restriction of a smooth function $\widetilde{f} : M \to \mathbb{R}$.
(2) The restriction $f|S$ is Morse (in the classical sense) for each stratum S.

A point p in a stratum S is then called a *critical point* for f if it is a critical point (in the classical sense) for $f|S$. The *index* of f at p is $c + \lambda$, where c is the complex codimension of S and λ is the (classical) index of $f|S$. (The motivation for this definition comes from Theorem C' below.)

(3) All critical values are distinct.
(4) For each critical point p and for each generalized tangent space Q at p other than T_pS, $d\widetilde{f}(p)(Q) \neq 0$.

Note that if $\{p\}$ is a zero-dimensional stratum, then p must be critical.

Pignoni [Pig79] proved that if X is closed in M then the subset of proper functions that restrict to Morse functions on X is open and dense in the set of all proper C^∞ functions $M \to \mathbb{R}$ and that such Morse functions are structurally stable.

Assuming f is proper, Theorem A generalizes to this setting verbatim.

Theorem A'. *As v varies within the open interval between two adjacent critical values, the topological type of $X_{\leq v}$ does not change.*

To generalize Theorem B we need the following important construction.

Let p be a critical point contained in a stratum S. Choose an analytic manifold $V \subset M$ that meets S transversely in the single point p. Let $B_\delta(p)$ denote the closed ball of radius δ with respect to some local coordinates for M. Then the *normal slice* N_δ is the intersection $B^\circ_\delta(p) \cap V \cap X$.

Theorem B'. *If $1 \gg \delta \gg \varepsilon > 0$ are sufficiently small, then the Morse data at p may be written:*[2]

$$\underbrace{(D^\lambda \times D^{a-\lambda}, \partial D^\lambda \times D^{a-\lambda})}_{\text{“tangential Morse data”}} \times \underbrace{((N_\delta)_{\leq f(p)+\varepsilon}, (N_\delta)_{\leq f(p)-\varepsilon})}_{\text{“normal Morse data”}}$$

where a is the real dimension of the stratum S and λ is the index of $f|S$.

The topological types of the normal slice and the normal Morse data are independent of V, δ, and ε. In fact, by a miracle of complex geometry, the topological type of the normal Morse data is also independent of f.

Theorems A' and B' are essential to what follows, but we do not prove them. And for good reason: although they emerge intuitively from examples (see for instance the (not complex analytic) “perverse donut” on pp. 6–9 of [GM88]), their full proofs occupy about one-hundred pages of [GM88].

So, how might we generalize the beautiful Theorem C? The following example illustrates that it will not be easy.

Example. Consider the (unreduced) suspension SY of some space Y. The projection $[y,t] \mapsto t$ is a Morse function $SY \to \mathbb{R}$. Its only critical points are the cone points. If something like Theorem C were true, it would imply (by excision) that the relative group $H_k(CY,Y)$ vanished in all but one dimension. But at the same time, the long exact sequence for the pair (CY,Y) gives isomorphisms $H_k(CY,Y) \cong \widetilde{H}_{k-1}Y$. Thus, if Y has nonvanishing reduced homology in several dimensions, we would arrive at a contradiction.

If we consider the complex cone over a complex algebraic variety Y embedded in projective space, we see that the problem persists even if we consider only complex analytic varieties. (In fact, for complex analytic varieties it can be shown ([GM88], p. 211) that $\mathrm{H}_k(X_{\leq f(p)-\varepsilon}, X_{\leq f(p)+\varepsilon}) \cong \widetilde{H}_{k-\lambda-1}(\mathscr{L})$, where $\mathscr{L}$ is the *complex link* associated to p—an important space which we define later.)

Fortunately Theorem C can nevertheless be generalized, but in place of singular homology we must use *intersection homology*. (The reader unfamiliar with intersection homology should consult the Appendix on p. 16, where we define intersection homology and state all its properties we need.)

We repeat the standing assumptions about X and f for emphasis.

Theorem C'. *Let X be a complex analytic variety of pure dimension (analytically embedded in an ambient complex analytic manifold M) and let $f : X \to \mathbb{R}$ be a proper Morse function (in the sense of Definition 2.3). If p is a critical point of index $i = c + \lambda$, where c is the complex codimension of the stratum S containing p and λ is the index of $f|S$, then:*

$$\mathrm{IH}_k(X_{\leq f(p)+\varepsilon}, X_{\leq f(p)-\varepsilon}) = \begin{cases} A_p & \text{for } k = i \\ 0 & \text{for } k \neq i \end{cases}$$

for sufficiently small $\varepsilon > 0$.

[2] The product of pairs $(A,B) \times (A',B')$ denotes the pair $(A \times A', A \times B' \cup A' \times B)$.

The group A_p is called the *Morse group* of the critical point p and, in contrast with Theorem C, is not necessarily $\mathbb{Z}$. We will see that it depends only on the stratum of X containing p—and *not* on f. In fact we will give a somewhat geometric description of the group.

The first step in proving Theorem C' is the following corollary of Theorem B'.

Corollary 2.4 (of Theorem B')**.** *If p is a critical point of index $i = c+\lambda$, then:*

$$\mathrm{IH}_k(X_{\le f(p)+\varepsilon}, X_{\le f(p)-\varepsilon}) \cong \mathrm{IH}_{k-\lambda}\underbrace{((N_\delta)_{\le f(p)+\varepsilon}, (N_\delta)_{\le f(p)-\varepsilon})}_{\textit{normal Morse data}} .$$

Proof. Excise $X_{<f(p)-\varepsilon}$ from the decomposition provided by Theorem B' and apply the Künneth Formula to the remaining pair. $\square$

A point is always a zero-dimensional stratum of its normal slice, so Corollary 2.4 reduces the proof of Theorem C' to the case of a zero-dimensional stratum $\{p\}$. Moreover, if δ is sufficiently small then we may use local coordinates in M to regard N_δ as sitting directly in $\mathbb{C}^m$.

