TOPOLOGY OF HYPERSURFACE SINGULARITIES

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Abstract. Kähler's paper *Über die Verzweigung einer algebraischen Funktion zweier Veränderlichen in der Umgebung einer singulären Stelle*" offered a more perceptual view of the link of a complex plane curve singularity than that provided shortly before by Brauner. Kähler's innovation of using a "square sphere" became standard in the toolkit of later researchers on singularities. We describe his contribution and survey developments since then, including a brief discussion of the topology of isolated hypersurface singularities in higher dimension.

1. Topology of plane curve singularities

The Riemann surface of an algebraic function on the plane represents a complex curve (real dimension 2) as a covering of the Riemann sphere, ramified over some finite collection of points. At the start of the 20th century, the study of complex surfaces (real dimension 4) was rapidly developing, and they too were often studied as "Riemann surfaces," — now of algebraic functions on the complex plane. The branching of such a "Riemann surface" is along a complex curve, and the only difficult case in understanding the local topology of this branching is at a singularity of the curve. The problem therefore arose, to understand the topology of a complex plane curve C near a singular point.

The first discussion of this appears to be in Heegaard's 1898 thesis [16] (see Epple [13, 14]). A small ball B around the singular point will intersect the curve C in a set that is homeomorphic to the cone on $C \cap \partial B$. This set $C \cap \partial B$, which is a link (disjoint union of embedded circles) in the 3-sphere ∂B, therefore determines the local topology completely. It thus suffices to understand the links that arise this way: *links of plane curve singularities*, as they are now called.

To understand the local branching of the "Riemann surface" one also needs the fundamental group of the complement of the link in the 3-sphere. The first comprehensive article on this topic is the 1928 paper [3] by Karl Brauner, who writes that he learned the problem from Wirtinger, who had spoken on it to the Mathematikervereinigung in Meran in 1905 and subsequently held a seminar on the topic in Vienna. In his paper Brauner follows Heegaard in using stereographic projection to move the link from S^3 to $\mathbb{R}^3$. He then describes the topology of the link in terms of repeated cabling, and gives an explicit presentation of the fundamental group of the complement of the link.

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Brauner's exposition is complicated by the stereographic projection, and Erich Kähler revisits the question in the article "*Über die Verzweigung einer algebraischen Funktion zweier Veränderlichen in der Umgebung einer singulären Stelle*" [20], promising and providing a more perceptual view than Brauner's. After choosing local coordinates centered at the singular point, he replaces the round sphere ∂B by a "rectangular sphere"

$$\{(x,y)\in\mathbb{C}^2 : |x|=\epsilon, |y|\le\delta \text{ or } |x|\le\epsilon, |y|=\delta\} = \partial(D^2(\epsilon)\times D^2(\delta)).$$

By choosing the coordinates suitably he also arranges that the curve meets this rectangular sphere only in the portion $|x|=\epsilon$. Since this portion is a solid torus that can be identified with a standard solid torus in $\mathbb{R}^3$, this makes the topology easier to visualize. Nowadays his technique is used routinely, but it is reasonable to guess that the timespan between Wirtinger's seminar and a general description of the topology was in part due to the lack of an easy visualization technique.

To be more specific about Kähler's approach, suppose our curve is given in local coordinates by an equation $f(x,y)=0$. Newton had already pointed out long before that one can give approximate solutions to this equation, giving y in terms of fractional powers of x. Assuming the y-axis is not tangent to the curve at $(0,0)$, Newton's successive approximations have the form

$$\begin{aligned}
y &= a_1 x^{\frac{q_1}{p_1}}\\
y &= x^{\frac{q_1}{p_1}}(a_1 + a_2 x^{\frac{q_2}{p_1p_2}})\\
y &= x^{\frac{q_1}{p_1}}(a_1 + x^{\frac{q_2}{p_1p_2}}(a_2 + a_3 x^{\frac{q_2}{p_1p_2p_3}}))\\
&\;\;\vdots
\end{aligned}$$

with p_i and q_i relatively prime positive integers. There may be several solutions of this type near $(x,y)=(0,0)$, corresponding to different branches of the curve at $(0,0)$. The fact that the curve is not tangent to the y-axis implies that $q_1\ge p_1$ for each branch, and by choosing δ and ϵ suitably (they should be small, and δ/ϵ should exceed the absolute value of the coefficient a_1 for each branch with $q_1=p_1$), one arranges that the curve intersects only the solid torus $x=\epsilon$ of Kähler's rectangular sphere.

