1. Sheaves: the lightning tour

1.1. Let R be a commutative ring (with 1). Let X be a topological space. A presheaf of R modules on X is a contravariant functor

$$S : \text{category of open sets and inclusions} \to \text{category of } R\text{-modules}.$$

This is a fancy way to say that $V \subset U$ gives $S(U) \to S(V)$ and $W \subset V \subset U$ gives a commutative diagram

$$\begin{tikzcd} S(U) \arrow[rr] \arrow[dr] & & S(V) \arrow[dl] \\ & S(W) & \end{tikzcd}$$

Also, $U \subset U$ gives $S(U) \to S(U)$ identity and $S(\phi) = 0$. Elements $s \in S(U)$ are called sections of S over U (for reasons that will become clear shortly). If $V \subset U$ and $s \in S(U)$, its image in $S(V)$ is denoted $s|V$ and is called the restriction of the section s to V. A morphism $S \to T$ of presheaves is a natural transformation of functors, that is, a collection of homomorphisms $S(U) \to T(U)$ (for every open set $U \subset X$) which commute with the restriction maps. This defines a category of presheaves of R-modules, and it is an abelian category with kernels and cokernels defined in the obvious manner, for example, the kernel presheaf of a morphism $f : S \to T$ assigns to each open set $U \subset X$ the R-module $\ker(S(U) \to T(U))$.

More generally, for any category $\mathcal{C}$ one may define, in a similar manner, the category of presheaves on X with coefficients in $\mathcal{C}$. If $\mathcal{C}$ is abelian then so is the category of presheaves with coefficients in $\mathcal{C}$.

A presheaf S is a *sheaf* if the following *sheaf axiom* holds: Let $\{U_\alpha\}_{\alpha \in I}$ be any collection of open subsets of X, let $U = \cup_{\alpha\in I} U_\alpha$ and let $s_\alpha \in S(U_\alpha)$ be a collection of sections such that

$$s_\alpha|(U_\alpha \cap U_\beta) = s_\beta|(U_\alpha \cap U_\beta)$$

for all $\alpha, \beta \in I$. Then there exists a unique section $s \in S(U)$ such that $s|U_\alpha = s_\alpha$ for all $\alpha \in I$. In other words, if you have a bunch of sections over open sets that agree on the intersections of the open sets then they patch together in a unique way to give a section over the union of those open sets. [An essential point in this definition is that the index set I may have infinite cardinality.]

The *category of sheaves* is the *full subcategory* of the category of presheaves, whose objects are sheaves. [This means that $\mathrm{Hom}_{\mathrm{Sh}}(S,T) = \mathrm{Hom}_{\mathrm{preSh}}(S,T)$.]

1.2. The *stalk* of a presheaf S at a point $x \in X$ is the R-module

$$S_x = \varinjlim_{U \ni x} S(U)$$

This means, in particular, that for any open set U and for any $x \in U$ there is a canonical mapping $S(U) \to S_x$ which we also refer to as "restriction" and denote by $s \mapsto s|S_x$.

The *leaf space* LS of S is the disjoint union

$$LS = \coprod_{x \in X} S_x \xrightarrow{\pi} X$$

with a topology that is discrete on each S_x and that makes π into a local homeomorphism, namely, each $U^{open} \subset X$ and each $s \in S(U)$ defines an open set

$$U_s = \{(x,t) | x \in U, t \in S_x \text{ and } t = s|S_x\} \subset \pi^{-1}(U) \subset LS.$$

Then $\pi : U_s \to U$ is a homeomorphism.

Let $\Gamma(U, LS)$ be the set of continuous sections of π over U, that is, the set of continuous mappings $h : U \to LS$ such that $\pi h =$ identity. The restriction maps of S are compatible, giving $S(U) \to S_x$ for any $U \ni x$ and therefore any $s \in S(U)$ defines a continuous section $h \in \Gamma(U, LS)$ by setting $h(x) = s|S_x$.

1.3. **Proposition.** (exercise) *The presheaf S is a sheaf if and only if the canonical mapping $S(U) \to \Gamma(U, LS)$ is an isomorphism for every open set $U \subset X$. If S and T are sheaves then there are canonical isomorphisms*

$$\mathrm{Hom}_{Sh}(S,T) \cong \mathrm{Hom}_X(LS, LT) \cong \mathrm{Hom}_{preSh}(S,T)$$

where $\mathrm{Hom}_X(LS, LT)$ denotes the R-module of continuous mappings $LS \to LT$ that commute with the projection to X and that consist of R-module homomorphisms $S_x \to T_x$ for all $x \in X$.

1.4. An immediate consequence is that if S is a presheaf then we obtain a sheaf $\widehat{S}$ by defining

$$\widehat{S}(U) = \Gamma(U, LS)$$

to be the R-module of continuous sections of the leaf space of S over the open set U. Then $\widehat{S}$ is called the *sheafification* of S. The *category of sheaves* is the full subcategory of the category of presheaves whose objects satisfy the above sheaf axiom, in other words,

$$\mathrm{Hom}_{\mathrm{Sh}}(A,B) = \mathrm{Hom}_{\mathrm{PreSh}}(A,B).$$

To simplify notation, if S is a sheaf we drop the notation LS and we write $s \in S(U) = \Gamma(U, S)$

1.5. **Proposition.** (exercise) *Sheafification is an exact functor from the category of presheaves to the category of sheaves. It is left adjoint to the inclusion functor i : Sheaves $\to$ Presheaves, that is, if A is a presheaf and if B is a sheaf (on X) then*

$$\mathrm{Hom}_{Sh}(\widehat{A}, B) \xrightarrow{\cong} \mathrm{Hom}_{preSh}(A, i(B)).$$

Here is an application of this formula. Following the identity morphisms $B \to B$ through this series of canonical isomorphisms

$$\mathrm{Hom}_{\mathrm{Sh}}(B,B) \cong \mathrm{Hom}_{\mathrm{preSh}}(i(B), i(B)) \cong \mathrm{Hom}_{\mathrm{Sh}}(\widehat{i(B)}, B)$$

gives a canonical isomorphism $\widehat{i(B)} \to B$, that is, if we take a sheaf B, look at it as a presheaf, then sheafifity it, the result is canonically isomorphic to the sheaf B that we started with.

1.6. **Caution.** If A, B are sheaves then the set of morphisms $A \to B$ is the same whether we consider A, B to be sheaves or presheaves. However, care must be taken when considering the kernel, image, or cokernel of such a morphism. If we consider $f : A \to B$ to be a morphism in the category of presheaves, then $\ker(f)$ is the presheaf which assigns to an open set U the kernel of $f(U) : A(U) \to B(U)$, and this turns out to be a sheaf. But the presheaf

$$U \mapsto \mathrm{Image}(A(U) \to B(U))$$

is (usually) not a sheaf, so it is necessary to define the sheaf $\mathrm{Image}(f)$ to be the sheafification of this presheaf. Similarly for the cokernel. Consequently, *a sheaf mapping $f : A \to B$ is injective (resp. surjective) iff the mapping $f_x : A_x \to B_x$ on stalks is injective (resp. surjective) for all $x \in X$.* In other words, the abelian category structure on the category of sheaves is most easily understood in terms of the leaf space picture of sheaves.

1.7. **Examples.**
(a.) Let M be an R-module. The constant presheaf (let us denote it by $\mathbf{M}$) assigns to every (nonempty) open set $U \subset X$ the module M. The leaf space is then $L\mathbf{M} = X \times M$ and the stalk at each point $x \in X$ is $\mathbf{M}_x = M$.

(b.) Let $C^0(0,1)$ be the presheaf that assigns to an open set $U \subset (0,1)$ the vector space of continuous functions $f : U \to \mathbb{R}$. Then this is a sheaf because a family of continuous functions defined on open sets that agree on the intersections of those sets clearly patch together to give a continuous function on the union. Similarly, smooth functions, holomorphic functions, algebraic functions etc. can be naturally interpreted as sheaves.

(c.) Let S be the sheaf on $(0,1)$ whose sections over an open set U are those C^∞ functions $f : U \to \mathbb{R}$ such that $\int_U f(x)^2 dx < \infty$. This presheaf is not a sheaf because it is possible to patch (infinitely many) L^2 functions (defined on smaller and smaller subintervals) together to obtain a function that grows too fast to be square integrable. In fact, the sheafification of this sheaf is the set of all smooth functions on $(0,1)$.

(d.) Let X be a connected topological space with universal cover $\widetilde{X}$. Let $x_0 \in X$ be a basepoint and let $\pi_1 = \pi_x(X, x_0)$ be the fundamental group of X. This group acts freely on $\widetilde{X}$ (from the right) with quotient X. Let M be an R-module and let $\rho : \pi_1 \to \mathrm{Aut(M)}$ be a homomorphism. (For example, if M is a vector space over the complex numbers then $\rho : \pi_1 \to \mathrm{GL}_n(\mathbb{C})$ is a representation of π_1). Define

$$\mathcal{L} = \widetilde{X} \times_{\pi_1} M$$

to be the quotient of $\widetilde{X} \times M$ by the equivalence relation $(yg, m) \sim (y, \rho(g)m)$ for all $y \in \widetilde{X}$, $m \in M$, and $g \in \pi_1$. The projection $\widetilde{X} \to X$ passes to a projection $\pi : \mathcal{L} \to X$ which makes $\mathcal{L}$ into the leaf space of a sheaf, which is called a *local system*, or *bundle of coefficients*. Its stalk at x_0 is canonically isomorphic to M and whose stalk other points $x \in X$ is isomorphic to M but not in a canonical way. If $U \subset X$ is a contractible subset then there exist trivializations

$$\pi^{-1}(U) \cong U \times M$$

which identify the leaf space over U with the constant sheaf. So the sheaf $\mathcal{L}$ is a *locally constant sheaf* and every locally constant sheaf of R modules on a connected space X arises in this way. If X is a simplicial complex then a simplicial r-chain with values in the local system $\mathcal{L}$ is a finite formal sum $\sum a_i\sigma_i$ where σ_i are (oriented) r-dimensional simplices and where $a_i \in \pi^{-1}(x_i)$ for some (and hence for any) choice of point $x_i \in \sigma_i$. So there is a chain group (or module) $C_r(X; \mathcal{L})$. One checks that the boundary map $\partial_r : C_r(X; \mathcal{L}) \to C_{r-1}(X; \mathcal{L})$ continues to make sense in this setting and so it is possible to define the simplicial homology group $H_r(X; \mathcal{L}) = \ker(\partial_r)/\mathrm{Image}(\partial_{\mathrm{r}+1})$. In other words, locally constant sheaves have homology.

(e.) Let R be a commutative ring (with 1). There is a topological space, $\mathrm{Spec}(R)$ which consists of all prime ideals in R. The topology on this set was constructed by O. Zariski. For any subset $E \subset R$ let $V(E) = \{\mathfrak{p} | E \subset \mathfrak{p}\}$ be the set of prime ideals, each of which contains E. These form the closed sets in a basis for a topology, that is, the open sets in this basis are the sets $X - V(E)$. The topology generated by these open sets is called the Zariski topology. If M is an R-module then it defines a sheaf on this space in the following way. [to be completed]

(f.) Let $K \subset \mathbb{R}$ be the Cantor set and let $\mathbb{Z}_K$ be the constant sheaf (with value equal to the integers, $\mathbb{Z}$) on K. Let $j : K \to \mathbb{R}$ be the inclusion. Then $j_*(\mathbb{Z}_K)$ is a sheaf on $\mathbb{R}$ (see the definition of f_* below) that is supported on the Cantor set. So "bad" sheaves exist on "good" spaces.

(g.) Fix $r \geq 0$. For any topological space Y let $C_r(Y; \mathbb{Z})$ be the group of singular r-dimensional simplices on Y. (It is the set of finite formal sums of pairs (σ, f) where σ is an oriented r-dimensional simplex and $f : \sigma \to Y$ is a continuous map.) Now let X be a topological space. The presheaf of r-dimensional singular cochains C^r on X assigns to any open set $U \subset X$ the group $C^r(U) := \mathrm{Hom}(C_r(U; \mathbb{Z}), \mathbb{Z})$. If $V \subset U$ then $C_r(V; \mathbb{Z})$ is included in $C_r(U; \mathbb{Z})$ which gives a (surjective) restriction mapping $C^r(U) \to C^r(V)$. This presheaf is also a sheaf.

1.8. **Sheaf Hom.** If A is a sheaf on X with leaf space $\pi : LA \to X$ and if $U \subset X$ is an open set let $A|U$ be the restriction of the sheaf A to the subset U, in other words, the sheaf on U whose leaf space is $\pi^{-1}(U) \to U$. In other words, If A, B are sheaves of R modules then $\mathrm{Hom}(A, B)$ is again an R module that consists of all sheaf mappings $A \to B$. However there is an associated presheaf, perhaps we will denote it by $\mathbf{Hom}(A, B)$, which assigns to any open set the R module of homomorphism

$$\mathrm{Hom}_{Sh(U)}(A|U, B|U))$$

of sheaf mappings $A|U \to B|U$. This presheaf is a sheaf (exercise) for which the group of global sections is the original module of all sheaf homomorphisms, that is,

$$\Gamma(X, \mathbf{Hom}(A, B)) = \mathrm{Hom}_{\mathrm{Sh}(X)}(A, B)$$

1.9. **Functoriality.** Let $f : X \to Y$ be a continuous map, let T be a sheaf on Y, lert S be a sheaf on X. Define $f_*(S)$ to be the presheaf on Y given by

$$f_*(S)(U) = S(f^{-1}(U)).$$

This presheaf is a sheaf (exercise). Define $f^*(T)$ to be the sheaf on X whose leaf space is the pull back of the leaf space of T, that is,

$$Lf^*(T) = f^*(LT) = X \times_Y LT = \{(x, \xi) \in X \times LT |\ f(x) = \pi(\xi)\}.$$

Then this defines a sheaf, and the sections of this sheaf are given by

$$\Gamma(U, f^*T) = \varinjlim_{V \supset f(U)} \Gamma(V, T).$$

(Although $f(U)$ may fail to be open, we take a limit over open sets containing $f(U)$.)

There is also a pushforward with proper support, $f_!S$ with sections $\Gamma(U, f_!S)$ consisting of all sections $s \in \Gamma(f^{-1}(U, S))$ such that the mapping

$$\text{closure}\{x \in U | s(x) \neq 0\} \to U$$

is proper (that is, the pre-image of every compact set is compact). It is not so clear what this means, but if $f : X \to Y$ is the inclusion of a subspace then $f_!S(U)$ consists of sections $s \in \Gamma(U \cap X, S)$ whose support is compact. This implies, in particular, that $f_!(S)$ vanishes outside X, even if X is open. So, in the case of an inclusion, the sheaf $f_!(S)$ is called the extension by zero of S.