To prove this special case we use a beautiful construction due to Milnor: near p we factor f through a fiber bundle over a punctured disk in the complex plane. The construction of this so-called Milnor fibration depends heavily on the assumption that X is complex analytic. (In contrast, the proofs of Theorems A' and B' make no use of this assumption and would in fact be true if we replaced X by a real analytic variety, or even a closed subanalytic subset of an analytic manifold—although in this more general setting the normal Morse data depends on f.)

3. The Local Structure of Complex Analytic Varieties

Let N be a Whitney stratified (not necessarily closed) complex analytic variety in $\mathbb{C}^m$ with a zero-dimensional stratum $\{p\}$.

Let $\pi : \mathbb{C}^m \to \mathbb{C}$ be a linear projection such that $\pi(p) = 0$ and such that $\mathrm{Re}(\pi)$ restricts to a Morse function on N near p. By condition (3) of Definition 2.3, this means that $d\pi(p)(Q) = 0$ for every generalized tangent space to N at p. The set of such projections is open and dense in the space of all linear maps $\mathbb{C}^m \to \mathbb{C}$. (See Pignoni [Pig79]).

To proceed further we must introduce several new spaces.

First, let N_δ denote the open ball in N of radius δ about p, and $\overline{N}_\delta$ its closure (in N). (Note that this notation is consistent with the earlier meaning of these symbols if N is a normal slice.) Let $D_\varepsilon \subset \mathbb{C}$ denote the closed disk of radius ε.

Definition 3.1. Let $1 \gg \delta \gg \varepsilon > 0$ be very small and let $\xi \in \partial D_\varepsilon$. The following definitions are illustrated schematically in Figures 1 to 4 on pp. 7–8 (modified slightly from [GM83b]).

(1) The *complex link* $\mathscr{L}$ and its boundary:

$$\mathscr{L} := \pi^{-1}(\xi) \cap N_\delta(p)$$
$$\partial\mathscr{L} := \pi^{-1}(\xi) \cap \partial N_\delta(p) .$$

(2) The *cylindrical neighborhood* C of p:

$$C := \pi^{-1}(D_\varepsilon) \cap \overline{N}_\delta(p) .$$

(3) The *horizontal* and *vertical parts of the real link*:

$$L_h := \pi^{-1}(\partial D_\varepsilon) \cap \overline{N}_\delta \qquad\qquad L_v := \pi^{-1}(D_\varepsilon) \cap \partial\overline{N}_\delta .$$

(4) The *real link*:

$$L := \partial C = L_h \cup L_v \ .$$

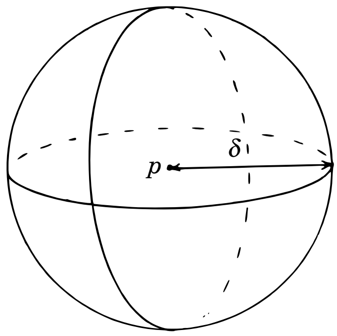

FIGURE 1. $\overline{N}_\delta(p)$, the closed ball in the normal slice. [GM83b]

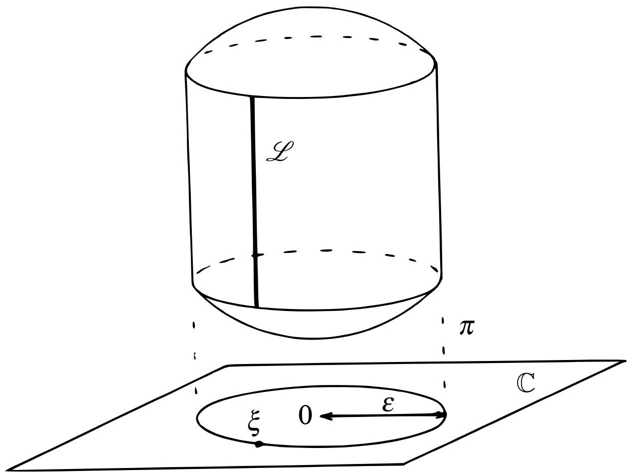

FIGURE 2. C, the cylindrical neighborhood, and its image under the projection π. [GM83b]

The following theorem is the key result of this section—and an indispensable ingredient in the proof of Theorem C'.

Theorem 3.2. *(a) The restricted projection:*

$$\pi \ : \ C - \pi^{-1}(0) \ \longrightarrow \ D_\varepsilon - \{0\}$$

is a topological fiber bundle (called the Milnor fibration) with fiber $\mathscr{L}$.

(b) (C,L) is a mapping cylinder neighborhood of $\{p\}$. More precisely, there is a homeomorphism:

$$\mathrm{Cone}(L) \to C$$

that takes the vertex of the cone to p, takes the base of the cone to L by the identity, and takes $\mathrm{Cone}(L_{<0})$ *homeomorphically to* $C_{<0} \cup \{p\}$. *(Here "$_{<0}$" refers to the Morse function* $\mathrm{Re}(\pi)$*.)*

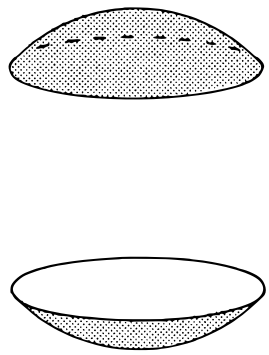

FIGURE 3. L_h, the horizontal part of the real link. [GM83b]

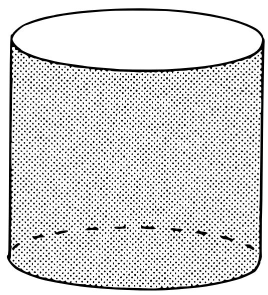

FIGURE 4. L_v, the vertical part of the real link. [GM83b]

(c) There is an embedding:

$$r : \partial\mathscr{L} \times [0,1) \times D_\varepsilon \to C$$

which takes $\partial\mathscr{L} \times \{0\} \times D_\varepsilon$ *homeomorphically to* L_h *and for which:*

$$\pi \circ r : \partial\mathscr{L} \times [0,1) \times D_\varepsilon \to D_\varepsilon$$

is projection to the third factor.