It is now easy to see that the first Newton approximation gives a link that is a (p_1,q_1) torus knot, represented by a closed braid with p_1 strands in this solid torus. The next approximation replaces this by the (p_2,q_2)-cable[1] on this knot, represented by a p_1p_2-strand braid, and so on. Thus each branch of the curve leads to a component of the link that is an iterated cabling on a torus knot. Such a link is called an *iterated torus link*.

Kähler actually used the more familiar expression of Newton's approximations as the partial sums of a fractional power series solution

$$y = b_1 x^{\frac{r_1}{p_1}} + b_2 x^{\frac{r_2}{p_1p_2}} + b_3 x^{\frac{r_3}{p_1p_2p_3}} + \dots$$

to $f(x,y)=0$, which had been introduced by Puiseux in the mid-nineteenth century. The pairs (p_i,r_i) occurring in the exponents of the Puiseux series are called *Puiseux*

[1]When talking of a (p,q) cable on a knot, q is only well defined after choosing a framing of the knot, that is, a choice of a parallel copy to call the $(1,0)$ cable. The framing we are using here is the "naive" framing of a cable knot, determined by choosing the parallel copy on the same torus that the cable knot naturally lies on. We return to the framing issue later.

pairs. They of course determine and are determined by the *Newton pairs* (p_i, q_i). The precise inductive relationship is $q_1 = r_1$, $q_i = r_i - p_i r_{i-1}$.

Not all Puiseux pairs are topologically significant[2]: a pair with $p_i = 1$ does not contribute to the topology of its link component, since it represents a $(1, q)$ cabling for some q, which simply replaces a knot by a parallel copy of itself. However this pair may, nevertheless, be topologically significant, in that it *can* contribute to the linking of different link components with each other. Thus care must be taken in attempting to retain only the topologically significant data. Kähler satisfies himself with describing typical cases that must be considered in an iterative understanding of the topology and fundamental group of any given example, but he gave no general solution to this issue, writing: "Es soll uns jedoch genügen, an den vorstehenden bereits sehr allgemeinen Beispielen die merkwürdigen Verzweigungsverhältnisse der Funktionen mehrerer Variablen dargetan zu haben" (It should suffice to have presented the remarkable branching behavior of functions of several variables by these already very general examples).

Although Kähler's presentation indeed provides the techniques to deal with any particular example, it gives no explicit closed form encapsulation of the topology. A question that was therefore addressed by many later authors was:

Question. *What invariant or collection of invariants completely determines the topology of a plane curve singularity?*

The implicit answer of Kähler's paper is simply to retain relevant parts of the Puiseux expansions for each branch, where "relevant" can be taken to mean: whenever two branches have identical Puiseux expansions up to some point, include the final term where they agree, and otherwise include only those terms that are topologically relevant to a branch.

A classical notion of equivalence of plane curve singularities, which, with hindsight, is the same as topological equivalence, is based on the the tree of infinitely near points, or, what is essentially equivalent, the resolution diagram (see [12]). Various characterizations of this equivalence are given in Zariski's investigation of equisingularity [34]. A classical characterization, according to Reeve [26], is that an equivalence class is determined by the sequence of characteristic Puiseux pairs for each branch and the pairwise intersection numbers of the branches (for a proof see [35] or [22]). Reeve shows that these intersection numbers are the linking numbers of the corresponding components of the link of the singularity. Thus:

Theorem 1.1. *The link of a plane curve singularity is determined by the sequences of characteristic Puiseux pairs of the individual components, and their pairwise linking numbers.*