1.10. **Lemma.** *Suppose the space X is locally compact. If $J : K \subset X$ denotes the inclusion of a closed subset, and if S is a sheaf on K then $j_*(S) \cong j_!(S)$.*

1.11. **Adjunction.** Let $f : X \to Y$ be a continuous mapping, let A be a sheaf on X, let B be a sheaf on Y. Then there exist natural sheaf morphisms

$$f^* f_*(A) \to A \ \text{ and } \ B \to f_* f^*(B).$$

To see this, for the first one, let us consider sections over an open set $U \subset X$. Then

$$\Gamma(U, f^* f_* A) = \varinjlim_{W \supset f(U)} \gamma(f^{-1}(W), A).$$

If $W \supset f(U)$ then $f^{-1}(W) \supset U$ so we get a mapping from this group to $\Gamma(U, A)$. One verifies that these mappings are compatible when we shrink U, and so this gives a sheaf morphism $f^* f_*(A) \to A$. For the second morphism, again we look at sections over an open set $V \subset Y$. If t is a section of LB over V then, pulling it back by f gives a section $f^*(t)$ of the leaf space of $f^*(B)$ over the set $f^{-1}(V)$, in other words, we have defined a map

$$\Gamma(V, B) \to \Gamma(f^{-1}(V), f^*(B)) = \Gamma(V, f_* f^*(B))$$

which again is compatible with restrictions to smaller open sets. In other words, this defines a sheaf morphism $B \to f_* f^* B$.

1.12. **Proposition.** *The adjunction maps determine a canonical isomorphism*

$$\mathrm{Hom}_{Sh(X)}(f^*B, A) \cong \mathrm{Hom}_{Sh(Y)}(B, f_*A)).$$

Given $f^*B \to A$, apply f_* and adjunction to obtain $B \to f_*f^*B \to f_*A$. Given $B \to f_*A$ apply f^* and adjunction to obtain $f^*B \to f^*f_*A \to A$. This gives maps back and forth between the Hom groups in the proposition. We omit the check that they are inverses to each other.

2. Lecture 2

2.1. **Two comments about categories.** In a category $\mathcal{C}$ the collection of morphisms $A \to B$ between two objects is assumed to form a set, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ and so we may speak of two morphisms being the same. However the collection of objects do not (in general) form a set and so the statement that "B is the same object as A" does not make sense. Rather, "the morphism $f : A \to B$ is an isomorphism" is the correct way to indicate an identification between two objects. This is especially important when the set of self-isomorphisms of A is nontrivial, for in this case there will be many distinct isomorphisms between A and B.

If $\mathcal{C}, \mathcal{D}$ are categories, $f : \mathcal{C} \to \mathcal{D}$ and $g : \mathcal{D} \to \mathcal{C}$ are functors, then f and g are said to be adjoint if there are "natural" isomorphisms $\mathrm{Hom}_{\mathcal{C}}(A, g(B)) \to \mathrm{Hom}_{\mathcal{D}}(f(A), B)$ for all A, B objects of $\mathcal{C}, \mathcal{D}$ respectively (where "natural" means that these isomorphism are compatible with all morphisms $C_1 \to C_2$ and $D_1 \to D_2$). We say that f is left adjoint to g and g is right adjoint to f. In this case, f essentially determines g and vice versa, that is, given f and two adjoint functors g_1, g_2 then there exists a natural equivalence of functors between g_1, g_2. In many cases, the functor f is something simple, and the functor g is something surprising (or vice versa), so it is a fun game to pick your favorite functor and ask whether it has an adjoint. For example, if H is a subgroup of a finite group G then "restriction" is a more or less obvious functor from the category of representations of G to the category of representations of H; its adjoint turns out to be "induction", a much more subtle construction, and the adjointness property is classically known as the Frobenius reciprocity theorem. In the previous lecture we had two examples of adjoint functors: sheafification is the adjoint of the inclusion (sheaves $\to$ presheaves). We also saw that if $f : X \to Y$ is a continuous mapping then f^* is left adjoint to f_*. For example, one might ask whether there is an adjoint to the functor $f_!$. This turns out to be a very subtle question.

2.2. **Simplicial sheaves.** This is a "toy model" of sheaves. Let K be a (finite, for simplicity) simplicial complex. Each (closed) simplex σ is contained in a naturally defined open set, $St^o(\sigma)$, the open star of σ. It has the property that $\sigma < \tau \implies St^o(\tau) \subset St^o(\sigma)$. Using these open sets to define a presheaf, and assigning the values of the presheaf to the simplex itself, gives the following definition:

2.3. **Definition.** A simplicial sheaf S (of abelian groups, or R-modules, etc.) on K is an assignment of an abelian group $S(\sigma)$ for the interior of each simplex and a group homomorphism $S(\sigma) \to S(\tau)$ whenever $\sigma < \tau$, in such a way that whenever $\sigma < \tau < \omega$ then the resulting triangle of groups and morphisms commutes. To make some explicit notation, let $j_{\sigma,\tau} : \sigma \to \tau$ denote the inclusion whenever $\sigma < \tau$ and let $S_{\sigma,\tau} : S(\sigma) \to S(\tau)$ be the corresponding homomorphism (so that S becomes a covariant functor from the category of simplices and inclusions to the category of abelian groups).

In this setting there is no distinction between a sheaf and a presheaf. However the leaf space of S is easily constructed as the union

$$LS = \coprod_{\sigma} \sigma^0 \times S(\sigma)$$

but as in the case of general sheaves, the topology must be constructed with some care so as to be a local homeomorphsim.

2.4. **Cohomology of simplicial sheaves.** Let K be a finite simplicial complex and let S be a simplicial sheaf (of abelian groups, or of R-modules). In order to define cohomology it is necessary to choose orientations of the simplices. (An orientation of a simplex is determined by an ordering of its vertices, two orderings giving the same orientation iff they differ by an even permutation.) The simplest method of orienting all the simplices, is to choose an ording of the vertices of K and to take the induced ordering on the vertices of each simplex. Assume this to be done.

Fix $r \geq 0$. An r-chain with values in the simplicial sheaf S is a function F that assigns, to each (oriented) r-dimensional simplex σ an element of $S(\sigma)$. The collection of all r-chains is denoted $C^r(K;S)$. The coboundary $\delta F \in C^{r+1}(K;S)$ is defined as follows. If τ is an $r+1$ simplex with vertices $v_0, v_1, \cdots, v_{r+1}$ (in ascending order) we write $\tau = \langle v_0, v_1, \cdots, v_{r+1}\rangle$ and we denote its i-th face by $\partial_i \tau = \langle v_0, \cdots, \widehat{v_i}, \cdots, v_{r+1}\rangle$. Then

$$\delta F(\tau) = \sum_{i=0}^{r+1} (-1)^i S_{\partial_i \tau, \tau}(F(\partial_i \tau)).$$

One checks that $\delta\delta F = 0$ (it is the same calculation that is involved in proving that $\partial\partial = 0$ for simplicial homology), so we may define the cohomology, $H^r(K;S) = \ker(\delta)/\mathsf{Im}(\delta)$ to be the cohomology of the cochain complex

$$\cdots \longrightarrow C^{r-1}(K;S) \xrightarrow{\delta} C^r(K;S) \xrightarrow{\delta} C^{r+1}(K;S) \longrightarrow \cdots$$

This combinatorial construction is easily implemented on a computer. By reversing the arrows one has the analogous notion of a simplicial cosheaf and a similar construction of the *homology* of a simplicial cosheaf.

2.5. **Historical interlude.** Many different techniques have been developed for exploring the properties of cohomology of sheaves, the most elegant being the methods associated with the derived category. Any one of these methods may be used as a "definition" of the cohomology of a sheaf, and although the historical methods are the most accessible, they are also the most cumbersome. We will use the method of injective resolutions, prefaced by a brief tour through its historical development.

(1873) B. Riemann and later, E. Betti, consider the number of "cuts" of varying dimensions that are needed in order to reduce a space into contractible pieces.

(1892) H. Poincaré, in *Analysis Situs* constructs homology of a "variété" using cycles that consist of the zeroes of smooth functions.

(1898) P. Heegaard publishes a scathing scriticism of Poincareś article for its lack of rigor.
(1900) H. Poincaré publishes *Supplement* to his *Analysis Situs* in which he essentially describes simplicial homology for a space that has been decomposed into simplices.
(1912) F. Hausdorff publishes the general definition of a topological space and interpret continuity purely in terms of the open sets.
(1925) H. Hopf develops the general notion of a chain complex.
(1926) Alexander, Hopf give precise definition of simplicial complex.
(1928) H. Hopf, E. Noether describe homology as a group.
(1930) E. Cartan, G. deRham formalize notion of differential forms, Poincaré lemma, de Rham theorem.
(1933) The drive to develop singular homology theory, with contributions by Dehn, Heegard, Lefschetz, others.
(1934) E. Čech develops his approach to cohomology using the open sets in a space; cohomology with coefficients in a ring.
(1935) H. Whitney develops the general theory of differentiable manifolds (and their embeddings into Euclidean space).
(1935) H. Reidemeister develops theory of homology with local coefficients.
(1935-40) Products in cohomology, modern formulation of Poincaré duality, Stiefel Whitney classes, differential forms. (Until this period, differential forms were "expressions".)
(1942) S. Eilenberg and S. MacLane: Category theory
(1945-46) J. Leray (while a prisoner of war): sheaves and their cohomology, spectral sequence of a map
(1946) S. S. Chern: Chern classes
(1950) Čech cohomology of sheaves
(1956) A. Borel, J. C. Moore, the dual of a complex of sheaves; Borel-Moore homology
(1956) H. Cartan, S. Eilenberg: injective resolutions and derived functors
(1957) A. Grothendieck: Tohoku paper
(1958) D. Kan: notion of adjoint functors
(1961) J. L. Verdier, derived categories, Verdier duality (published in 1996).

Given the complexity of this history, we will describe several ways to define the cohomology of a sheaf, and leave the proof that they all give the same answer until later when we have efficient machinery for doing so.

2.6. Let X be a topological space and let S be a sheaf (of abelian groups, or of R-modules) on X. Let $\sigma \in \Gamma(X, S)$, and consider σ to be a section of the leaf space $LS \to X$. The *support* $spt(\sigma)$ of σ is the closure of the set of points $x \in X$ such that $\sigma(x) \neq 0$. Let $\Gamma_c(X, S)$ denote the group of sections with compact support. If $K \subset X$ is a closed subset, let $\Gamma_K(S)$ denote the group of sections whose support is contained in K. This may also be identified with the sections $\Gamma(K, j^*(S))$ where $j : K \to X$ denotes the inclusion.

The sheaf S is *injective* if the following holds: Suppose $f : A \to B$ is an injective morphism of sheaves. Then any morphism $h : A \to S$ extends to a morphism $\tilde{h} : B \to S$.

$$\begin{tikzcd} B \arrow[dr, dotted, "\tilde{h}"] & \\ A \arrow[u, hook, "f"] \arrow[r, "h"'] & S \end{tikzcd}$$

The sheaf S is *flabby* if $S(U) \to S(V)$ is surjective, for all open subsets $V \subset U$. The sheaf S is *soft* if $\Gamma(X, S) \to \Gamma_K(S)$ is surjective, for every closed subset $K \subset X$. The sheaf S is *fine* if, for every open cover $X = \cup_{\alpha \in I} U_\alpha$ there exists a family of morphisms $h_i : S \to S$ such that $\mathrm{spt(h_\alpha) \subset U_\alpha}$ and $\sum_\alpha h_\alpha = I$. (Usually the h_α are just a partition of unity with respect to the coefficient ring.) For completeness we include here the following definition, which actually requires having previously defined cohomology: The sheaf S is *acyclic* if $H^r(X, S) = 0$ for all $r \geq 1$.

The following fact will not be used

2.7. **Proposition.** *For any sheaf S on a locally compact space X,*

$$\text{injective} \implies \text{flabby} \implies \text{soft} \implies \text{acyclic} \ \textit{and} \ \text{fine} \implies \textit{soft} \implies \textit{acyclic.}$$

Of these notions, *injective* and *acyclic* are categorical, and we will concentrate on them. However, in order that a sheaf be injective, it must have certain topological properties and certain algebraic properties. For example, if the ring R is an integral domain, then the constant sheaf on a single point is injective iff R is a field. The ring $\mathbb{Z}$ is not injective but it has an injective resolution $\mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$.

2.8. **Definition.** An injective resolution of a sheaf S on X is an exacT sequence

$$0 \to S \to I^0 \to I^1 \to \cdots$$

where each I^r is an injective sheaf.

A abelian category $\mathcal{C}$ has *enough injectives* if every object can be embedded in an injective. In this case, every object A admits an injective resolution: just embed $A \to I^0$ and let K^0 be the cokernel of this map. Then embed $K^0 \to I^1$ and let K^1 be the cokenerl of this map. Then embed $K^1 \to I^2$ and so on. The resulting sequence $0 \to A \to I^0 \to I^1 \cdots$ is exact. The category of modules over a commutative ring R has enough injectives, and the category of sheaves of R modules on any topological space X has enough injectives. However, there is a canonical and functorial injective resolution of any sheaf, namely the *Godement resolution*. It will be described later.

First definition of cohomology Let S be a sheaf on a topological space X and let $0 \to S \to I^0 \to \cdots$ be the Godement injective resolution of S. Then the cohomology of the complex of global sections

$$0 \to \Gamma(X, I^0) \to \Gamma(X, I^1) \to \Gamma(X, I^2) \cdots$$

is called the cohomology of S, denoted $H^r(X,S)$.

Second definition of cohomology In fact: You will get the same answer, up to unique isomorphism, if you use any injective resolution, or any fine, flabby, soft, or acyclic resolution instead of an injective resolution. (See Lecture 4: injectives and Čech cohomology)

3. Lecture 3: Complexes of sheaves

3.1. Sheaves tend to occur in complexes. Familiar examples include the de Rham complex of differential forms on a smooth manifold M,

$$\Omega^0(M,\mathbb{R}) \to \Omega^1(M,\mathbb{R}) \to \Omega^2(M,\mathbb{R}) \cdots$$

or the $\bar\partial$ complex on a complex manifold; the sheaf of singular cochains, etc. Less obvious examples include the push-forward $f_*(S^\bullet)$ of a complex by a (continuous, smooth, or algebraic) mapping $f : X \to Y$.