(d) There is a homeomorphism of pairs:

$$((N_\delta)_{\le\varepsilon}, (N_\delta)_{\le-\varepsilon}) \cong (C_{\le\eta}, C_{\le-\eta})$$

for $0 < \eta \ll \varepsilon$. *(Here "*$_{\le\varepsilon}$*" etc again refer to the Morse function* $\mathrm{Re}(\pi)$*.)*

The proof of Theorem 3.2 involves delicate arguments in stratification theory using controlled vector fields (see [GM88]). We do not prove it. We encourage the reader to see Milnor's delightful book [Mil68] for further information about the Milnor fibration.

We also omit the proof of the following important technical result.

Theorem 3.3. *The topological types of the spaces and maps defined in this section—including the cylindrical neighborhood and the complex link—are independent of the projection* π *and the numbers* δ, ε, ξ, *and* η. *If* N *is the normal slice at a point* p *in a stratum* S, *then the topological types of these spaces and maps are also independent of* N *and* $p \in S$.

4. The Intersection Homology of the Cylindrical Neighborhood

We need the following three corollaries of Theorem 3.2 to prove Theorem C'.

Corollary 4.1. *There is an isomorphism:*

$$\mathrm{IH}_k(\underbrace{(N_\delta)_{\le f(p)+\varepsilon},(N_\delta)_{\le f(p)-\varepsilon}}_{\textit{normal Morse data}}) \cong \mathrm{IH}_k(C, C_{<0}) \quad \textit{for all } k.$$

Proof. Theorem 3.2(d) gives an isomorphism:

$$\mathrm{IH}_k((N_\delta)_{\le\varepsilon},(N_\delta)_{\le-\varepsilon}) \cong \mathrm{IH}_k(C_{\le\eta}, C_{\le-\eta})$$

for sufficiently small $\eta > 0$.

By Theorem 3.2(a) the inclusions $C_{\le\eta} \hookrightarrow C$ and $C_{\le-\eta} \hookrightarrow C_{<0}$ are isotopic through inclusions to homeomorphisms and hence induce isomorphisms on intersection homology. The Five Lemma, applied to the inclusion-induced maps between the long exact sequences for these pairs, then yields an isomorphism:

$$\mathrm{IH}_k(C_{\le\eta}, C_{\le-\eta}) \cong \mathrm{IH}_k(C, C_{<0})\ .$$

□

Corollary 4.2. *There is a canonical isomorphism:*

$$\mathrm{IH}_k(C_{<0}) \cong \mathrm{IH}_k(\mathscr{L}) \quad \textit{for all } k.$$

Proof. By Theorem 3.2(a) the projection π restricts to $C_{<0}$ as a fiber bundle over the contractible space $D_{<0}$. Therefore $C_{<0} \cong \mathscr{L} \times D_{<0}$, and the Künneth Formula gives a (canonical) isomorphism on intersection homology. □

We have saved the best for last.

Corollary 4.3. *There is a canonical isomorphism:*

$$\mathrm{IH}_k(C - \{p\}, C_{<0}) \cong \mathrm{IH}_{k-1}(\mathscr{L}, \partial\mathscr{L}) \quad \textit{for all } k.$$

Proof. There are four steps.

Step 1. Theorem 3.2(b) gives a homeomorphism:

$$(C - \{p\}, C_{<0}) \cong (L, L_{<0}) \times [0,1)\ ,$$

so the Künneth Formula gives an isomorphism:

$$\mathrm{IH}_k(C - \{p\}, C_{<0}) \cong \mathrm{IH}_k(L, L_{<0})\ . \tag{1}$$

Step 2. Consider the following diving-helmet-like subset of L (illustrated in Figure 6 on p. 11):

$$L_c := L_{<0} \cup L_h \cup r(\partial\mathscr{L} \times [0, \tfrac{1}{2}) \times \partial D_\varepsilon)$$

where r is the embedding provided by Theorem 3.2(c). We show that $(L, L_{<0})$ is homeomorphic to (L, L_c).

Consider the following neighborhood $U \subset L$ of L_h:

$$U := L_h \ \cup\ r(\partial\mathscr{L} \times [0,1) \times \partial D_\varepsilon)\ .$$

The homeomorphism $L_h \cong \partial\mathscr{L} \times \{0\} \times D_\varepsilon$ provided by Theorem 3.2(c) extends to a homeomorphism $U \cong \partial\mathscr{L} \times D^\circ_{3\varepsilon}$ sending collaring lines to radial lines in $\mathbb{C}$. Composing this homeomorphism with the homeomorphism between the open subsets of $\mathbb{C}$ drawn in Figure 5 on the next page yields the desired homeomorphism $(L, L_{<0}) \cong (L, L_c)$ and thus induces an isomorphism:

$$\mathrm{IH}_k(L, L_{<0}) \cong \mathrm{IH}_k(L, L_c)\ . \tag{2}$$

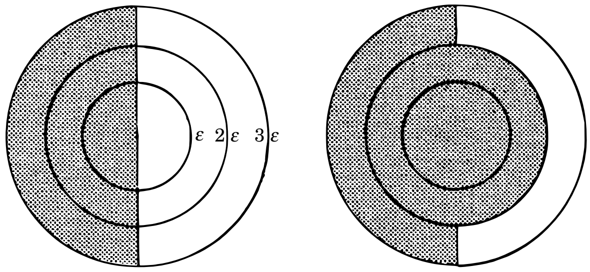

FIGURE 5. These two open subsets of $\mathbb{C}$ are related by a homeomorphism fixing a neighborhood of the circle of radius 3ε. [GM83b]