This presupposes agreement of classical and topological equivalence, first proved for one branch in 1932 by Burau [5] and Zariski [33] independently, and then for two branches in 1934 by Burau [6], who points out that the general case follows. The connection with classical equivalence is not explicit in [6], but was presumably understood. It is explicit in Reeve's exposition. Both Burau and Zariski recover the Puiseux data for the link from its Alexander polynomial, and they use that the

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[2] The terminology *characteristic pair* is often used to single out the topologically significant Puiseux pairs, although this is with hindsight; the characteristic pairs were originally singled out for geometric reasons.

link is the link of a plane curve singularity. In 1953 H. Schubert showed that one can unravel the cabling numbers from the topology of any cabled link [27].

Generalizing the work of Burau and Zariski, Evers [15] and Yamamoto [32] independently showed:

Theorem 1.2. *The multi-variable Alexander polynomial of the link of a plane curve singularity is a complete invariant for its topology.*

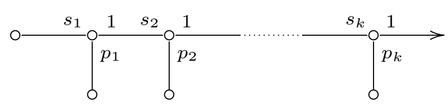
R. Waldi, in his Regensburg dissertation [31] showed:

Theorem 1.3. *The value semigroup of a plane curve singularity is a complete invariant for its topology.*

Attractive as the above results are, they are not entirely satisfactory as an encoding of the topology: each does so in terms of a redundant set of data from which other useful invariants are not necessarily easy to compute.

In the 1970's there was a revolution in 3-manifold topology, brought in part[3] by the JSJ decomposition theorem for 3-manifolds (foreseen by Waldhausen [30] in a little-noticed paper, and proved by Jaco-Shalen [18] and Johannson [19]). In particular, this canonical decomposition of any 3-manifold provides a general framework for (and radical generalization of) Schubert's results for links mentioned above, and hence a new view of the fact that classical and topological equivalence of plane curve singularities are the same.

In the early 1980's Eisenbud and the author used JSJ decomposition to provide a new combinatorial encoding of the topology of a plane curve singularity: the *splice diagrams* of [11] (adapted[4] from a concept due to Siebenmann [28]). On a superficial level, the splice diagram is just another way of encoding the cabling information, i.e., the Puiseux data. The Puiseux pairs are replaced by new pairs that have global topological meaning. For instance a single branch with Puiseux pairs $(p_1, r_1), (p_2, r_2), \dots, (p_k, r_k)$ is encoded by a splice diagram



with $s_1 = r_1$ and, for $i \geq 1$, $s_{i+1} = r_{i+1} - r_i p_{i+1} + p_i p_{i+1} s_i$. The pairs (p_i, s_i) describe the repeated cabling in terms of the natural topological framings of knots[5]. In particular, they do not change under coordinate change or when topologically irrelevant pairs are omitted.

In this splice diagram the arrowhead represents the component of the link and the weights along and adjacent to the path to this arrowhead give the sequence of cabling pairs.

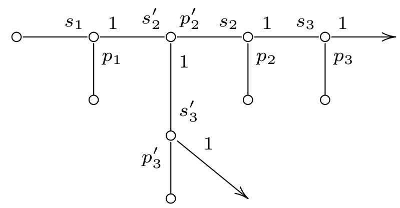
To understand the placement issue for more than one branch, suppose, for example, that $k = 3$ above, so there are just three characteristic pairs. Suppose also that our curve has a second branch whose first pair is also (p_1, s_1) but whose later pairs are different, say $(p_2', s_2'), (p_3', s_3')$. Assume also that $\frac{p_2'}{s_2'}$ is the larger of $\frac{p_2}{s_2}$ and

[3]The other part was Thurston's geometrization conjecture, which is, however, irrelevant to the 3-manifolds that arise in algebraic geometry.

[4]Actually, a case of convergent evolution.

[5]This is the framing in which a parallel copy of a knot has zero linking number with the knot.