3.2. A complex $S^\bullet$ (in an abelian category) is a sequence

$$\cdots \longrightarrow S^{r-1} \xrightarrow{d^{r-1}} S^r \xrightarrow{d^r} S^{r+1} \xrightarrow{d^{r+1}} \cdots$$

where $d \circ d = 0$. (We will assume that all of our complexes are bounded from below, that is $S^r = 0$ if r is sufficiently small, usually if $r < 0$). The *cohomology* $H^r(S^\bullet)$ of the complex is $\ker d^r / \operatorname{Im} d^{r-1}$. So the complex $S^\bullet$ is *exact* (meaning that it forms an exact sequence) iff $H^r(S^\bullet) = 0$ for all r. A *complex of sheaves* is a complex in which each S^r is a sheaf. In this case, the *cohomology sheaf* $\mathbf{H}^r(S^\bullet)$ is the sheaf $\ker d / \operatorname{Im} d$. The stalk of the cohomology sheaf coincides with the cohomology of the stalks (exercise), that is,

$$\mathbf{H}^r_x(S^\bullet) := \mathbf{H}^r(S^\bullet)_x \cong H^r(S^\bullet_x).$$

A morphism (of complexes in an abelian category) $S^\bullet \to T^\bullet$ of complexes is a collection of morphisms that commute with the differentials. Two morphisms $f, g : S^\bullet \to T^\bullet$ are *chain homotopic* if there is a collection of mappings $h : S^r \to T^{r-1}$ (for all r) so that $d_T h + h d_S = f - g$. A morphism $\phi : S^\bullet \to T^\bullet$ is a *quasi-isomorphism* if it induces isomorphisms on the cohomology objects $H^r(S^\bullet) \to H^r(T^\bullet)$ for all r. We will see that a quasi-isomorphism of complexes of sheaves induces isomorphisms on cohomology.
Exercise: For the complex $\mathbf{\Omega}^\bullet_M$ of sheaves of differential forms on a smooth manifold M show that the cohomology sheaves are zero in all degrees except zero, and that $\mathbf{H}^0(\mathbf{\Omega}^\bullet_M) \cong \mathbb{R}$ is the constant sheaf.

3.3. **Magic Triangles.** The *mapping cone* $C^\bullet = C(\phi)$ of a morphism $\phi : A^\bullet \to B^\bullet$ is the complex $C^r = A^{r+1} \oplus B^r$ with differential $d_C(a,b) = (d_A(a), (-1)^{\deg(a)}\phi(a) + d_B(b))$. It is the total complex of the double complex

$$\begin{tikzcd} \ & \ \\ A^2 \arrow[u, "d"] \arrow[r, "\phi"] & B^2 \arrow[u, "d"'] \\ A^1 \arrow[u, "d"] \arrow[r, "\phi"] & B^1 \arrow[u, "d"'] \\ A^0 \arrow[u, "d"] \arrow[r, "\phi"] & B^0 \arrow[u, "d"'] \end{tikzcd}$$

from which we see that there are obvious morphisms $\beta : B^\bullet \to C^\bullet$ and $\gamma : C^\bullet \to A^\bullet[1]$. It is customary to denote this situation as a triangle of morphisms

$$\begin{tikzcd} A^\bullet \arrow[rr, "\phi"] & & B^\bullet \arrow[dl] \\ & C(\phi) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

3.4. **Lemma.** *If ϕ is injective then there is a natural quasi-isomorphism* $\operatorname{coker}(\phi) \cong C(\phi)$. *If ϕ is surjective then there is a natural quasi-isomorphism $C(\phi) \cong \ker(\phi)[1]$. There are natural quasi-isomorphisms $A^\bullet[1] \cong C(\beta)$ and $B^\bullet[1] \cong C(\gamma)$. Moreover, there is a long exact sequencer on cohomology*

$$\cdots \to \mathbf{H}^{r-1}(B^\bullet) \to \mathbf{H}^{r-1}(C^\bullet) \to \mathbf{H}^r(A^\bullet) \to \mathbf{H}^r(B^\bullet) \to \mathbf{H}^r(C^\bullet) \to \cdots .$$

The proof is an exercise, and it works in any abelian category.

3.5. **Double complexes.** A *double complex* is an array C^{pq} with horizontal and vertical differentials $d_h : C^{pq} \to C^{p+1,q}$ and $d_v : C^{pq} \to C^{p,q+1}$ such that $d_v d_v = 0$, $d_h d_h = 0$, and $d_v d_h = d_h d_v$. [Some authors assume instead that $d_v d_h = -d_h d_v$ which eliminates the necessity for a sign in the definition below of the total complex.] For convenience we will always assume that $C^{pq} = 0$ unless $p \geq 0$ and $q \geq 0$. The *associated single complex*, or *total complex* $T^\bullet$ is defined by $T^r = \oplus_{p+q=r} C^{pq}$ with "total differential" $d_T : T^r \to T^{r+1}$ defined by $d_T(c_{pq}) = (d_h + (-1)^q d_v) c_{pq}$ for $c_{pq} \in A^{pq}$ (that is, change 1/4 of the signs to obtain that $d_T d_T = 0$).

3.6. **Lemma.** *Let $C^{\bullet\bullet}$ be a first quadrant double complex and suppose the rows are exact and that the zeroth horizontal arrows d_h^{0q} are injections (which is the same as saying that an extra zero may be added to the left end of each row, without destroying the exactness of the rows). Then the total complex T is exact.*

Proof. A proper proof involves indices, signs, and an induction that is totally confusing and is best worked out in the privacy of your own home. To see how the argument goes, let us show that the total complex $T^0 \to T^1 \to T^2 \cdots$ is exact at T^2. Let $x = x_{02} + x_{11} + x_{21} \in T^2$ and suppose that $d_T x = 0$.

$$\begin{tikzcd}
\cdot & & & \\
x_{02} \arrow[u] \arrow[r] & \cdot & & \\
y_{01} \arrow[u] \arrow[r] & x_{11} \arrow[u] \arrow[r] & \cdot & \\
\cdot \arrow[u] \arrow[r] & y_{10} \arrow[u] \arrow[r] & x_{20} \arrow[u] \arrow[r] & \cdot
\end{tikzcd}$$

- Since $d_T x = 0$ we have:

$$d_v x_{02} = 0;\ d_v x_{11} + d_h x_{02} = 0;\ d_v x_{20} - d_h x_{11} = 0;\ d_h x_{20} = 0.$$

- Since the bottom row is exact there exists y_{10} so that $d_h y_{10} = x_{20}$.
- Now consider $x'_{11} = x_{11} - d_v y_{10}$. Check that $d_h(x'_{11}) = 0$.
- Since the first row is exact there exists y_{01}so that $d_h y_{01} = x'_{11}$.
- Now consider $x'_{02} = x_{02} + d_v y_{01}$. Check that $d_h x'_{02} = 0$.
- But now we are in the left column, so this last operation, d_h was an injective mapping. This implies that $x'_{02} = 0$.
- Now just check that we have the right answer, $y = y_{01} + y_{10} \in T^1$. □

3.7. **Corollary.** *Let $\{C^{pq}\}$ be a first quadrant double complex $(p, q \geq 0)$ with exact rows. Let $T^\bullet$ denote the total complex, $T^r = \oplus_{p+q=r} C^{pq}$. Let $A^r = \ker(d^{0,r} : C^{0r} \to C^{1r})$ denote the subcomplex of the zeroth column, with its vertical differential d_v. Then the morphism $A^\bullet \to T^\bullet$ (given by the inclusion $A^r \to C^{0r} \to T^r$) is a quasi-isomorphism.*

Proof. Let us consider the extended double complex obtained by considering the complex $A^\bullet$ to be the -1-st column, that is, $C^{-1,r} = A^r$, with vertical differential $d_v = d_A$ and with horizontal differential the inclusion $A^r \to C^{0,r}$.

$$\begin{tikzcd}
A^2 \arrow[r, hook] & C^{02} \arrow[r] & C^{12} \arrow[r] & C^{22} \arrow[r] & \cdots \\
A^1 \arrow[u, "d_A"'] \arrow[r, hook] & C^{01} \arrow[u] \arrow[r] & C^{11} \arrow[u] \arrow[r] & C^{21} \arrow[u] \arrow[r] & \cdots \\
A^0 \arrow[u, "d_A"'] \arrow[r, hook] & C^{00} \arrow[u] \arrow[r] & C^{10} \arrow[u] \arrow[r] & C^{20} \arrow[u] \arrow[r] & \cdots
\end{tikzcd}$$

Let $S^\bullet$ denote the total complex of this extended double complex. It is precisely the mapping cone of the morphism $A^\bullet \to T^\bullet$ so we have a magic triangle

$$\begin{tikzcd} A^\bullet \arrow[rr, hook] & & T^\bullet \arrow[dl] \\ & S^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Moreover, the extended double complex has exact rows and the leftmost horizon maps are injective so the previous lemma applies and we conclude that the cohomology of $S^\bullet$ vanishes. By the long exact sequence on cohomology this implies that $H^r(A^\bullet) \to H^r(T^\bullet)$ is an isomorphism for all r. □

3.8. **Cohomology of a complex of sheaves.** We defined the cohomology of a sheaf to be the (global section) cohomology of an injective resolution of the sheaf. So one might expect an injective resolution of a complex of sheaves $A^\bullet$ to be a double complex, and in fact, such a double complex can always be constructed. If the coefficient ring is a field, then the simplest way is to use the Godement resolution because it is functorial (see the next lecture). For more general coefficient rings, the Godement resolution can be tensored with an injective resolution of the ring. More generally, a double complex resolution was constructed by Cartan and Eilenberg, and it is known as a Cartan-Eilenberg resolution of the complex. It is easy to take an injective resolution of each of the sheaves in the complex, but not so easy to see how to fit them together so that the differentials satisfy $d_v^2 = 0$, but this can be done (see the Stacks Project, or Wikipedia on Cartan-Eilenberg resolutions). In any case, let us assume that we have a double complex, the r-th row of which resolves A^r.

But that is not the end. Given an injective resolution as a double complex $A^\bullet \to I^{\bullet\bullet}$ with horizontal and vertical differentials d_h, d_v respecitvely, we form the associated total complex $T^r = \oplus_{p+q=r} I^{pq}$ and with $d_r = d_h + (-1)^q d_v$.

3.9. **Lemma.** *The resulting map $A^\bullet \to T^\bullet$ is a quasi-isomorphism.*

This follows immediately from Corollary 3.7, which says that there is a long exact sequence of cohomology sheaves,

$$\cdots \to \mathbf{H}^r(A^\bullet) \to \mathbf{H}^r(T^\bullet) \to \mathbf{H}^r(S^\bullet) \to \mathbf{H}^{r+1}(A^\bullet) \to \cdots$$

where the cohomology sheaves of $S^\bullet$ vanish.

In summary, we have replaced the double complex I^{pq} with a single complex $T^\bullet$ so we arrive at the following definition, which works in any abelian category:

3.10. **Definition.** An injective resolution of a complex $A^\bullet$ is a quasi-isomorphism $A^\bullet \to T^\bullet$ where each T^r is an injective object. The cohomology $H^r(X, A^\bullet)$ (also known as the hypercohomology of $A^\bullet$) of a complex of sheaves is defined to be the cohomology of the complex of global sections

$$\Gamma(X, T^{r-1}) \to \Gamma(X, T^r) \to \Gamma(X, T^{r+1})$$

of any injective resolution $A^\bullet \to T^\bullet$

[A messy technical point: a complex of injective objects is not necessarily an injective object in the category of complexes. So the somewhat ambiguous terminology of "injective resolution" could be misleading and some authors refer to these as "K-injective resolutions". Fortunately we will not be required to consider injective objects in the category of complexes.]

This fits beautifully with the notion of a resolution of a single sheaf S: it is a quasi-isomorphism,

$$\begin{tikzcd} 0 \arrow[r] & S \arrow[r] \arrow[d] & 0 & & \\ 0 \arrow[r] & I^0 \arrow[r] & I^1 \arrow[r] & I^2 \arrow[r] & \cdots \end{tikzcd}$$

As before, flabby, soft or fine resolutions may be used instead of injective resolutions. For example, let M^n be a smooth manifold. Let $x \in M$ and let U_x be a neighborhood of x that is diffeomorphic to an n-dimensional ball. The Poincaré lemma says that if ξ is a closed (i.e. $d\xi = 0$) differential r-form ($r \geq 1$) defined in U_x then there is a differential $r-1$ form η so that $d\eta = \xi$. The sheaf of smooth differential forms is fine, so the Poincaré lemma says that this complex of sheaves is a fine resolution of the constant sheaf,

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r] \arrow[d] & 0 & & \\ 0 \arrow[r] & \Omega^0_M \arrow[r] & \Omega^1_M \arrow[r] & \Omega^2_M \arrow[r] & \cdots \end{tikzcd}$$

Therefore the cohomology $H^r(M, \mathbb{R})$ is canonically isomorphic to the cohomology of the complex of global sections of Ω^*_M, that is, the de Rham cohomology.

In fact, injective resolutions may be constructed directly without resorting to the Godement double complex or the Cartan-Eilenberg double complex. [I hope to write down the argument for this but involves some serious latex hacking with diagrams.]

3.11. **Proposition.** *Let $A^\bullet \to B^\bullet$ be a quasi-isomorphism of complexes of sheaves on a topological space X. Then it induces an isomorphism on cohomology $H^r(U, A^\bullet) \cong H^r(U, B^\bullet)$ for any open set $U \subseteq X$.* (Proof in the next lecture.)

4. Lecture 4: Godement and Čech

4.1. Examples.
1. Let X be the 2-dimensional simplex. In the category of simplicial sheaves, suppose that S is a sheaf on X that assigns the value $\mathbb{Q}$ to the interior of the 2-simplex and assigns 0 to simplices on the boundary. Find an injective resolution of S. Determine the global sections of each step in the resolution. Show that the cohomology of S is $\mathbb{Q}$ in degree 2 and is 0 in all other degrees.

2. Let X be a triangulation of S^2, which may be taken, for example to be the boundary of a 3-simplex. Let S be the constant sheaf on X. Find an injective resolution of S in the category of simplicial sheaves, and compute the cohomology of its global sections.

3. Let X be a triangulation of S^2 with 10^8 simplices. Describe an injective resolution of X.

4. Let X be a topological space, let $x_0 \in X$ and let $S = S(x_0, \mathbb{Q})$ be the presheaf that assigns to any open set U

$$S(U) = \begin{cases} \mathbb{Q} & \text{if } x_0 \in U \\ 0 & \text{else} \end{cases}$$

Show that S is injective and that its leaf space LS is a skyscraper, that is, it consists of a single group $\mathbb{Q}$ at the point x_0 and zero everywhere else.

5. In the above example, fix $x_0 \in X$ and let $T^\bullet$ be the complex of sheaves $S(x_0, \mathbb{Q}) \to S(x_0, \mathbb{Q}/\mathbb{Z})$. Show that this complex is an injective resolution of the skyscraper sheaf that is $\mathbb{Z}$ at the point x_0.