Step 3. Now we excise all but the helmet's visor (and its rim). To define these subspaces formally we introduce two subsets of ∂D_ε:

$$I := \{z \in \partial D_\varepsilon : -\tfrac{\varepsilon}{2} < \mathrm{Re}(z)\} \quad \text{and}$$
$$J := \{z \in \partial D_\varepsilon : -\tfrac{\varepsilon}{2} < \mathrm{Re}(z) < 0\}\ .$$

Since I is contractible, the restriction of π to $\pi^{-1}(I)\cap L$ is a trivial fiber bundle. In fact, according to Theorem 3.2(c), there is a trivialization:

$$T : \mathscr{L} \times I \xrightarrow{\cong} \pi^{-1}(I)\cap L$$

that takes $(\partial\mathscr{L}\times[0,1))\times I$ to $r(\partial\mathscr{L}\times[0,\tfrac{1}{2})\times I)$.

The visor (and its rim) may then be written:

$$(R,R') \quad := \quad T(\ (\mathscr{L},\partial\mathscr{L}\times[0,1))\times(I,J)\ ) \quad \subset \quad (L,L_c)\ .$$

Excising the smoothly enclosed set $L_c - R$ we obtain:

$$\mathrm{IH}_k(L,L_c) \cong \mathrm{IH}_k(R,R')\ . \tag{3}$$

Step 4. By the Künneth Formula:

$$\mathrm{IH}_k(R,R') \cong \mathrm{IH}_{k-1}(\mathscr{L},\partial\mathscr{L}\times[0,1)) \cong \mathrm{IH}_{k-1}(\mathscr{L},\partial\mathscr{L})\ . \tag{4}$$

Composing the isomorphisms (1)-(4), we obtain the desired isomorphism $\mathrm{IH}_k(C-\{p\},C_{<0}) \cong \mathrm{IH}_{k-1}(\mathscr{L},\partial\mathscr{L})$. A careful analysis shows that this isomorphism is in fact canonical. □

5. THE VARIATION MAP

Now we introduce a homomorphism whose image is the Morse group A_p.

Definition 5.1. We call the composition:

$$\mathrm{IH}_k(\mathscr{L},\partial\mathscr{L}) \overset{\text{Cor. 4.3}}{\cong} \mathrm{IH}_{k+1}(C-\{p\},C_{<0}) \xrightarrow{\partial_*} \mathrm{IH}_k(C_{<0}) \overset{\text{Cor. 4.2}}{\cong} \mathrm{IH}_k(\mathscr{L})$$

the *variation map*, and we denote it $\mathrm{Var} : \mathrm{IH}_k(\mathscr{L},\partial\mathscr{L}) \to \mathrm{IH}_k(\mathscr{L})$.

Equivalently, the variation map may be constructed geometrically as follows. Restrict π to $\pi^{-1}(\partial D_\varepsilon)\cap C$. This fiber bundle is classified by a *monodromy homeomorphism* $\mu : \mathscr{L} \to \mathscr{L}$ (which, together with an orientation for the base circle ∂D_ε, describes how to recover the fiber bundle from $\mathscr{L}\times[0,1]\to[0,1]$ by identifying $(q,0)$ with $(\mu(q),1)$ for $q\in\mathscr{L}$).

By Theorem 3.2(c), this monodromy homeomorphism may be chosen to be constant in some neighborhood of $\partial\mathscr{L}$. Therefore if $[c]\in\mathrm{IH}_k(\mathscr{L},\partial\mathscr{L})$ then the chain $c-\mu(c)$ determines an element of $\mathrm{IH}_k(\mathscr{L})$—and this element is in fact

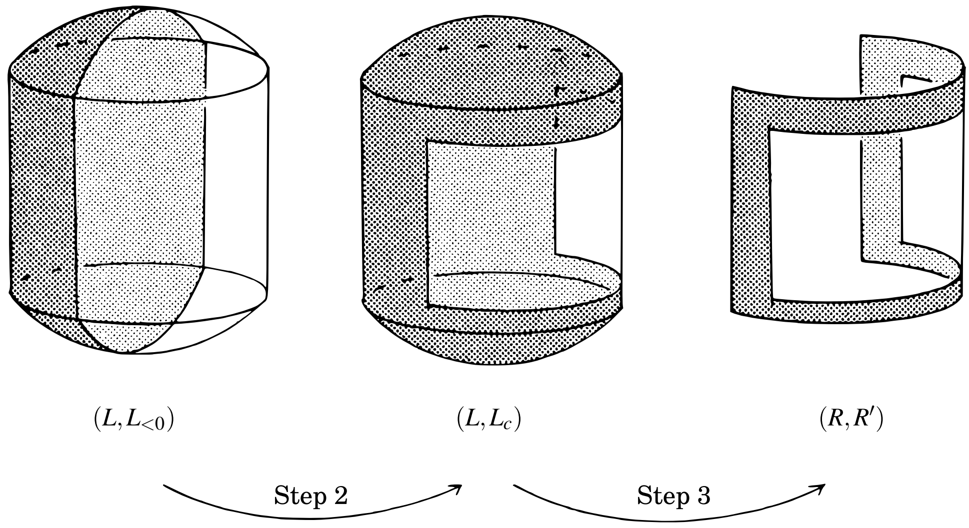

FIGURE 6. Steps 2 and 3 in the proof of Corollary 4.3. [GM83b]

Var($[c]$) as defined above. (The reader can convince himself of this by recalling the geometric steps in the proofs of Corollaries 4.3 and 4.2.)