$\frac{p'_2}{s'_2}$. The splice diagram may then be



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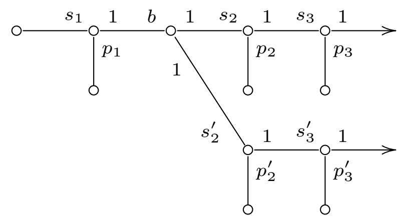

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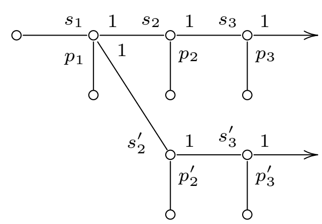

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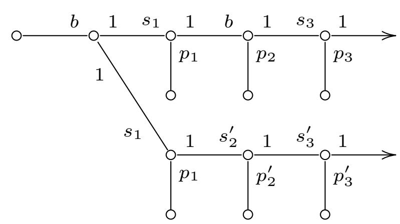

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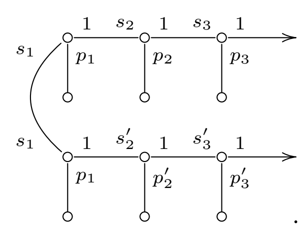

There are conditions on splice diagrams that are necessary and sufficient for a diagram to occur for the link of some plane curve singularity (and be the unique smallest diagram representing its topology). The edge weights around any node (vertex of valence > 1) are pairwise coprime. Modification procedures on a splice diagram can sometimes create an edge weight adjacent to a leaf (vertex of valence 1), but if this happens the weight should just be deleted. The other conditions are:

- all edge weights are positive;

- each *edge determinant* is positive (the edge determinant, defined for each edge connecting two nodes, is the product of the two weights on the edge minus the product of the weights directly adjacent to the edge);
- An edge to a leaf should not have weight 1 (if it does, remove the edge);
- no vertex should have valence 2 (eliminate such a vertex by replacing it and its two adjacent edges by a single edge);
- if all arrowheads are replaced by vertices, the diagram should collapse to a single vertex or single edge using the moves just described.

The linking number of two link components is particularly easy to compute in terms of the splice diagram: it is the product of the weights adjacent to but not on the path that connects the corresponding arrowheads. For example, in the first of the above diagrams it is $s_2'p_2p_3p_3'$. The positive edge determinant condition implies immediately that this linking number is strictly decreasing as one runs through the five possible placements of the two branches in the above example. It is easy to turn this into a quick general splice diagram proof of Theorem 1.1.

With different conditions, splice diagrams can encode many other objects of interest. For example, the last condition above just assures that we are looking at a point of a curve at a non-singular point of a surface. One advantage of splice diagrams is that invariants such as fundamental group, Alexander polynomial (single-variable and multi-variable), Milnor fiber, value semigroup, etc., can be computed quite easily and uniformly from the splice diagram in any situation where the invariant makes sense. Such situations include the study of the global topology of plane curves (work of Neumann, Neumann and Norbury, Pierrette Cassou-Noguès, and others; here the rôle of Milnor fiber is played by the generic fiber of the defining polynomial of the curve), and the study of surface singularities with homology sphere links. Recently, splice diagrams in which the coprimality condition is relaxed have been used in the study of universal abelian covers of surface singularities (Neumann and Wahl).

We have surveyed here the *topology* of plane curve singularities and intentionally not ventured into the large and active literature on algebraic/analytic aspects such as deformation and moduli spaces, curves over fields of finite characteristic, etc. Even with this restriction we have had to leave much out. The 1981 book [4] of Brieskorn and Knörrer is a delightful and readable survey from ancient times to 1980. But the subject has not stopped there. Very recent papers include A'Campo's beautiful construction of the Milnor fibration from a real morsification [1], and an intriguing and surprising geometric formula for the Alexander polynomial in [17].