These examples show that injective sheaves must be sums of sheaves with tiny support. This leads one to the following:

4.2. Godement resolution. Given a sheaf A on a topological space X it embeds in a flabby sheaf $\mathrm{God}(A)$ with sections

$$\Gamma(U, \mathrm{God}(A)) = \prod_{x \in U} A_x$$

the product of all the stalks at points in U. It is sometimes called the sheaf of totally discontinuous sections. If we start with the constant sheaf $\underline{\underline{\mathbb{Z}}}$ then a section $s \in \Gamma(U, \mathrm{God}(\underline{\underline{\mathbb{Z}}}))$ assigns to each point $x \in U$ an integer, without any regard to continuity or compatibility. It is the sort of sheaf that you definitely do not want to meet in a dark alley. The Godement *resolution* $\underline{\underline{\underline{\mathrm{God}}}}^\bullet(A)$ is obtained by applying this construction to the cokernel of $A \to \mathrm{God}(A)$ and iterating:

$$\begin{tikzcd}
A \arrow[r, hook] & \mathrm{God}(A) \arrow[rr] \arrow[dr] & & \mathrm{God}(\mathrm{coker}) \arrow[rr] \arrow[dr] & & \mathrm{God}(\mathrm{coker}) \\
& & \mathrm{coker} \arrow[ur, hook] & & \mathrm{coker} \arrow[ur, hook] &
\end{tikzcd}$$

If A_x is injective for all $x \in X$ (for example, if the coefficient ring R is a field) then this is an injective resolution of A. For many rings there are functorial injective resolutions that can be used, together with the Godement construction to make a double complex, the associated total complex of which is then a canonical injective resolution. The Godement resolution is functorial: a morphism $f : A \to B$ induces a morphism of complexes $\mathrm{God}(f) : \underline{\underline{\mathrm{God}}}^\bullet(A) \to \underline{\underline{\mathrm{God}}}^\bullet(B)$ in such a way that $\mathrm{God}(f \circ g) = \mathrm{God}(f) \circ \mathrm{God}(g)$.

In summary, injective sheaves are huge, horrible objects and maybe we use them to prove things but never to compute with. A much more efficient computational tool is the Čech cohomology.

4.3. **Čech cohomology of sheaves.** Let A be a sheaf on X. Let $\mathfrak{U} = \{U_\alpha\}_{\alpha \in I}$ be a collection of open sets in X. For any subset $J \subset I$ let $U_J = \cap_{\alpha \in J} U_\alpha$ be the corresponding intersection. The Čech cochain complex is

$$\begin{array}{ccc} \check{C}^r(X, A) & = & \prod_{|J|=r+1} \Gamma(U_J, A) \\ \downarrow d & & \downarrow d \\ \check{C}^{r+1}(X, A) & = & \prod_{|K|=r+2} \Gamma(U_K, A) \end{array}$$

where d is defined as follows: suppose that $K = \{k_0, k_1, \cdots, k_{r+1}\} \subset I$ and $\sigma \in \check{C}^r(X, A)$. Then

$$d\sigma(U_K) = \sum_{i=0}^{r+1} \sigma(U_{k_0} \cap \cdots \cap \widehat{U}_{k_i} \cap \cdots \cap U_{k_{r+1}})|U_K.$$

For example, if $\sigma \in \check{C}^1$ and if $U = U_0 \cap U_1 \cap U_2$ then

$$d\sigma(U) = \sigma(U_1 \cap U_2)|U - \sigma(U_0 \cap U_1)|U + \sigma(U_0 \cap U_2)|U.$$

Then one checks that $d \circ d = 0$ so the cohomology of this resulting complex is defined:

$$\check{H}^r_{\mathfrak{U}}(X, A) = \ker d / \operatorname{Im} d.$$

Notice, in particular, that $\check{H}^0_{\mathfrak{U}}(X, A)$ consists of sections σ_α over U_α that agree on each intersection $U_\alpha \cap U_\beta$ so it coincides with the global sections: $\check{H}^0_{\mathfrak{U}}(X, A) = \Gamma(X, A)$ for any covering $\mathfrak{U}$.

4.4. **Theorem.** *Suppose the open cover $\mathfrak{U}$ has the property that $H^r(U_J, A) = 0$ for every $J \subset I$ and for all $r > 0$. Then there is a canonical isomorphism $\check{H}^i(X, A) \cong H^i(X, A)$ for all i. In particular, $H^0(X, A) \cong \Gamma(X, A)$.*

This is an incredibly useful result because it says that we can use possibly very few and very large open sets when calculating sheaf cohomology, and it even tells us how to tailor the open sets to take advantage of the particular sheaf, whereas the original theorem of Čech assumed that all the multi-intersections of the open sets were contractible (and it applied only to the constant sheaf). On the other hand, given the machinery that we have developed, the proof is very simple.

Proof. Let A be a sheaf on a topological space and let $\mathcal{U} = \{U_\alpha\}_{\alpha\in K}$ be an open covering of X. The Čech complex $\check{C}^\bullet(A) = \check{C}_{\mathcal{U}}{}^\bullet(A)$ is

$$\cdots \to \prod_{|J|=r+1} \Gamma(U_J, A) \to \prod_{|J|=r+2} \Gamma(U_J, A) \to \cdots .$$

Sheafify this construction by defining

$$\mathbf{C}^p(A) = \mathbf{C}^p_{\mathcal{U}}(A) = \prod_{|J|=p+1} i_*(A|U_J) \to \mathbf{C}^{p+1}(A) \to \cdots$$

so that $\mathbf{C}^p(A)(V) = \prod_{|J|=p+1} \Gamma(V \cap U_J, A)$. This is functorial in A. There is a little combinatorial argument to show that

$$0 \to A \to \mathbf{C}^0(A) \to \mathbf{C}^1(A) \to \cdots$$

is exact. (To check exactness of the stalks at a single point x, we need to consider the combinatorics of having $p+1$ open sets whose multi-intersection contains x. Then exactness comes down to proving that the homology of the p-simplex is trivial.)

Now let $A \to I^0 \to I^1 \to I^2 \to \cdots$ be an injective resolution of A, and apply the Čech resolution to each term in this sequence, which gives a double complex of sheaves:

$$\begin{array}{ccccccc}
\mathbf{C}^0(I^2) & \longrightarrow & \mathbf{C}^1(I^2) & \longrightarrow & \mathbf{C}^2(I^2) & \longrightarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbf{C}^0(I^1) & \longrightarrow & \mathbf{C}^1(I^1) & \longrightarrow & \mathbf{C}^2(I^1) & \longrightarrow & \\
\uparrow & & \uparrow & & \uparrow & & \\
\mathbf{C}^0(I^0) & \longrightarrow & \mathbf{C}^1(I^0) & \longrightarrow & \mathbf{C}^2(I^0) & \longrightarrow &
\end{array}$$

whose rows are exact. Let $\mathbf{T}^\bullet$ denote the associated single complex (of sheaves). If we augment the left column with the column $I^0 \to I^1 \to I^2 \to \cdots$ then Corollary 3.7 says that the resulting map on cohomology sheaves $\mathbf{H}^*(I^\bullet) \to \mathbf{H}^*(T^\bullet)$ is an isomorphism, which is to say that $I^\bullet \to T^\bullet$ is a quasi-isomorphism. So it induces an isomorphism on cohomology, $H^*(I^\bullet) = H^*(X, A) \cong H^*(T^\bullet)$.

On the other hand, let us take global sections to obtain a double complex of groups. The r-th column now reads (from the bottom up)

$$\prod_{|J|=r+1} \Gamma(U_J, I^0) \to \prod_{|J|=r+1} \Gamma(U_J, I^1) \to \cdots$$

which is a complex that computes the product of hypercohomology groups $\prod_{|J|=r+1} H^*(U_J, A) = 0$ by hypothesis. The kernel of the zeroth vertical map is exactly the Čech cochains $\check{C}^r(A)$. Therefore, if we augment the bottom row with the complex $\check{C}^0(A) \to \check{C}^1(A) \to \check{C}^2(A) \cdots$ of Čech cochains then Corollary 3.7 says that we will obtain a quasi-isomorphism of this complex with the total complex of this double complex, namely $\Gamma(X, T^\bullet)$. Hence, the cohomology of the Čech complex of groups coincides with the cohomology of this total complex, which was shown above to coincide with the hypercohomology of the sheaf A as computed using injective resolutions. $\square$

5. Lecture 5: Homotopy and injectives

5.1. **The sheaf of chains.** Most of the complexes of sheaves that were discussed until now have the property that their cohomology sheaves live only in degree zero. The sheaf of chains, however, is a naturally occurring entity with complicated cohomology sheaves. But it is not so obvious how to construct a sheaf on X that corresponds to the singular chains. One might define a presheaf C_r with sections $\Gamma(U, C_r) = C_r(U)$ to be the group of singular chains on U and with restriction mapping $C_r(U) \to C_r(V)$ that assigns zero to every singular simplex that is not completely contained in V. If we sheafify this then we are forced to consider singular chains that are (possibly) infinte sums of simplices. If the space X is paracompact then we may restrict to chains that are locally finite. The result is called Borel-Moore homology. For field coefficients k, Borel and Moore defined a sheaf with sections

$$\Gamma(U, C_r^{BM}) = \mathrm{Hom}(C_c^r(U, k), k)$$

where C_c^r denotes the cochains with compact support. For more general rings R it is necessary to replace $\mathrm{Hom}(\cdot, k)$ with a complex $\mathrm{Hom}(\cdot, I^\bullet)$ where $R \to I^\bullet$ is an injective resolution of the ring. The resulting double complex is converted into a single complex by the usual diagonal sum trick. If X is compact then the Borel-Moore homology coincides with the usual (e.g. singular) homology.

Exercise. In the simplicial sheaf setting find a complex of injective sheaves that gives the Borel-Moore homology for a finite simplicial complex K.

Let us say that a topological space X has *finite type* if it is homeomorphic to $K - L$ where K is a finite simplicial complex and L is a closed subcomplex. In this case $H_r^{BM}(X) \cong H_r(K, L)$ coincides with the relative homology which can then be expressed as the homology of a chain complex formed by the simplices that are contained only in K, and by defining the differential so as to ignore all components of the boundary that may lie in L. This gives a simple combinatorial construction of Borel-Moore homology for spaces of finite type.

Exercise. Find a complex of injective sheaves that gives the Borel-Moore homology for a space $X = K - L$ of finite type, where K is a finite simplicial complex and L is a closed subcomplex.

5.2. **Homotopy theory.** Two morphisms $f, g : A^\bullet \to B^\bullet$ of complexes are said to be *homotopic* if there is a collection of mappings $h : A^r \to B^{r-1}$ so that $hd_A + d_B h = f - g$. This is an equivalence relation. Equivalence classes are referred to as homotopy classes of maps; the set of which is denoted $[A^\bullet, B^\bullet]$. Define the complex of abelian groups (or R modules)

$$\mathrm{Hom}^n(A^\bullet, B^\bullet) = \prod_s \mathrm{Hom}(A^s, B^{s+n})$$

with differential $df = d_B f + (-1)^{n+1} f d_A$ where $f : A^s \to B^{s+n}$.

5.3. **Lemma.** *Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes and let $C(f)$ be the cone of f. For any complex $S^\bullet$ we have a morphism of complexes $f_* : \mathrm{Hom}^\bullet(S^\bullet, A^\bullet) \to \mathrm{Hom}^\bullet(S^\bullet, B^\bullet)$. Then there is* *a canonical isomorphism of complexes of abelian groups,*

$$C(f_*) \cong \mathrm{Hom}^\bullet(S^\bullet, C(f)).$$

For any complex $T^\bullet$ we have a morphism of complexes $f^* : \mathrm{Hom}(B^\bullet, T^\bullet) \to \mathrm{Hom}(A^\bullet, T^\bullet)$. Suppose the cohomology of $S^\bullet, T^\bullet$ is bounded, that is, $H^r(S^\bullet) = H^r(T^\bullet) = 0$ if $|r|$ is sufficiently large. Then there is a canonical quasi-isomorphism of complexes of abelian groups,

$$C(f^*)[-1] \cong \mathrm{Hom}^\bullet(C(f), T^\bullet).$$

The first statement is obvious because $\mathrm{Hom}(S^s, A^{t+1} \oplus B^t) = \mathrm{Hom}(S^s, A^{t+1}) \oplus \mathrm{Hom}(S^s, B^t)$. The second statement is similar. The following exercise is crucial .

5.4. **Proposition.** $H^n(\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)) = [A^\bullet, B^\bullet[n]]$. ☐

In particular given $f : A^\bullet \to B^\bullet$ let $C^\bullet = C(f)$ be the cone. Then for any complex $S^\bullet$ there is a long exact sequence

$$\cdots [S^\bullet, A^\bullet] \to [S^\bullet, B^\bullet] \to [S^\bullet, C^\bullet] \to [S^\bullet, A^\bullet[1]] \to [S^\bullet, B^\bullet[1]] \to \cdots$$

We can do the same with sheaf-Hom. Recall that $\underline{\underline{\underline{\mathrm{Hom}}}}(A, B)(U) = \mathrm{Hom}_{Sh(U)}(A|U, B|U))$. We obtain a complex of sheaves,

$$\underline{\underline{\underline{\mathrm{Hom}}}}^n(A^{^\bullet}, B^{^\bullet}) = \prod_s \underline{\underline{\underline{\mathrm{Hom}}}}(A^s, B^{s+n})$$

with the property that

$$H^0(X, \underline{\underline{\underline{\mathrm{Hom}}}}^\bullet(A^{^\bullet}, B^{^\bullet})) = \Gamma(X, \underline{\underline{\underline{\mathrm{Hom}}}}^\bullet(A^{^\bullet}, B^{^\bullet})) = H^0(\mathrm{Hom}^\bullet(A^\bullet, B^\bullet)) = [A^\bullet, B^\bullet]$$

5.5. The bounded *homotopy category* $K^b(X)$ of sheaves on X is the category whose objects are complexes of sheaves whose cohomology sheaves are bounded (meaning that $H^r(A^\bullet) = 0$ for sufficiently large r), and whose morphisms are homotopy classes of morphisms, that is,

$$\mathrm{Hom}_{K^b(X)}(A^\bullet, B^\bullet) = [A^\bullet, B^\bullet] = H^0(X, \underline{\underline{\underline{\mathrm{Hom}}}}^\bullet(A^\bullet, B^\bullet)).$$

5.6. **Wonderful properties of injective sheaves.** Roughly speaking, when we restrict to injective objects, then quasi-isomorphisms become homotopy equivalences. For sheaf theory, this is important because a homotopy of complexes of sheaves also gives a homotopy on global sections. In this way, quasi-isomorphisms of complexes of injective sheaves give isomorphisms on hypercohomology.