6. PROOF OF THEOREM C'

We now prove Theorem C' by induction: to prove Theorem C' when X has dimension n, we apply Theorem C' to the complex link $\mathscr{L}$, which has dimension strictly less than n. To be completely explicit, we prove the following theorem inductively for $n \geq 0$.

Theorem C'$_n$. *Let X be a purely n-dimensional complex analytic variety (analytically embedded in some ambient complex analytic manifold M) and let $f : X \to \mathbb{R}$ be a proper Morse function. Suppose p is a critical point of index $i = c + \lambda$, where c is the complex codimension of the stratum S containing p and λ is the index of $f|S$. Then:*

$$\mathrm{IH}_k(X_{\leq f(p)+\varepsilon}, X_{\leq f(p)-\varepsilon}) = \begin{cases} A_p & \text{for } k = i \\ 0 & \text{for } k \neq i \end{cases}$$

for sufficiently small $\varepsilon > 0$.

The Morse group A_p is isomorphic to the image of the variation map:

$$\mathrm{Var} : \mathrm{IH}_{c-1}(\mathscr{L}, \partial\mathscr{L}) \to \mathrm{IH}_{c-1}(\mathscr{L})$$

where $(\mathscr{L}, \partial\mathscr{L})$ is the complex link (and its boundary) associated to the stratum S containing p.

Proposition 6.1. *If Theorem C'$_c$ holds for $c < n$, then it also holds for $c = n$.*

Proof. Let X, M, f, p, S, i, c, and λ be as in the statement of Theorem C'$_n$.

Let N_δ be the normal slice of X at p. Note that N_δ has pure complex dimension c (because X has pure dimension n). Corollary 2.4 says that:

$$\mathrm{IH}_k(X_{\leq f(p)+\varepsilon}, X_{\leq f(p)-\varepsilon}) \cong \mathrm{IH}_{k-\lambda}\underbrace{((N_\delta)_{\leq f(p)+\varepsilon}, (N_\delta)_{\leq f(p)-\varepsilon})}_{\text{normal Morse data}}$$

for sufficiently small $1 \gg \delta \gg \varepsilon > 0$.

If δ is small enough then we can use local coordinates in M to regard N_δ as sitting directly in $\mathbb{C}^m$. Let $\pi : \mathbb{C}^m \to \mathbb{C}$ be a linear projection whose real part $\mathrm{Re}(\pi)$ restricts to a Morse function on N_δ. (Such projections exist by the results of Pignoni cited earlier.)

The last part of Theorem B' says that the normal Morse data for f and $\mathrm{Re}(\pi)$ are homeomorphic.[3] Composing the resulting isomorphism on intersection homology groups with the isomorphism provided by Corollary 4.1 gives an isomorphism:

$$\mathrm{IH}_{k-\lambda}\big(\underbrace{(N_\delta)_{\le f(p)+\varepsilon}, (N_\delta)_{\le f(p)-\varepsilon}}_{\textit{normal Morse data}}\big) \cong \mathrm{IH}_{k-\lambda}(C, C_{<0}) ,$$

where C is the cylindrical neighborhood constructed from π.

To compute the groups $\mathrm{IH}_k(C, C_{<0})$ we examine the triple of spaces:

$$C \supset C - \{p\} \supset C_{<0} ,$$

which we denote $Y \supset A \supset B$ in the following diagrams for simplicity.

First, write the long exact sequence for this triple as a sweeping wave:

$$\begin{tikzcd}[column sep=small]
{} \arrow[rr, bend left] & & \mathrm{IH}_k(A,B) \arrow[dr] & & & & & & \mathrm{IH}_{k-1}(Y,A) \arrow[rr, bend left] & & {} \\
& & & \mathrm{IH}_k(Y,B) \arrow[dr] & & & & \mathrm{IH}_{k-1}(Y,B) \arrow[ur] & & & \\
& & & & \mathrm{IH}_k(Y,A) \arrow[rr, bend right] & & \mathrm{IH}_{k-1}(A,B) \arrow[ur] & & & &
\end{tikzcd}$$

Superpose the long exact sequence for the pair $(Y,B) = (C, C_{<0})$:

$$\begin{tikzcd}[column sep=small]
{} \arrow[rr, bend left] & & \mathrm{IH}_k(A,B) \arrow[dr] & & \mathrm{IH}_{k-1}(B) \arrow[rr, bend left] & & \mathrm{IH}_{k-1}(Y) \arrow[dr] & & \mathrm{IH}_{k-1}(Y,A) \arrow[rr, bend left] & & {} \\
& & & \mathrm{IH}_k(Y,B) \arrow[ur] \arrow[dr] & & & & \mathrm{IH}_{k-1}(Y,B) \arrow[ur] \arrow[dr] & & & \\
{} \arrow[rr, bend right] & & \mathrm{IH}_k(Y) \arrow[ur] & & \mathrm{IH}_k(Y,A) \arrow[rr, bend right] & & \mathrm{IH}_{k-1}(A,B) \arrow[ur] & & \mathrm{IH}_{k-1}(B) \arrow[rr, bend right] & & {}
\end{tikzcd}$$

Similarly superpose the long exact sequences for the pairs $(Y,A) = (C, C - \{p\})$ and $(A,B) = (C - \{p\}, C_{<0})$ to obtain a commuting cascade of exact sequences:

[3]The homeomorphism can be made canonical by choosing a projection π with covector $d(\mathrm{Re}(\pi))(p)$ sufficiently close to $df(p)$.

$$
\begin{tikzcd}[column sep=small]
& \mathrm{IH}_k(A,B) \arrow[dr] \arrow[rr, bend left] & & \mathrm{IH}_{k-1}(B) \arrow[dr] \arrow[rr, bend left] & & \mathrm{IH}_{k-1}(Y) \arrow[dr] \arrow[rr, bend left] & & \mathrm{IH}_{k-1}(Y,A) \arrow[dr] & \\
{} \arrow[ur] \arrow[dr] & & \mathrm{IH}_k(Y,B) \arrow[ur] \arrow[dr] & & \mathrm{IH}_{k-1}(A) \arrow[ur] \arrow[dr] & & \mathrm{IH}_{k-1}(Y,B) \arrow[ur] \arrow[dr] & & {} \\
& \mathrm{IH}_k(Y) \arrow[ur] \arrow[rr, bend right] & & \mathrm{IH}_k(Y,A) \arrow[ur] \arrow[rr, bend right] & & \mathrm{IH}_{k-1}(A,B) \arrow[ur] \arrow[rr, bend right] & & \mathrm{IH}_{k-1}(B) \arrow[ur] &
\end{tikzcd}
$$

We prove the proposition by examining the $\overset{\frown}{\mathrm{X}}$-shaped subdiagram of the cascade centered at $\mathrm{IH}_k(Y,B) = \mathrm{IH}_k(C, C_{<0})$ for each k:

$$
\begin{tikzcd}
\mathrm{IH}_k(C-\{p\}, C_{<0}) \arrow[dr] \arrow[rr, bend left] & & \mathrm{IH}_{k-1}(C_{<0}) \\
& \mathrm{IH}_k(C, C_{<0}) \arrow[ur] \arrow[dr] & \\
\mathrm{IH}_k(C) \arrow[ur] & & \mathrm{IH}_k(C, C-\{p\})
\end{tikzcd}
$$

Composing with the isomorphisms provided by Corollaries 4.3 and 4.2, we obtain the exact commuting diagram:

$$
(*) \qquad
\begin{tikzcd}
\mathrm{IH}_{k-1}(\mathscr{L}, \partial\mathscr{L}) \arrow[dr] \arrow[rr, bend left, "\mathrm{Var}"] & & \mathrm{IH}_{k-1}(\mathscr{L}) \\
& \mathrm{IH}_k(C, C_{<0}) \arrow[ur] \arrow[dr] & \\
\mathrm{IH}_k(C) \arrow[ur] & & \mathrm{IH}_k(C, C-\{p\})
\end{tikzcd}
$$

Local calculation (cf the Appendix) shows that:

$$
\begin{aligned}
\mathrm{IH}_k(C) = 0 \ \text{ for } k \geq c \quad \textit{and} \\
\mathrm{IH}_k(C, C-\{p\}) = 0 \ \text{ for } k \leq c \ .
\end{aligned}
$$

Therefore when $k = c$ both groups at the base of the diagram vanish, and exactness gives a commuting triangle:

$$
\begin{tikzcd}
\mathrm{IH}_{k-1}(\mathscr{L}, \partial\mathscr{L}) \arrow[dr, "\text{surjective}"'] \arrow[rr, bend left, "\mathrm{Var}"] & & \mathrm{IH}_{k-1}(\mathscr{L}) \\
& \mathrm{IH}_k(C, C_{<0}) \arrow[ur, "\text{injective}"'] &
\end{tikzcd}
$$

By commutativity, the injection $\nearrow$ maps onto the image of Var. At the same time this injection cannot hit anything but the image of Var, since the homomorphism $\searrow$ is surjective. Therefore the injection $\nearrow$ in fact gives an isomorphism from the Morse group to the image of Var, as desired.

It remains to show that all other groups $\mathrm{IH}_k(Y,B) = \mathrm{IH}_k(C,C_{<0})$ vanish. We do this by showing that opposing pairs of groups in the diagram $(*)$ always vanish when $k \neq c$. In light of the local calculation above, it suffices to prove the following lemma.

Lemma.

$$\mathrm{IH}_{k-1}(\mathscr{L}) = 0 \ \ \textit{for } k > c \quad \text{and}$$
$$\mathrm{IH}_{k-1}(\mathscr{L},\partial\mathscr{L}) = 0 \ \ \textit{for } k < c\,.$$

Proof. We apply Theorem $\mathrm{C}'_{\dim_{\mathbb{C}}\mathscr{L}}$ to the complex link $\mathscr{L}$. Recall that the normal slice N_δ has pure complex dimension $c \leq n$, so $\mathscr{L}$ has pure complex dimension $c-1 < n$. Thus, our inductive hypothesis makes Theorem $\mathrm{C}'_{\dim_{\mathbb{C}}\mathscr{L}}$ available.

In what follows, let $\mathscr{L}$ denote the complex link over a fixed $\xi \in D_\varepsilon - \{0\}$.

For the first conclusion, choose a point p' very close to p so that the proper function $g : \mathbb{C}^m \to \mathbb{R}$ defined by:

$$g(q) := \|p'-q\|^2$$

restricts to a Morse function on $\mathscr{L}$. Just as in the proof of the Lefschetz hyperplane theorem (see [Mil63]), we conclude that if A is a stratum of $\mathscr{L}$, then the index of $g|A$ at any critical point is at most the *complex* dimension of A (ie at most *half* its real dimension). Theorem C'_{c-1} applied to g therefore implies that for each critical value v:

$$\mathrm{IH}_k(\mathscr{L}_{\leq v+\varepsilon'},\mathscr{L}_{\leq v-\varepsilon'}) = 0 \ \text{ for } k > \dim_{\mathbb{C}}A + \mathrm{cod}_{\mathbb{C}}A = c-1$$

provided $\varepsilon' > 0$ is sufficiently small. (Here the symbols "${}_{\leq v+\varepsilon'}$" etc of course refer to the Morse function g, not f.)