2. Topology of hypersurface singularities in higher dimension

The study of the topology of isolated singularities of complex hypersurfaces in higher dimensions received a considerable boost in the late 1960's from Brieskorn's construction of exotic spheres as singularity links (of what are now known as Brieskorn-Pham singularities), and from Milnor's monograph [24]. Durfee [10] gives an excellent history of this period. Milnor's fibration theorem is now a fundamental tool in the subject. It says that the link of an isolated hypersurface singularity is a fibered link (that is, the complement of the link can be fibered over S^1 with fibers which are the interiors of submanifolds of the sphere with the link as boundary). Milnor proved that on the standard sphere $\partial D^{2n}(\epsilon)$ the fibration of the complement of the link is given by $f/|f|$, but this is rarely needed, so we will sketch a version of Milnor's proof that omits this fact, but has some of the spirit of Kähler's paper.

Suppose $f : \mathbb{C}^n \to \mathbb{C}$ is such that $f(0) = 0$ and f has an isolated singularity at $0 \in \mathbb{C}^n$. Then for δ and ϵ sufficiently small, and $\delta << \epsilon$, a vector field argument shows that $D := f^{-1}(D^2(\delta)) \cap D^{2n}(\epsilon)$ is isotopically equivalent to the ball $D^{2n}(\epsilon)$. We can thus consider the link of the singularity in the boundary of this (somewhat twisted) "rectangular ball" D. The function f restricted to ∂D makes the desired fibered structure evident.

The fiber of Milnor's fibration is highly connected, with homology only in its middle dimension. It is a Seifert surface for the link, and the Seifert linking form with respect to it is a natural algebraic invariant. (The Seifert form evaluates linking numbers of cycles in the Seifert surface with cycles in a parallel copy of the surface; it is a non-singular integral bilinear form.) Durfee observed in [8] that work of Levine [23] implies that the Seifert form is a complete invariant for the topology of a link of an isolated hypersurface in $\mathbb{C}^n$ if $n > 3$.

In [11] it was asked if the Seifert form is a complete topological invariant for plane curve singularities ($n = 2$). Counterexamples were found by Du Bois and Michel [7], and used by Artal-Bartolo [2] to give a counterexample also for $n = 3$.

Isolated hypersurface singularities in $\mathbb{C}^3$ thus remain the topologically least well understood. The link is a 3-manifold in S^5. Although one knows what 3-manifolds can be links of isolated surface singularities (by the work of Grauert combined with standard 3-manifold theory—see, e.g., [25]), it is not known which of them occur for hypersurface singularities, and the possible embeddings in S^5 as links of hypersurface singularities are even less understood. If the 3-manifold is S^3 then there is no singularity and the embedding is standard, but other 3-manifolds may have several embeddings as singularity links.

This is not to say that the topology *is* understood in dimensions $n > 3$. The Durfee-Levine theorem says that the Seifert form tells all, but it is unknown what forms L actually occur as Seifert forms of singularity links in dimension n. Some restrictions are known. For example, there is a basis that makes the form L upper triangular with diagonal entries $(-1)^{\frac{n(n-1)}{2}}$ (Durfee [8]). The eigenvalues of L^tL^{-1} are roots of unity with maximal Jordan block size n by the monodromy theorem of Grothendieck and Deligne (see [29] for a slight sharpening). It is conjectured (problem 3.31 of [21], attributed to Durfee [9]) that $(-1)^{\frac{n(n-1)}{2}}(L + L^t)$ always has positive signature. Durfee's original conjecture was only for $n = 2$, but even this is unknown.

Since "suspending" a singularity (replacing the hypersurface $f(x_1, \ldots, x_n) = 0$ by $f(x_1, \ldots, x_n) + x_{n+1}^2 = 0$) just multiplies the Seifert form by $(-1)^n$, if we adjust sign of the Seifert form by replacing L by $(-1)^{\frac{n(n-1)}{2}}L$, then the set of realized forms grows with dimension. This graded set of forms is closed, in a graded sense, under tensor product. It fully describes the topology of all isolated hypersurface singularities in ambient dimensions > 3, but it remains mysterious.

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