5.7. **Lemma.** *Let $C^\bullet$ be a (bounded below) complex of sheaves and suppose that the cohomology sheaves $\mathbf{H}^r(C^\bullet) = 0$ for all r. Let $J^\bullet$ be a complex of injective sheaves. Then any morphism $f : C^\bullet \to J^\bullet$ is homotopic to zero, meaning that there exists $h : C^\bullet \to J^\bullet[-1]$ such that $d_J h + h d_C = f$.*

Proof. It helps to think about the diagram of complexes:

$$\begin{tikzcd}
0 \arrow[r] & C^0 \arrow[r, "d^0"] \arrow[d, "f"'] & C^1 \arrow[r, "d^1"] \arrow[d, "f"'] \arrow[dl, "h^1"'] & C^2 \arrow[r, "d^2"] \arrow[d, "f"'] \arrow[dl, "h^2"'] & C^3 \arrow[r] \arrow[d, "f"'] \arrow[dl, "h^3"'] & \cdots \\
0 \arrow[r] & J^0 \arrow[r, "d^0"'] & J^1 \arrow[r, "d^1"'] & J^2 \arrow[r, "d^2"'] & J^3 \arrow[r] & \cdots
\end{tikzcd}$$

The first step is easy, since $C^0 \to C^1$ is an injection and since J^0 is injective there exists $h^1 : C^1 \to J^0$ that makes the triangle commute, that is, $h^1 d^0 = f$. Now let us define $h^2 : C^2 \to J^1$. Consider the map $(f - d^0 h^1) : C^1 \to J^1$. It vanishes on $\mathsf{Im}(d^0) = \ker(d^1)$ because

$$(f - d^0h^1)d^0 = fd^0 - d^0h^1d^0 = fd^0 - d^0f = 0.$$

Therefore it passes to a vertical mapping in this diagram:

$$\begin{tikzcd}
C^1 \arrow[r] \arrow[dr] & C^1/\ker(d^1) \arrow[r, hook] \arrow[d, "f - d^0h^1"] & C^2 \\
& J^1 &
\end{tikzcd}$$

where the second horizontal mapping is an injection. Since J^1 is injective we obtain an extension $h^2 : C^2 \to J^1$ such that $h^2 \circ d^1 = f - d^0h^1$ so that $h^2d^1 + d^0h^1 = f$. Continuing in this way, the other h^r can be constructed inductively. □

Exactly the same argument may be used to prove the following:

5.8. **Lemma.** *Let $f : A^\bullet \to B^\bullet$ be a quasi-isomorphism of (bounded below) complexes. Then for any complex $J^\bullet$ of injectives, the induced map $[B^\bullet, J^\bullet] \to [A^\bullet, J^\bullet]$ on homotopy classes is an isomorphism.* □

This result can also be proven by applying the previous lemma to the cone $C(f)$ and using the long exact sequence on cohomology.

5.9. **Corollary.** *The following statements hold.*

(1) *Suppose $J^\bullet$ is a complex of injective sheaves and $H^n(J^\bullet) = 0$ for all n. Then $J^\bullet$ is homotopy equivalent to the zero complex.*
(2) *Let $\phi : X^\bullet \to Y^\bullet$ be a quasi-isomorphism of sheaves of* injective *complexes. Then ϕ admits a homotopy inverse $g : Y^\bullet \to X^\bullet$ (meaning that $g\phi \sim I_X$ and $\phi g \sim I_Y$).*
(3) *Let $A^\bullet \to I^\bullet$ and $B^\bullet \to J^\bullet$ be injective resolutions of complexes $A^\bullet, B^\bullet$. Then any morphism $f : A^\bullet \to B^\bullet$ admits a lift $\tilde{f} : I^\bullet \to J^\bullet$ and any two such lifts are homtopic.*
(4) *Let $f : A^\bullet \to B^\bullet$ be a quasi-isomorphism of complexes of sheaves. Then f induces an isomorphism on hypercohomology $H^r(X, A^\bullet) \cong H^r(X, B^\bullet)$ for all r.*

Proof. For (1) consider the identity mapping $J^\bullet \to J^\bullet$ For (2) , mapping $(X^\bullet \xrightarrow{\phi} Y^\bullet)$ to $X^\bullet$ and using the lemma gives an isomorphism $[Y^\bullet, X^\bullet] \to [X^\bullet, X^\bullet]$, the map given by $f \mapsto f \circ \phi$. So the identity $X^\bullet \to X^\bullet$ corresponds to some f such that $f \circ \phi \sim Id$, implying that ϕ has a left homotopy-inverse. Now consider mapping $Y^\bullet$ into the triangle $X^\bullet \to Y^\bullet \to C(\phi) \to \cdots$, giving an exact sequence $\cdots \to [Y^\bullet, X^\bullet] \to [Y^\bullet, Y^\bullet] \to [Y^\bullet, C(\phi)] \to \cdots$. Since $C(\phi)$ is injective and its cohomology vanishes, the identity is homotopic to zero, hence $[Y^\bullet, C(\phi)] = 0$ so that $[Y^\bullet, X^\bullet] \cong [Y^\bullet, Y^\bullet]$ with the map given by $g \mapsto \phi \circ g$. Therefore there exists $g : Y^\bullet \to X^\bullet$ so that $\phi \circ g \sim Id$ meaning that ϕ has a right inverse in the homotopy category. If a mapping has both a left inverse and a right inverse then it has an inverse (in other words, f and g are homotopic, so either of them will behave as a homotopy inverse to ϕ). For (3), the lemma gives an isomorphism $[A^\bullet, J^\bullet] \to [I^\bullet, J^\bullet]$. For (4), the hypercohomology is defined in terms of the global sections of an injective resolution. So we may assume that $A^\bullet$ and $B^\bullet$ are injective. By the lemma, the cone $C(f)$ is homotopic to zero. Let h be such a homotopy. Now take global sections. The global sections of the cone coincides with the cone on the global sections, that is, we have a triangle of groups:

$$\begin{tikzcd}
\Gamma(X, A^\bullet) \arrow[rr] & & \Gamma(X, B^\bullet) \arrow[dl] \\
& \Gamma(X, C(f)) \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

The homotopy h also gives a homotopy on the global sections so that $H^n(\Gamma(X, C(f))) = 0$. So the long exact sequence on cohomology implies that $H^n(X, A^\bullet) \to H^n(X, B^\bullet)$ is an isomorphism. $\square$

6. Lecture 6: The derived category

There are several different "models' for the derived category. The first definition we give is easy to understand and useful for proofs but the objects themselves are not very natural. The second model is less intuitive but the objects occur naturally.

6.1. **The derived category: first definition.** Let X be a topological space. The bounded derived category $D^b(X)$ is the category whose objects are complexes of injective sheaves whose cohomology sheaves are bounded (meaning that $H^r(A^\bullet) = 0$ for sufficiently large r and for sufficiently small r). The morphisms are homotopy classes of morphisms (of complexes of sheaves), so that $D^b(X)$ is the *homotopy category* of (complexes of) injective sheaves.

If $\mathcal{S}(X)$ denotes the category of sheaves on X, if $C^b(X)$ denotes the category of complexes of sheaves with bounded cohomology and if $K^b(X)$ denotes the homotopy category of (complexes of) sheaves with bounded cohomology then we have a canonical functors

$$\mathcal{S}(X) \longrightarrow C^b(X) \longrightarrow K^b(X) \underset{\longleftarrow}{\xrightarrow{\text{God}}} D^b(X)$$

that associates to any sheaf S the corresponding complex concentrated in degree zero, and to any complex $A^\bullet$ its Godement injective resolution. [This construction makes sense if we replace the category of sheaves with any abelian category $\mathcal{C}$ provided it has enough injectives. In this way we define the bounded derived category $D^b(\mathcal{C})$ with functors $\mathcal{C} \to K^b(\mathcal{C}) \to D^b(\mathcal{C})$.]

From the previous lecture on "properties of injective sheaves" we therefore conclude:

- The mapping $K^b(X) \to D^b(X)$ is a functor (that is, a morphism between complexes determines a morphism in the derived category also).
- If $A^\bullet \to B^\bullet$ is a quasi-isomorphism of complexes of sheaves then it becomes an isomorphism in $D^b(X)$.
- if $A^\bullet$ is a complex of sheaves such that $H^m(A^\bullet) = 0$ for all m then $A^\bullet$ is isomorphic to the zero sheaf.

6.2. **Definition.** Let $T : \mathcal{S}(X) \to \mathcal{B}$ be a covariant (and additive) functor from the category of sheaves to some other abelian category with enough injectives. For the moment, let us also assume that it takes injectives to injectives. Define the *right derived functor* $RT : D^b(X) \to D^b(B)$ by $RT(A^\bullet)$ to be the complex $T(I^0) \to T(I^1) \to T(I^2) \to \cdots$ where $A^\bullet \to I^\bullet$ is the canonical (or the chosen) injective resolution of $A^\bullet$. Define $R^mT(A)$ ("the m-th derived functor", an older terminology) to be the cohomology object of this complex, $H^m(RT(A^\bullet))$.

Let $A^\bullet \to I^\bullet$ and $B^\bullet \to I^\bullet$ be the canonical injective resolutions of $A^\bullet, B^\bullet$. Then any morphism $f : A^\bullet \to B^\bullet$ has a lift $\tilde{f} : I^\bullet \to J^\bullet$ that is unique up to homotopy, which is to say that $\tilde{f}$ is a uniquely defined morphism in the category $D^b(X)$, and we obtain a well defined morphism

$$RT(f) = T(\tilde{f}) : RT(A^\bullet) = T(I^\bullet) \to T(J^\bullet) = RT(B^\bullet).$$

In other words, the right derived functor of T is obtained by replacing each complex $A^\bullet$ by its injective resolution $I^\bullet$ and then applying T to that complex.

6.3. **Examples.**
1. If $f : X \to Y$ is a continuous map between topological spaces, then we show (below) that $f_* : \mathcal{S}(X) \to \mathcal{S}(Y)$ takes injectives to injectives. Taking $Y = \{pt\}$ we get that the global sections functor Γ takes injectives to injectives. Then, for any complex of sheaves $A^\bullet$,

$$R^m\Gamma(A^\bullet) = H^m(\Gamma(X, I^\bullet))$$

where $A^\bullet \to I^\bullet$ is the canonical injective resolution. (This is how we defined the hypercohomology of the complex of sheaves A in §3.10.)

2. Let $f : X \to Y$ be a continuous map and let $\underline{\underline{\mathbb{Z}}}$ be the constant sheaf on X. If f is surjective and its fibers are connected then $f_*(\underline{\underline{\mathbb{Z}}})$ is again the constant sheaf, because as a presheaf, $f_*(\underline{\underline{\mathbb{Z}}}(U) = \underline{\underline{\mathbb{Z}}}(f^{-1}(U)) = \mathbb{Z}$ for any connected open set $U \subset Y$. Although we cannot hope to understand $Rf_*(\underline{\underline{\mathbb{Z}}})$ we can understand its cohomology sheaves:

$$R^m f_*(\underline{\underline{\mathbb{Z}}})(U) = H^m(\Gamma(f^{-1}(U), I^\bullet))$$

where $I^\bullet$ is an injective resolution (or perhaps the canonical injective resolution) of the constant sheaf. But this is exactly the definition of the hypercohomology $H^m(f^{-1}(U), \underline{\underline{\mathbb{Z}}}) = H^m(f^{-1}(U, \mathbb{Z}))$. If f is proper then the stalk cohomology (of the cohomology sheaf of $Rf_*(\underline{\underline{\mathbb{Z}}})$) at a point $y \in Y$ is equal to $H^m(f^{-1}(y); \mathbb{Z})$, the cohomology of the fiber. In other words, the sheaf $\underline{\underline{\mathbf{H}}}^m(Rf_*(\underline{\underline{\mathbb{Z}}}))$ is a sheaf on Y which, as you move around in Y, displays the cohomology of the fiber.

3. We can also determine the global cohomology of the complex $Rf_*(\underline{\underline{\mathbb{Z}}})$, for it is the cohomology of the global sections $\Gamma(X, I^\bullet)$, that is, the cohomology of X. More generally, the same argument shows that: for any complex of sheaves $A^\bullet$ on X, the complex $Rf_*(A^\bullet)$ is a sheaf on Y whose global cohomology is

$$H^*(Y, Rf_*(A^\bullet)) \cong H^*(X, A^\bullet).$$

This complex of sheaves therefore provides data on Y which allows us to compute the cohomology of X. It is called the Leray Sheaf (although historically, Leray really considered only its cohomology sheaves $R^m f_*(A^\bullet)$). In particular we see that *the functor Rf_* does not change the hypercohomology*. For $f : X \to \{\text{pt}\}$, if S is a sheaf on X then $f_*(S) = \Gamma(X, S)$ is the functor of global sections (or rather, it is a sheaf on a single point whose value is the global sections), so $R^i f_*(A^\bullet) = H^i(X, A^\bullet)$ is the hypercohomology.

4. The m-th derived functor of Hom is called Ext^m, i.e., it is the group

$$\text{Ext}^m(A^\bullet, B^\bullet) = H^m(\text{RHom}(A^\bullet, B^\bullet)) = H^m(\text{Hom}^\bullet(A^\bullet, J^\bullet)) = H^0(\text{Hom}^\bullet(A^\bullet, J^\bullet[m]))$$

where $B^\bullet \to J^\bullet$ is an injective resolution. (We consider $\text{Hom}(A^\bullet, B^\bullet)$ to be a functor of the $B^\bullet$ variable and derive it by injectively resolving. It turns out, as we will see later, that the same result can be obtained by *projectively* resolving $A^\bullet$.) As before, there is a sheaf version of Hom, which also gives a sheaf version of Ext:

$$\underline{\underline{\mathrm{Ext}}}^m(A^\bullet, B^\bullet)) = \underline{\underline{H}}^m(\underline{\underline{\mathrm{RHom}}}(A^\bullet, B^\bullet))$$

Exercise. Let G, H be abelian groups, considered as complexes in degree zero. Show that $\mathrm{Ext}^1(G, H)$ coincides with the usual definition of $\mathrm{Ext}_{\mathbb{Z}}(G, H)$,

6.4. **The derived category: second construction.** The derived category can be constructed as a sort of quotient category of the homotopy category $K^b(X)$ of complexes, by inverting quasi-isomorphisms. Let $E^b(X)$ be the category whose objects are complexes of sheaves on X, and where a morphism $A^\bullet \to B^\bullet$ is an equivalence class of diagrams

$$\begin{tikzcd} & C^\bullet \arrow[dl, "qi"'] \arrow[dr] & \\ A^\bullet & & B^\bullet \end{tikzcd}$$

where $C^\bullet \to A^\bullet$ is a quasi-isomorphism, and where two such morphisms $A^\bullet \leftarrow C_1^\bullet \to B^\bullet$ and $A^\bullet \leftarrow C_2^\bullet \to B^\bullet$ are considered to be equivalent if there exists a diagram

$$\begin{tikzcd} & C_1^\bullet \arrow[dl] \arrow[dr] & \\ A^\bullet & C_3^\bullet \arrow[l, "qi"'] \arrow[r] \arrow[u] \arrow[d] & B^\bullet \\ & C_2^\bullet \arrow[ul] \arrow[ur] & \end{tikzcd}$$

that is commutative up to homotopy. (Exercise: figure out how to compose two morphisms and then check that the result is well defined with respect to be above equivalence relation.)