By induction on the critical values v, using long exact sequences of pairs at each step, we conclude that:

$$\mathrm{IH}_{k-1}(\mathscr{L}) = \mathrm{IH}_{k-1}(\mathscr{L}_{\leq \delta+\varepsilon'}) = \mathrm{IH}_{k-1}(\mathscr{L}_{\leq 0-\varepsilon'}) = \mathrm{IH}_{k-1}(\emptyset) = 0 \ \text{ for } k > c$$

as desired.

The second conclusion is dual to the first. Consider the Morse function $-g : \mathscr{L} \to \mathbb{R}$. If the reader stands on his head, he will see that if A is a stratum of $\mathscr{L}$, then the index of $(-g)|A$ at any critical point is *at least* the complex dimension of A. Theorem C'_{c-1} applied to $-g$ therefore implies that for each critical value v:

$$\mathrm{IH}_k(\mathscr{L}_{\leq v+\varepsilon'},\mathscr{L}_{\leq v-\varepsilon'}) = 0 \ \text{ for all } k < \dim_{\mathbb{C}}A + \mathrm{cod}_{\mathbb{C}}A = c-1$$

provided $\varepsilon' > 0$ is sufficiently small. (Here the symbols "${}_{\leq v+\varepsilon'}$" etc refer to the Morse function $-g$, not f or g.)

By induction on the critical values $v_1 < v_2 < \cdots < v_r$, using long exact sequences of triples at each step, we conclude that:

$$\begin{aligned} 0 &= \mathrm{IH}_{k-1}(\mathscr{L}_{\leq v_1+\varepsilon'},\mathscr{L}_{\leq v_1-\varepsilon'}) \\ &= \mathrm{IH}_{k-1}(\mathscr{L}_{\leq v_2+\varepsilon'},\mathscr{L}_{\leq v_1-\varepsilon'}) \\ &\ \vdots \\ &= \mathrm{IH}_{k-1}(\mathscr{L}_{\leq v_r+\varepsilon'},\mathscr{L}_{\leq v_1-\varepsilon'}) \ \text{ for } k < c \end{aligned}$$

provided $\varepsilon' > 0$ is sufficiently small. By Theorem A', the latter group is isomorphic to $\mathrm{IH}_{k-1}(\mathscr{L}, \partial\mathscr{L})$ (provided p is so close to p' that $\mathscr{L}_{=v}$ lies in the collaring of $\partial\mathscr{L}$ for some $v < v_1$). □

This completes the proof of Proposition 6.1 and hence also Theorem C'. □

7. Complex Algebraic Curves

Complex algebraic curves are beautiful examples where the Morse group A_p can be computed intuitively.

Let $C \subset \mathbb{C}^2$ be the complex curve cut out by a nonconstant polynomial $P(w,z)$ with complex coefficients, ie $C = P^{-1}(0)$. Assume without loss of generality that P has no repeated factors. Let $\{p\}$ be a zero-dimensional stratum of C.

A classical theorem (see for example [Mil68] §3.3) says that p has a neighborhood consisting of finitely many *branches* $B_1, \ldots, B_b$ that meet only at the point p. Topologically, each branch is a 2-disc, but within $\mathbb{C}^2$ they may "twist around" p several times and fail at to be smooth submanifolds of $\mathbb{C}^2$ at p. The *multiplicity* of C at p measures this twisting; it is the least integer m such that, for some $i + j = m$:

$$\frac{\partial^m P}{\partial w^i \partial z^j}(p) \neq 0 \, .$$

Note that $m = 1$ if and only if C is a smooth submanifold of $\mathbb{C}^2$ near p.

It can be shown that the complex link of C at p consists of precisely m points. (For the proof of a more general fact, see [Mil68] §7.) Each branch contains at least one of these points, and the monodromy homeomorphism permutes the points within each branch cyclically (the boundary of each branch is a circle). In this way, the variation map restricted to the (zero-dimensional) intersection homology of each branch has 1-dimensional kernel $\{(k,k,\ldots,k) : k \in \mathbb{Z}\}$. As a result, the variation map has total rank $m - b$, and so the Morse group A_p is isomorphic to $\mathbb{Z}^{m-b}$.

Examples. Consider the curve cut out by the polynomial $P(w,z) = wz$. It has two branches at the origin corresponding to the factors w and z. Its multiplicity at the origin is 2 (because $\partial^2 P/\partial w \partial z = 1$ and all lower derivatives vanish at the origin). So the complex link consists of two points, one in each branch. The monodromy homeomorphism fixes each point, so the variation map has total rank 0, and the Morse group A_0 is 0.

At the other extreme is the "cusp" cut out by the polynomial $Q(w,z) = w^2 - z^3$. Its multiplicity at the origin is 2, but since Q is irreducible there is only one branch. (So the origin is not singular from the topological viewpoint—just from the differential viewpoint.) So the complex link consists of two points, both in the same branch. The monodromy homeomorphism exchanges these points, so the variation map has total rank 1, and the Morse group A_0 is $\mathbb{Z}$.

Synthesizing these examples, we see that the complex link at the origin of the curve cut out by the polynomial $R(w,z) = w(w^2 - z^3)(w^4 - z^5)$ consists of 7 points, distributed in 3 branches. The monodromy homeomorphism fixes the first point, exchanges the next two, and permutes the last four in a cycle. In this way, the variation map has total rank 4, and the Morse group A_0 is $\mathbb{Z}^4$.

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APPENDIX A. INTERSECTION HOMOLOGY

Let Y be a (not necessarily compact) purely n-dimensional complex analytic variety (analytically embedded in some ambient complex analytic manifold M) with a fixed Whitney stratification. A theorem (see [Gor78]) says that Y admits a triangulation that is compatible with its stratification. Let $(\mathrm{C}_k(Y), \partial)$ denote the complex of simplicial chains on Y with $\mathbb{Z}$ coefficients.

Definition A.1. The *intersection homology of* Y, denoted $\mathrm{IH}_*(Y)$, is the homology of the sub-chain-complex:

$$\begin{aligned}\mathrm{IC}_k(Y) := \{c \in \mathrm{C}_k(Y) : &\dim(\,|c|\,\cap S) < k - c \text{ and} \\ &\dim(|\partial c| \cap S) < k - c - 1 \text{ for each stratum } S \\ &\text{of complex codimension } c > 0\}\,.\end{aligned}$$

(The second condition ensures that ∂ restricts to this sub-chain-complex.)

Goresky and MacPherson [GM80] introduced intersection homology in order to extend Poincaré Duality to stratified spaces; for Y compact and oriented they constructed an intersection pairing:

$$IH_i(Y) \times IH_{n-i}(Y) \to H_0(Y) \to \mathbb{Z}$$

which is nondegenerate when tensored with $\mathbb{Q}$.

Note that although we defined intersection homology with reference to a fixed stratification, it is independent of choice of stratification. In fact it is a topological invariant,[4] but *not* a homotopy invariant. Intersection homology thus defies the most basic stereotype of a homology theory. (It also lacks induced maps in general—although large classes of maps, such as so-called placid maps, do induce homomorphisms.)

Nevertheless, intersection homology has the following familiar properties.

Relative Groups. If $U \subset Y$ is open then U inherits a stratification from Y, and relative intersection homology $\mathrm{IH}_k(Y,U)$ can be defined just as for ordinary homology. These relative groups fit into the familiar long exact sequence for pairs. Furthermore, if $V \subset U$ is open, then the resulting relative groups fit into the familiar long exact sequence for triples.

More generally, $\mathrm{IH}_k(Y,U)$ can be defined if $U \subset Y$ is *smoothly enclosed*—this means that $U = Y \cap M'$, where M' is a complex analytic submanifold of M of $\dim M' = \dim M$ whose boundary $\partial M'$ intersects each stratum of Y transversely.[5] If $V \subset U$ is smoothly enclosed, then the resulting relative groups again fit into the familiar long exact sequences for pairs and triples.

If Y has a collared boundary ∂Y, then the collaring is smoothly enclosed and gives rise to relative groups $\mathrm{IH}_k(Y, \partial Y)$.

Excision. If $V \subset U^\circ$ is smoothly enclosed, then the inclusion:

$$(Y - V, U - V) \hookrightarrow (Y, U)$$

induces an isomorphism on intersection homology.

---

[4]To prove topological invariance, Goresky and MacPherson [GM83a] (following a suggestion of Deligne and Verdier) recast their work in sheaf-theoretic language; they defined a differential complex of sheaves $\mathbf{IC}^\bullet(Y)$ whose hypercohomology groups are the intersection homology groups of Y.

[5]In this way, if $f : M \to \mathbb{R}$ restricts to a Morse function on X, then the relative groups $\mathrm{IH}_k(X_{\leq v+\varepsilon}, X_{\leq v-\varepsilon})$ are defined since $X_{\leq v-\varepsilon} = X \cap f^{-1}(-\infty, v-\varepsilon]$ is smoothly enclosed, the boundary $f^{-1}(v-\varepsilon)$ being the pre-image of a regular value and thus a submanifold of M transverse to each stratum of X.

Künneth Formula. Let $(D^a, \partial D^a)$ denote the (closed) a-dimensional disk and its boundary sphere. Then for all k there are canonical isomorphisms:

$$\mathrm{IH}_k(Y,U) \cong \mathrm{IH}_k(\ (Y,U) \times D^a\ ) \text{ and}$$
$$\mathrm{IH}_k(Y,U) \cong \mathrm{IH}_{k-a}(\ (Y,U) \times (D^a, \partial D^a)\ )\ .$$

Local Calculation. This property highlights an essential difference between intersection homology and ordinary (homotopy invariant) homology theories.

Suppose a point $p \in Y$ lies in a stratum S of complex codimension c. Then S has real dimension $a = 2(n-c)$. A consequence of Whitney's conditions (cf Definition 2.2 on p. 3) is that p has a neighborhood:

$$U \cong D^a \times \mathrm{Cone}(L)$$

where L is the boundary of (the closure of) a sufficiently small normal slice to S at p.

Local calculation states that:

$$\mathrm{IH}_k(U) = \begin{cases} 0 & \text{for } k \geq c \\ \mathrm{IH}_k(L) & \text{for } k < c \end{cases}$$

and dually that:

$$\mathrm{IH}_k(U, \partial U) = \mathrm{IH}_k(U, U - \{p\}) = \begin{cases} \widetilde{\mathrm{IH}}_{k-a-1}(L) & \text{for } k \geq 2n-c+1 \\ 0 & \text{for } k < 2n-c+1 \end{cases}$$

where reduced intersection homology $\widetilde{\mathrm{IH}}_k(Y)$ is defined as $\mathrm{IH}_k(Y, y_0)$ where y_0 is a point in the top-dimensional stratum of Y.

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