6.5. **Theorem.** *The natural functor $D^b(X) \to E^b(X)$ is an equivalence of categories.*

Proof. To show that a functor $F : \mathcal{C} \to \mathcal{D}$ is an equivalence of categories it suffices to show (a) that it is essentially surjective, meaning that every object in $\mathcal{D}$ is isomorphic to an object $F(C)$ for some object C in $\mathcal{C}$, and (b) that F induces an isomorphism on Hom sets. The first part (a) is clear because we have injective resolutions. Part (b) follows immediately from the fact that a quasi-isomorphism of injective complexes is a homotopy equivalence and has a homotopy inverse. $\square$

This gives a way of referring to elements of the derived category without having to injectively resolve. Each complex of sheaves is automatically an object in the derived category $E^b(X)$. Here are some applications.

6.6. **T-acyclic resolutions.** A functor $T : \mathcal{C} \to \mathcal{D}$ between abelian categories is *exact* if it takes exact sequences to exact sequences. It is *left exact* if it preserves kernels, that is, if $f : X \to Y$ and if $Z = \ker(f)$ (so $0 \to Z \to X \to Y$ is exact) then $T(Z) = \ker(T(f))$ (that is, $0 \to T(Z) \to T(X) \to T(Y)$ is exact). An object X is T-acyclic if $R^iT(X) = 0$ for all $i \neq 0$. This means: take an injective resolution $X \to I^\bullet$, apply T, take cohomology, the result should be zero except possibly in degree zero.

The great advantage of T-acyclic objects is that they may be often used in place of injective objects when computing the derived functors of T, that is

6.7. **Lemma.** *Let T be a left exact functor from the category of sheaves to some abelian category with enough injectives. Let $A^\bullet$ be a complex of sheaves and let $A^\bullet \to X^\bullet$ be a quasi-isomorphism, where each of the sheaves X^r is T-acyclic. Then $R^rT(A^\bullet)$ is canonically isomorphic to the r-th cohomology object of the complex*

$$T(X^0) \to T(X^1) \to T(X^2) \to \cdots .$$

If T is exact then there is no need to take a resolution at all: $RT(A^\bullet)$ is canonically isomorphic to $T(A^\bullet)$.

The proof is the standard double complex argument: Suppose T is left exact. Let $I^{\bullet\bullet}$ be a double complex of injective sheaves, the r-th row of which is an injective resolution of X^r. Let $Z^\bullet$ be the total complex of this double complex. It follows that $A^\bullet \to X^\bullet \to Z^\bullet$ are quasi-isomorphisms and so the complex $Z^\bullet$ is an injective resolution of $A^\bullet$. Now augment the double complex by attaching $X^\bullet$ to the zeroth column, and apply the functor T to the augmented complex. Since each X^r is T-acyclic, each of the rows remains exact except possibly at the zeroth spot. Since T is left exact, the rows are also exact at the zeroth spot. So our lemma says that $T(X^\bullet) \to T(Z^\bullet) = RT(A^\bullet)$ is also a quasi-isomorphism, which gives an isomorphism between their cohomology objects. If the functor T is exact then every object A^r is T-acyclic (exercise), so the original complex $A^\bullet$ may be used as its own T-acyclic resolution.

6.8. **Key exercises.** Show that injective objects are T-acyclic for any left exact functor T. If $f : X \to Y$ is a continuous map, show that $f^* : \mathcal{S}(Y) \to \mathcal{S}(X)$ is exact (and so it does not need to be derived). Using this and the adjunction formula $\mathrm{Hom}_{\mathcal{S}(Y)}(B, f_*(I)) \cong \mathrm{Hom}_{\mathcal{S}(X)}(f^*(B), I)$ show that f_* is left exact and takes injectives to injectives. Show that fine, flabby, and soft sheaves are Γ-acyclic. In particular, *the cohomology of a sheaf (or of a complex of sheaves) may be computed with respect to any injective, fine, flabby, or soft resolution.*

6.9. **More derived functors.** This also gives us a way to define "the" derived functor $RT : D^b(X) \to D^b(\mathcal{C})$ for any left exact functor $T : Sh(X) \to \mathcal{C}$ provided the category $\mathcal{C}$ has enough injectives, namely, if $A^\bullet$ is a complex of sheaves on X, take an injective resolution $A^\bullet \to I^\bullet$, then apply the functor T to obtain a complex $T(I^\bullet)$ of objects in the category $\mathcal{C}$, then injectively resolve this complex by the usual method of resolving each $T(I^r)$ to obtain a double complex, then forming the associated total complex. Let RT denote the resulting complex. Different choices of resolutions give isomorphic complexes RT.

6.10. **The sheaf of chains in the derived category.** Suppose X is a finite simplicial complex. If σ is a (closed) simplex let $\underline{\mathbb{Q}}_\sigma$ denote the constant sheaf on σ. It is injective in the category of simplicial sheaves on X, and every injective simplicial sheaf (of rational vector spaces) is a direct sum of such *elementary injectives*. The sheaf of chains can be realized as the (injective) complex $\underline{\mathbf{C}}_\bullet$:

$$\bigoplus_{\dim(\sigma)=0} \underline{\mathbb{Q}}_\sigma \longleftarrow \bigoplus_{\dim(\sigma)=1} \underline{\mathbb{Q}}_\sigma \longleftarrow \bigoplus_{\dim(\sigma)=2} \underline{\mathbb{Q}}_\sigma \longleftarrow \cdots$$

in degrees $0, -1, -2, \cdots$ respectively. (We place the chains in negative degrees so that the differentials will increase degree; it is a purely formal convention.) The global sections of this complex equals the usual complex $C_\bullet(X)$ of simplicial chains. If we use the constant sheaf $\mathbb{Z}_\sigma$ everywhere, then the resulting sheaves are soft, rather than injective, but they may still be used to compute the homology of X.

Now consider the limit over all subdivisions of the simplicial complex X. We define a "topological" sheaf on X that is, in some sense, the sheafification of the direct limit of these sheaves. To be precise, let $U \subset X$ be an open subset and let T be a locally finite triangulation of U, and let $C_r^T(U)$ be the group of r-dimensional simplicial chains with respect to this triangulation. Then the *sheaf of piecewise linear chains* is the sheaf $\underline{\mathbf{C}}_{PL}^\bullet$ with sections

$$\Gamma(U, \underline{\mathbf{C}}_{PL}^{-r}) = \varinjlim_{T} C_r^T(U)$$

for $r \geq 0$. It is a *soft* sheaf, and the resulting complex

$$\underline{\mathbf{C}}_{PL}^{0} \longleftarrow \underline{\mathbf{C}}_{PL}^{-1} \longleftarrow \underline{\mathbf{C}}_{PL}^{-2} \longleftarrow \cdots$$

is quasi-isomorphic to the sheaf of Borel-Moore chains. If the space X has a real analytic (or semi-analytic or subanalytic or $\mathcal{O}$-minimal) structure then one similarly has the sheaf of locally finite subanalytic or $\mathcal{O}$-minimal chains, which gives another quasi-isomorphic "incarnation" of the sheaf of chains.

Evidently, the derived category would be more useful if such complexes could be considered to be objects in the derived category (without having to resort to taking injective resolutions).

6.11. **The bad news.** The derived category is not an abelian category. In fact, the homotopy category of complexes is not an abelian category. Kernels and cokernels do not make sense in these categories. The saving grace is that the cone operation still makes sense and in fact, it passes to the homotopy category. So we have to replace kernels and cokernels with triangles.

6.12. **Definition.** A triangle of morphisms

$$\begin{tikzcd} A^\bullet \arrow[rr] & & B^\bullet \arrow[dl] \\ & C^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

in $K^b(X)$ or in $D^b(X)$ is said to be a *distinguished triangle* if it is homotopy equivalent to a triangle

$$\begin{tikzcd} X^\bullet \arrow[rr, "\phi"] & & Y^\bullet \arrow[dl] \\ & C(\phi) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

where $C(\phi)$ denotes the cone on the morphism ϕ.

This means that there are morphisms between corresponding objects in the triangles such that the resulting squares commute up to homotopy. (In the homotopy categories $K(X)$ and $D^b(X)$ homotopy equivalences are isomorphisms, so people often define a distinguished triangle to be a triangle in $K(X)$ that is isomorphic to a mapping cone.)

6.13. **Lemma.** *The natural functor $K^b(X) \to D^b(X)$ takes distinguished triangles to distinguished triangles. If*

$$\begin{tikzcd} A^\bullet \arrow[rr] & & B^\bullet \arrow[dl] \\ & C^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

is a distinguished triangle and if $X^\bullet$ is a complex (bounded from below) then there are distinguished triangles

$$\begin{tikzcd} \mathrm{RHom}(X^\bullet, A^\bullet) \arrow[rr] & & \mathrm{RHom}(X^\bullet, B^\bullet) \arrow[dl] \\ & \mathrm{RHom}(X^\bullet, C^\bullet) \arrow[ul, "{[1]}"] & \end{tikzcd} \quad \textit{and} \quad \begin{tikzcd} \mathrm{RHom}(A^\bullet, X^\bullet) \arrow[dr, "{[1]}"'] & & \mathrm{RHom}(B^\bullet, X^\bullet) \arrow[ll] \\ & \mathrm{RHom}(C^\bullet, X^\bullet) \arrow[ur] & \end{tikzcd}$$

The proof is the observation (from the last lecture) that Hom into a cone is equal to the cone of the Homs. We stress again that the hypercohomology of RHom is exactly the group of homomorphisms in the derived category: $H^0(X, \mathrm{RHom}(A^\bullet, B^\bullet)) = \mathrm{Hom}_{D^b(X)}(A^\bullet, B^\bullet)$.

6.14. **Exact sequence of a pair.** Let Z be a closed subspace of a topological space X, and let $U = X - Z$, say $Z \xrightarrow{i} X \xleftarrow{j} U$.

If S is a sheaf on X then there is a short exact sequence of sheaves $0 \to j_!j^*S \to S \to i_*i^*S \to 0$. The morphisms are obtained by adjunction, and exactness can be checked stalk by stalk: If $x \in Z$ then the sequence reads $0 \to 0 \to S_x \to S_x \to 0$. If $x \in U$ then the sequence reads $0 \to S_x \to S_x \to 0 \to 0$. Consequently if $A^\bullet$ is a complex of sheaves then there is a distinguished triangle

$$\begin{tikzcd} Rj_!j^*(A^\bullet) \arrow[rr] & & A^\bullet \arrow[dl] \\ & Ri_*i^*(A^\bullet) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

(recall the exercise that the cokernel of an injective morphism is quasi-isomorphic to the mapping cone). For the constant sheaf the triangle gives an exact sequence

$$H^r(X) \to H^r(Z) \to H_c^{r+1}(U) \to \cdots$$

but observe that $H_c^*(U) = H^*(X, Z)$ is the cohomology of cochains on X that vanish on Z. For the sheaf of chains this gives an exact sequence $H_r(U) \to H_r(X) \to H_r(X, U) \to \cdots$ because $i^*(\text{chains})$ is the limit over open sets containing Z of Borel-Moore chains on that open set.

If S is a sheaf on X define $i^!(S)$ to be the restriction to Z of the presheaf with sections supported in Z, that is

$$i^!(S) = i^*(S^Z) \text{ where } \Gamma(V, S^Z) = \{s \in \Gamma(V, S) |\ \operatorname{spt}(s) \subset Z\}.$$

Thus, if $W \subset Z$ is open then

$$\Gamma(W, i^!(S)) = \varinjlim_{V \supset W} \Gamma_Z(V, S)$$

(the limit is over open sets $V \subset X$ containing W). The functor $i^!$ is a right adjoint to the push-forward with compact support $i_!$, that is, $\mathrm{Hom}_X(i_!A, B) = \mathrm{Hom}_Z(A, i^!B)$. In fact, $\mathrm{Hom}_Z(A, i^!B)$ consists of mappings of the leaf space $LA \to LB|Z$ that can be extended by zero to a neighborhood of Z in X, and this is the same as $\mathrm{Hom}_X(i_!A, B)$. (In particular we obtain a canonical morphism $i_!i^!B \to B$.)

6.15. **Proposition.** *Let $A^\bullet$ be a complex of sheaves on X. There is another distinguished triangle*

$$\begin{tikzcd} Ri_*i^!(A^\bullet) \arrow[rr] & & A^\bullet \arrow[dl] \\ & Rj_*j^*(A^\bullet) \arrow[ul, "{[1]}"] & \end{tikzcd}$$

Later we will prove this using Verdier duality. For now, observe that in the case of the constant sheaf this gives an exact sequence

$$H^r(X) \to H^r(U) \to H^r(X, U) \to \cdots$$

For the sheaf of chains, $i^!C_{-r}$ will give the homology of a tiny neighborhood of Z in X which, for most nice spaces, will be homotopy equivalent to Z itself. The sheaf $j^*(C^{-r})$ will give the Borel-Moore homology of U, which is the relative homology $H_r(X, Z)$. So this triangle gives the long exact sequence for the homology of the pair (X, Z).

7. Lecture 7: Stratifications

7.1. For some historical comments see:
`http://www.math.ias.edu/~goresky/math2710/ThomMather.pdf`

7.2. The plan is to decompose a reasonable space into a locally finite union of smooth manifolds (called strata) which satisfy the axiom of the frontier: the closure of each stratum should be a union of lower dimensional strata. If $Y \subset \overline{X}$ are strata we write $Y < X$. But this should be done in a locally trivial way. In Whitney's example below, it is not enough to divide this figure

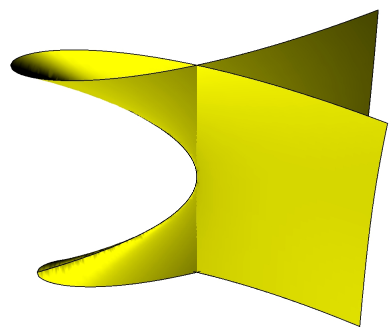


Figure 1. Three strata needed

into 1- and 2-dimensional strata, even though this gives a decomposition into smooth manifolds. If the origin is not treated as another stratum then the stratification fails to be "locally trivial". Whitney proposed a condition that identifies the origin as a separate stratum in this example. Let us say that a stratification of a closed subset W of some smooth manifold M is a locally finite decomposition $W = \coprod_\alpha S_\alpha$ into locally closed smooth submanifolds $S_\alpha \subset M$ (called strata) so as to satisfy the axiom of the frontier. A *stratified homeomorphism* $h : W_1 \to W_2$ between two stratified sets is a homeomorphism that takes strata to strata and is smooth on each stratum.

7.3. **Definition.** Let $Y \subset \overline{X}$ be strata in a stratification of a closed set $W \subset M$. The pair (X, Y) *satisfies Whitney's condition B* at a point $y \in Y$ if the following holds. Suppose that $x_1, x_2, \cdots \in X$ is a sequence that converges to y, and suppose that $y_1, y_2, \cdots \in Y$ is a sequence that also converges to y. Suppose that (in some local coordinate system near y) the secant lines $\ell_i = \overline{x_i, y_i}$ converge to some limiting line ℓ. Suppose that the tangent planes $T_{x_i}X$ converge to some limiting plane τ. Then $\ell \subset \tau$.

We say the pair (X, Y) satisfies condition B if it does so at every point $y \in Y$. The decomposition into strata is a *Whitney stratification* if every pair of strata $Y < X$ satisfies condition B at every point in the smaller stratum Y.

(If condition B is satisfied, and if the tangent planes $T_{y_i}Y$ also converge to some limiting plane η then $\eta \subset \tau$ as well, which Whitney had originally proposed as an additional condition, which he called Condition A.) It turned out that Whitney's condition B was just the right condition to guarantee that a stratification is locally trivial, but the verification involved the full development of stratification theory by René Thom and John Mather. The problem is that stratifications satisfying condition B may still exhibit certain pathologies, such as infinite spirals, so there is a very delicate balance between proving that local triviality holds while avoiding a host of counterexamples to similar sounding statements.

Suppose $W \subset M$ has a stratification that satisfies condition B. Let Y be a stratum and let $y \in Y$. Let $N_y \subset M$ be a *normal slice*, that is, a smooth submanifold of dimension $\dim(N_y) = \dim(M) - \dim(Y)$ that intersects Y transversally in the single point $\{y\}$. Define the *link of the stratum* Y,

$$L_Y = L_Y(y, \epsilon) = (\partial B_\epsilon(y)) \cap N_y \cap W$$

where $B_\epsilon(y)$ is a ball of radius ϵ (measured in some Riemannian metric on M) centered at the point y.

7.4. **Theorem.** *(R. Thom, J. Mather) If ϵ is chosen sufficiently small then*

(1) the closed set L_Y is stratified by its intersection with the strata Z of W such that $Z > Y$
(2) this stratification satisfies condition B
(3) the stratified homeomorphism type of L_Y is independent of the choice of N_y, ϵ, and the Riemannian metric
(4) if the stratum Y is connected then the stratified homeomorphism type of L_Y is also independent of the point y.

Moreover, the point y has a basic neighborhood $U_y \subset W$ *and a stratified homeomorphism*

$$U_y \cong c^o(L_Y) \times B$$

where $c^o(L_Y)$ denotes the open cone on L_Y (with its obvious stratification) and where B denotes the open ball of radius 1 in $\mathbb{R}^{\dim(Y)}$.

This homeomorphism preserves strata in the obvious way: it takes

(1) $\{y\} \times B \to Y \cap U$ with $\{y\} \times \{0\} \to \{y\}$
(2) $(L_Y \cap X) \times (0, 1) \times B \to X \cap U$ for each stratum $X > Y$

This result says that the set W does not have infinitely many holes or infinitely much topology as we approach the singular stratum Y and it says that the normal structure near Y is locally trivial as we move around in Y. In particular, the collection of links $L_Y(y)$ form the fibers of a stratified fiber bundle over Y.

It also implies that (for any $r \geq 0$) the local homology $H_r(W, W - y; \mathbb{Z})$ forms a local coefficient system on Y with stalk

$$H_r(W, W - y; \mathbb{Z}) \cong H_{r-\dim(Y)-1}(L_Y; \mathbb{Z}).$$

7.5. In fact, Thom and Mather proved that a Whitney stratified W set admits a system of *control data* consisiting of a triple (T_Z, π_Z, ρ_Z) for each stratum Z, where T_Z is a neighborhood of Z in W, where $\pi_Z : T_Z \to Z$ is a “tubular projection”, $\rho_Z : T_Z \to [0, \epsilon)$ is a “tubular distance function” so that the pair $(\pi_Z, \rho_Z)|Y \cap TZ : Y \cap T_Z \to Z \times (0, \epsilon)$ is a smooth submersion for each stratum $Y > Z$, and where $\pi_Z \pi_Y = \pi_Z$ in $T_Z \cap T_Y$ and $\rho_Z \pi_Y = \rho_Z$. (picture).

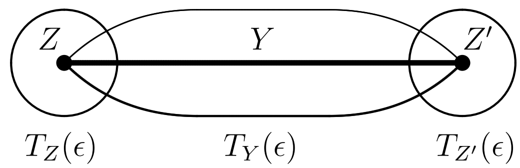


FIGURE 2. Tubular neighborhoods

7.6. This data was then used to construct *controlled vector fields* that trace out the local triviality of the stratification.

7.7. Whitney himself outlined a procedure for proving that any closed subset W of Euclidean space defined by analytic equations admits a Whitney stratification. The idea is to start with the open, nonsingular part W^0 of W as the "top" stratum, and then to look at the set of points in the singular set $\Sigma = W - W^0$ where condition B *fails*. He proves that this is an analytic subset of codimension two, whose complement in Σ is therefore the first singular stratum, W^1. Now, carry both W^0 and W^1 along, looking at the set of points (in what remains) where condition B fails, and continue in this way inductively. Since Whitney's early work, many advances have been made in the subject. The following statement is at best a partial summary of the work of many people.

7.8. **Theorem.** *The following sets admit Whitney stratifications: real and complex algebraic varieties, real and complex analytic varieties, semi-algebraic and semi-analytic varieties, subanaltyic sets, and sets with o-minimal structure. Given such an algebraic (resp. analytic etc.) variety W and a locally finite union Z of algebraic (resp. analytic etc.) subvarieties, the stratification of W can be chosen so that Z is a union of strata. Given an algebraic (resp. analytic etc.) mapping $f : W \to W'$ of algebraic (resp. analytic etc.) varieties, it is possible to find Whitney stratifications of W, W' so that the mapping f takes strata to strata, and so that for each stratum X of W the mapping $X \to f(X)$ is a smooth submersion onto a stratum of W'. Whitney stratified sets can be triangulated by a triangulation that is smooth on each stratum, such that that the closure of each stratum is a subcomplex of the triangulation.*

[A subanalytic set is the image under a projection (for example, a linear projection $\mathbb{R}^m \to \mathbb{R}^n$) of an analytic or a semi-analytic set. O-minimal structures allow for dertain non-analytic functions to be included in the definition of the set. Whitney stratifications also make sense for algebraic varieties defined over fields of finite characteristic. Given an algebraic mapping $f : W \to W'$ between complex algebraic varieties, it is not generally possible to choose triangulations of W, W' so that f becomes a simplicial mapping.]

7.9. **Pseudomanifolds and Poincaré duality.** A pseudomanifold of dimension n is a purely n dimensional (Whitney) stratified space that can be triangulated so that every $n-1$ dimensional simplex is a face of exactly two n-dimensional simplices. This implies that the $n-1$ dimensional simplices can be combined with the n-dimensional simplices to form an n-dimensional manifold, and that the remainder (hence, the "singularity set") has dimension $\leq n-2$. If this manifold is orientable then an orientation defines a fundamental class $[W] \in H_n(W;\mathbb{Z})$. Cap product with the fundamental class defines the Poincaré duality map $H^r(W;\mathbb{Z}) \to H_{n-r}(W;\mathbb{Z})$ which is an isomorphism if W is a manifold (or even a homology manifold) but which, in general, is not an isomorphism.

There is a sheaf-theoretic way to say this. If W is oriented and n-dimensional then a choice of orientation determines a sheaf map $\underline{\mathbb{Z}}_W \to \underline{C}_n$ to the sheaf of n-chains. On any open set U, choose a triangulation of U and map $m \in \mathbb{Z}$ to m times the sum of all the n-dimensional simplices in U. (Recall that a PL chains are identified under subdivision.) Therefore, if W is an n-dimensional homology manifold, that is, if $H_r(W, W-x) = 0$ for all $0 \leq r < n$ and $H_n(W, W-x;\mathbb{Z}) = \mathbb{Z}$ then the map

$$\underline{\mathbb{Z}}_W[n] \to \underline{C}^{\bullet}_W$$

is a quasi-isomorphism. This simple statement is the Poincaré duality theorem. For, it says that this quasi-isomorphism induces an isomorphism on cohomology, that is,

$$H^r(W;\mathbb{Z}) \cong H^{BM}_{n-r}(W;\mathbb{Z})$$

and an isomorphism on cohomology with compact supports, that is,

$$H^r_c(W;\mathbb{Z}) \cong H_{n-r}(W;\mathbb{Z}).$$

[Actually, from this point of view, the deep fact is that $H^i(W;k)$ and $H_i(W;k)$ are dual over any field k, but this is not a fact about manifolds. Rather, it is a fact about the sheaf of chains.]

More generally if W is not necessarily orientable then the *orientation sheaf* $\mathcal{O}_W$ is the local system whose stalk at $x \in W$ is the top local homology $H_n(W, W-x)$ and the mapping $O_W \to \underline{C}^{\bullet}$ is a quasi-isomorphism. So, for any local coefficient system $\mathcal{L}$ on W the mapping $\mathcal{L} \otimes \mathcal{O}_W \to \underline{C}^{\bullet}(\mathcal{L})$ is a quasi-isomorphism, giving an isomorphism on cohomology,

$$H^r(W;\mathcal{L} \otimes \mathcal{O}_W) \cong H^{BM}_{n-r}(W;\mathcal{L})$$

and on cohomology with compact supports,

$$H^r_c(W;\mathcal{L} \otimes \mathcal{O}_W) \cong H_{n-r}(W;\mathcal{L}).$$

So this quasi-isomorphism statement includes the Poincaré duality theorem for orientable and non-orientable manifolds, for non-compact manifolds, and for manifolds with boundary, and with possibly nontrivial local coefficient systems.

8. Lecture 8: Constructible sheaves

8.1. **Constructible sheaves.** Fix a Whitney stratification of a closed subset $W \subset M$ of some smooth manifold. A sheaf S (of abelian groups, or of R modules) on W is *constructible with respect to this stratification* if the restriction $S|X$ to each stratum X is a locally constant sheaf and the stalks S_x are finitely generated. A complex of sheaves $A^\bullet$ on W is *cohomologically constructible* with respect to this stratification if its cohomology sheaves are bounded (that is, $\underline{H}^r(S^\bullet) = 0$ for $|r|$ sufficiently large) and constructible.

If W is an complex algebraic (resp. complex analytic, resp. real algebraic etc.) variety then a complex of sheaves $S^\bullet$ on W is *algebraically construcible* (resp. analytically constructible, etc.) if its cohomology sheaves are bounded and constructible with respect to some algebraic (resp. analytic etc.) Whitney stratification.

In each of these constructibility settings (that is, constructible with respect to a fixed stratification, or algebraically constructible, etc.) the two constructions of the derived category make sense (as the homotopy category of injective complexes, or as the category of complexes and equivalence classes of roofs), which is then referred to as the bounded constructible derived category and denoted $D^b_c(W)$.

8.2. **Theorem.** *Suppose that W is a compact subset of some smooth manifold M and suppose that $S^\bullet$ is a complex of sheaves that is cohomologically constructible with respect to some Whitney stratification of W. Then the hypercohomology groups $H^r(W, S^\bullet)$ are finitely generated. If U_x is a basic neighborhood of $x \in W$ then the stalk cohomology $\underline{H}^r(S^\bullet)_x$ coincides with the cohomology $H^r(U_x S^\bullet)$ for all r (and so the limit over open sets containing x is essentially constant). If $i : Z \to X$ is a closed union of strata with open complement $j : U \to X$ then $Ri_*i^*(S^\bullet)$ and $Rj_*j^*(S^\bullet)$ are also cohomologically constructible. If $A^\bullet, B^\bullet$ are cohomologically constructible then so is* $\mathrm{RHom}^\bullet(A^\bullet, B^\bullet)$. *If $f : W \to W'$ is a proper stratified mapping and $A^\bullet$ is CC on W then $Rf_*(A^\bullet)$ is CC on W'.*

Proof. Let X be the top stratum and let $\Sigma = W - X$ be the singular set. Let U be the union of the tubular neighborhoods of the strata in Σ. Then $X - (X \cap U)$ is compact, and as U shrinks these form a sequence of diffeomorphic compact manifolds with corners that exhaust X. If $\mathcal{L}$ is a local system on X then (since W is compact)

$$H^*_c(X, \mathcal{L}) = H^*(X - (X \cap U), \mathcal{L})$$

is finitely generated. From the spectral sequence for cohomology of a complex, the same holds for $H^*_c(X, A^\bullet)$ since the cohomology sheaves of $A^\bullet$ are local systems on X.

Now consider the exact triangle

$$Rj_!j^*A^\bullet \to A^\bullet \to Ri_*i^*(A^\bullet) \to \cdots .$$

The cohomology of $Rj_!j^*A^\bullet$ is $H^*_c(X, A^\bullet)$ which is finitely generated as just shown. The complex $i^*(A^\bullet)$ is a constructible complex on Σ, which has smaller dimension, so its cohomology is finitely generated by induction. The long exact sequence implies the cohomology of $A^\bullet$ is finitely generated.

The stalk cohomology coincides with the cohomology of U_x because the family of these basic neighborhoods are cofinal in the set of all neighborhoods of x but as they shrink there are stratified isomorphisms $h : U_x \to U'_x$ with the inverse given by inclusion. Since the cohomology sheaves of $A^\bullet$ are locally constant on each stratum there is a quasi-isomorphism $h_*(A^\bullet) \to A^\bullet$ which induces isomorphisms on cohomology. In other words, $H^*(U_x, A^\bullet)$ is independent of the choices, so the limit stabilizes. Constructibility of $Ri_*i^*A^\bullet$ is obvious but constructibility of $Rj_*j^*A^\bullet$ takes some work. Here is the key point

8.3. **Lemma.** *Let $A^\bullet$ be a cohomologically constructible complex of sheaves on W. Let $Z \subset W$ be a closed subset with complement $j : V \to W$. Let X be the largest stratum of Z. Then the stalk cohomology at $x \in X$ of $Rj_*j^*(A^\bullet)$ is*

(8.3.1) $$H^i(Rj_*j^*A^\bullet)_x \cong H^i(L_x, A^\bullet).$$

The Lemma follows from the fact that the stalk cohomology is $H^i(Rj_*j^*A^\bullet)_x = H^i(j^{-1}(U_x \cap V); A^\bullet)$ where $U_x \cong c^o(L_x) \times B^{\dim(X)}$ is a basic open neighborhood. Since X is the largest stratum of Z, we have:

$$U_x \cap V \cong L_x \times (0,1) \times B^{\dim(X)},$$

and the cohomology sheaves of $A^\bullet$ are constant in the Euclidean directions of this product.

Since L_x is compact this cohomology is finitely generated. It is locally constant as $x \in X$ varies because the same is true of L_x and the cohomology sheaves of $A^\bullet|L_x$. □

In summary, the derived category $D^b_c(W)$ of complexes whose cohomology sheaves are bounded and constructibe forms a "paradise", in the words of Verdier, who had assured us (when we were writing IH II) that such a category, in which all these operations made sense, and was closed under pullback, proper push forward, Hom and Verdier duality, did not exist.

8.4. **Attaching sheaves.** Let us examine the triangle for $Rj_*j^*A^\bullet$ for $i \le \mathrm{cod(X)}-1$ and its stalk cohomology:

(8.4.1)
$$\begin{tikzcd}[column sep=small] Ri_*i^!(A^\bullet) \arrow[rr] & & A^\bullet \arrow[dl, "\alpha"] \\ & Rj_*j^*(A^\bullet) \arrow[ul, "\beta"', "{[1]}"] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=small] H_c^{i-\dim(X)}(U_x; A^\bullet) \arrow[rr] & & \underline{H}^i(A^\bullet)_x \arrow[dl, "\alpha"] \\ & H^i(L_x, A^\bullet) \arrow[ul, "\beta"', "{[1]}"] & \end{tikzcd}$$

The *attaching map* α goes from information ($\underline{H}^i(A^\bullet)_x$) living on the small stratum to information ($H^i(L_x, A^\bullet)$) living completely in the larger strata and so it represents the degree to which the sheaf $A^\bullet$ is "glued" across the strata.

Exercise. Suppose $W = X < U$ consists of two strata. Let $B^\bullet, C^\bullet$ be sheaves on X and on U respectively and let $A^\bullet = Ri_*(B^\bullet) \oplus Rj_!(C^\bullet)$ so that $A^\bullet$ consists of just these two sheaves with no relation between them. Show that the attaching homomorphism α is zero. For example, if $B^\bullet = \mathbb{Z}_X$ and if $C^\bullet = \mathbb{Z}_U$ then $A^\bullet$ is a sheaf whose stalk at each point is $\mathbb{Z}$ however it is not the constant sheaf. Show that if $A^\bullet = \mathbb{Z}_W$ is the constant sheaf then the attaching homomorphism α is injective.

8.5. **A motivating example.** Consider $W = \Sigma T^3$, the suspension of the 3-torus with singular points denoted $\{N\}, \{S\}$. We have natural cycles, point, T^1, ΣT^1, T^2, ΣT^2, T^3, ΣT^3. Some of these hit the singular points, some do not. The ones that do not are homologous to zero by a homology that hits the singular point. If we restrict cycles and homologies by not allowing them to hit the singular point, this will change the resulting homology groups. For $p = 0, 1, 2$ define $C_i^p(W) = \{\xi \in C_i(W) |\ \xi \cap \{N, S\} = \phi$ unless $i \geq 4 - p\}$ Here are the resulting homology groups.

	$p=0$	$p=1$	$p=2$
$i=4$	ΣT^3	ΣT^3	ΣT^3
$i=3$	0	ΣT^2	ΣT^2
$i=2$	T^2	0	ΣT^1
$i=1$	T^1	T^1	0
$i=0$	$\{pt\}$	$\{pt\}$	$\{pt\}$

	$p=0$	$p=1$	$p=2$
$i=4$	T^3	T^3	T^3
$i=3$	0	T^2	T^2
$i=2$	0	0	T^1
$i=1$	0	0	0
$i=0$	0	0	0

FIGURE 3. Intersection homology and stalk homology of ΣT^3

The larger the number p the more cycles are allowed into the singular points. If there are more strata we can assign such numbers to each stratum separately.

9. Lecture 9: Intersection Homology

9.1. **Digression on transversality.** Let $K \subset \mathbb{R}$ be the Cantor set. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth (C^∞) function that vanishes precisely on K. Let $A \subset \mathbb{R}^2$ denote the graph of f and let B denote the x-axis. Then A, B are smooth submanifolds of $\mathbb{R}^2$ but their intersection is the Cantor set. This sort of unruly behavior can be avoided using transversality.

Two submanifolds $A, B \subset M$ of a smooth manifold are said to be transverse at a point of their intersection $x \in A \cap B$ if $T_xA + T_xB = T_xM$. If A and B are transverse at every point of their intersection then $A \cap B$ is a smooth submanifold of M of dimension $\dim(A) + \dim(B) - \dim(M)$. Arbitrary submanifolds $A, B, \subset M$ can be made to be transverse by moving either one of them, say $A' = \phi_\epsilon(A)$ by the flow, for an arbitrarily small time, of a smooth vector field on M. If V is a finite dimensional vector space of vector fields on M which span the tangent space T_xM at every point $x \in M$ then there is an open and dense subset of V consisting of vector fields v such that the time $= 1$ flow ϕ_1 of v takes A to a submanifold $A' = \phi_1(A)$ that is transverse to B. [This is a very powerful result. It says, for example, that two submanifolds of Euclidean space can be made transverse by an arbitrary small translation. The proof, due to Marston Morse, is so elegant, that I decided to include it here `http://www.math.ias.edu/~goresky/math2710/Trans.pdf`.]

Two Whitney stratified subsets $W_1, W_2 \subset M$ are said to be transverse if each stratum of W_1 is transverse to each stratum of W_2, in which case the intersection $W_1 \cap W_2$ is also Whitney stratified. Whitney stratified sets can also be made to be transverse by the application of the flow, for an arbitrarily small time, of a smooth vector field on M.

9.2. **Intersection homology.** Let W be a compact n-dimensional Whitney stratified pseudo-manifold with strata S_α (α in some index set I, partially ordered by the closure relations between strata with S_0 being the stratum of dimensioin n) and let $0 \le p_\alpha \le \mathrm{cod(S_\alpha)} - 2$ be a collection of integers which we refer to as [trigger warning] a perversity. Define the intersection chains,

9.3. **Definition.**

$$IC_i^{\bar{p}}(W) = \left\{ \xi \in C_i(W) \;\middle|\; \begin{array}{l} \dim(\xi \cap S_\alpha) \le i - \mathrm{cod(S_\alpha)} + \mathrm{p}_\alpha \\ \dim(\partial\xi \cap S_\alpha) \le i - 1 - \mathrm{cod(S_\alpha)} + \mathrm{p}_\alpha \end{array} \text{ for } \alpha > 0 \right\} \tag{9.3.1}$$

Having placed the same restrictions on the chains as on their boundaries, we obtain a chain complex, in fact a complex of (soft) sheaves $\underline{\underline{IC}}^p$ with resulting cohomology groups $H_i^p(W)$. (As usual, "chains" could refer to PL chains, singular chains, subanalytic chains, etc.) Because W is a pseudomanifold the singular strata have codimension at least 2. The condition $p_\alpha \le \mathrm{cod(S_\alpha)} - 2$ implies that most of the chain, and most of its boundary are completely contained within the top stratum S_0. So a cycle ($\partial\xi = 0$) in $IC_i^{\bar{p}}$ is also a cycle for ordinary homology and we have a homomorphism $IH_i^{\bar{p}}(W) \to H_i(W)$. Moreover, if $\xi \in IC_i^{\bar{p}}(W)$ and if $\eta \in IC_j^{\bar{q}}(W)$ and if we can arrange that $\xi \cap S_\alpha$ and $\eta \cap S_\alpha$ are transverse within each stratum S_α then we will have an intersection

$$\xi \cap \eta \in IC_{i+j-n}^{\bar{p}+\bar{q}}(W)$$

which is well defined provided that $p_\alpha + q_\alpha \le \mathrm{cod}(\mathrm{S}_\alpha) - 2$ for all $\alpha > 0$.

The first problem with this construction is that it is obviously dependent on the stratification. Moreover, if we are not careful, large values of p_α for small strata $S_\alpha < S_\beta$ will have the effect of allowing chains into S_α but not into S_β thereby "locking" the chain into passing through a small stratum. This issue can be avoided by requiring that p_α depend only on $\mathrm{cod}(\mathrm{S}_\alpha)$ and that $\beta > \alpha \implies p_\beta \ge p_\alpha$.

The second problem involves the effect of refining the stratification. For a simple case, suppose W consists only of two strata, S_0 and S_c, the singular stratum having codimension $c \ge 3$, to which we assign a perversity p_c. Now suppose we refine this stratum by introducing a "fake" stratum, S_r of codimension $r > c$. Chains in $IC_i^p(W)$ may intersect S_c in dimension $\le i - c + p_c$ and for all we know, they may lie completely in S_r, meaning that the chain will have "perversity" $p_r = p_c + c - r$. On the other hand if we assume, as before, that we can arrange for this chain to be transverse to the fake stratum S_r within the stratum S_c then its intersection with S_r will have dimension $\le i - c + p_c - (r - c) = i - r + p_c$ which is to say that it has "perversity" $p_r = p_c$. This argument shows (or suggests) that in this case we have natural isomorphisms between the intersection homology $IH_i^{p_c}(W)$ as computed before the refinement, and the intersection homology $IH_i^{p_c,p_r}$ after refinement, for any p_r with $p_c \le p_r \le p_c + r - c$, that is,

$$IH_i^{p_c,p_c}(W) \cong IH_i^{p_c,p_c+1}(W) \cong \cdots \cong IH_i^{p_c,p_c+r-c}(W)$$

In summary, assuming that $p_c \le p_r \le p_c + r - c$ the resulting homology group $IH_i^{p_c,p_r}$ is unchanged after refinement. This leads us to the formal definition of intersection homology.

9.4. **Definition.** A *perversity* is a function $\bar{p} = (p_2, p_3, \cdots)$ with $p_2 = 0$ and with $p_c \le p_{c+1} \le p_c + 1$. The complex of sheaves of intersection chains is the complex with sections

$$\Gamma(U, \underline{IC}_{\bar{p}}^{-i}) = \left\{ \xi \in C_i(U) \;\middle|\; \begin{array}{l} \dim(\xi \cap S_c) \le i - c + p_c \\ \dim(\partial\xi \cap S_c) \le i - 1 - c + p_c \end{array} \text{ for } c \ge 2 \right\} \tag{9.4.1}$$

where S_c denotes the union of all strata of codimension $c \ge 2$.

Intersection homology with coefficients in a local system is defined similarly, however something special happens in this case. For any triangulation of a chain $\xi \in IC_{\bar{p}}^{-i}$ all of its i-dimensional simplices and all of its $i-1$ dimensional simplices will be completely contained within the top stratum (or "nonsingular part") of W. So if $\mathcal{L}$ is a local coefficient system *defined only on the top stratum of* W, we can still construct the sheaf of intersection chains $\underline{IC}_{\bar{p}}^{\bullet}(\mathcal{L})$ exactly as above.

Let $\bar{0}$ be the perversity $0_c = 0$ and let $\bar{t}$ be the perversity $t_c = c - 2$.

9.5. **Theorem.** *Let W be an oriented stratified pseudomanifold. For any choice of perversity $\bar{p}$ equation (9.4.1) defines a complex of soft sheaves $\underline{IC}_{\bar{p}}^{\bullet}$ on W and the following holds.*

(1) The cohomology sheaves $\underline{IH}_{\bar{p}}^{-m}$ and the hypercohomology groups $IH_i^{\bar{p}}(W)$ are well defined and are independent of the stratification;

(2) in fact they are topological invariants.

(3) There are canonical maps

$$H^{n-i}(W) \to IH_i^{\bar{p}}(W) \to H_i(W)$$

that factor the Poincaré duality map,

(4) if $\bar{p} \le \bar{q}$ then there are also compatible mappings $IH^{\bar{p}} \to IH^{\bar{q}}$. In sheaf language we have natural maps

$$\underline{\mathbb{Z}}_W[n] \to \underline{IC}_W^{\bar{p}} \to \underline{IC}_W^{\bar{q}} \to \underline{C}_W^{\bullet}.$$

(5) If the link L_x of each stratum is connectred then for $\bar{p} = \bar{0}$ the first of these maps is a quasi-isomorphism, and for $\bar{q} = \bar{t}$ the second map is a quasi-isomorphism.

(6) If $p_c + q_c \le t_c = c - 2$ for all c then the intersection of transversal *chains determines a pairing*

$$IH_i^{\bar{p}}(W) \times IH_j^{\bar{q}}(W) \to IH_{i+j-n}^{\bar{p}+\bar{q}}(W)$$

(7) If $\bar{p} + \bar{q} = \bar{t}$ then the resulting pairing

$$IH_i^{\bar{p}}(W) \times IH_{n-i}^{\bar{q}}(W) \to H_0(W) \to \mathbb{Z}$$

is nondegenerate over $\mathbb{Q}$ (or over any field).

The last statement in Theorem 9.5, Poincaré duality, was the big surprise when intersection homology was discovered for it is a duality statement that applies to singular spaces. Especially, if the stratification of W consists only of even codimension strata then there is a "middle" choice for p, that is, $p_c = (c-2)/2$ for which $IH^{\bar{p}}(W;k)$ is self-dual for any field k.

There is a technical problem with moving chains within a Whitney stratified set W, so as to be transverse within each stratum of W. This can be accomplished with piecewise-linear chains within a piecewise-linear stratified set W, and has recently been accomplished using semi-analytic chains within a semi-analytic stratified set, but to my knowedge, it has not been accomplished in any other setting. This is one of the many problems that is avoided with the use of sheaf theory. The proof of topological invariance depends entirely on sheaf theory. Other results such as the proof of Poincaré duality, that can be established using chain manipulations, are incredibly awkward, requiring a choice of model for the chains, and delicate manipulations with individual chains. These constructions are easier, but less geometric, if they are all made using sheaf theory. For this purpose we need to identify the quasi-isomorphism class of the complex of sheaves $\underline{IC}^{\bar{p}}$.

9.6. **Proposition.** *Let W be a Whitney stratified pseudomanifold and let $\mathcal{L}$ be a local coefficient system defined on the top stratum. Fix a perversity $\bar{p}$, and let $x \in S_c$ be a point in a stratum of codimension c. Then the stalk of the intersection homology sheaf at x is*

$$\underline{H}^{-i}(\underline{IC}^{\bar{p}}(\mathcal{L}))_x = IH_i^{\bar{p}}(W, W-x;\mathcal{L}) = \begin{cases} 0 & \text{if } i < n - p_c \\ IH_{i-n+c-1}(L_x;\mathcal{L}) & \text{if } i \ge n - p_c \end{cases}$$

and the stalk cohomology with compact supports is

$$H_c^{-i}(U_x;\underline{IC}^{\bar{p}}(\mathcal{L})) = IH_i^{\bar{p}}(U_x) = \begin{cases} H_i(L_x;\mathcal{L}) & \text{if } i \le c - p_c \\ 0 & \text{if } i > c - p_c \end{cases}$$

Proof. Use the local product structure of a neighborhood $U_x \cong c^o(L_x) \times \mathbb{R}^{n-c}$ and the Künneth formula

$$IH_i^{\bar{p}}(U, \partial U; \mathcal{L}) \cong IH_{i-(n-c)}^{\bar{p}}(c(L_x), L_x; \mathcal{L}).$$

If $\xi \in IC_i^{\bar{p}}$ and if $(i-n+c)-c+p_c \ge 0$ then the chain ξ is allowed to hit the cone point, otherwise it is not. When it is allowed to hit the cone point, we may assume (using a homotopy argument) that it locally coincides with the cone over a chain in L_x which satisfies the same allowability conditions. Similar remarks apply to $\partial\xi$. On the other hand, a compact i-dimensional chain ξ in the link L_x can be coned down to the cone point it vanishes in cohomology, and such a cone $c(\xi)$ is allowed provided $0 \le (i+1) - c + p_c$, that is, if $i > c - p_c$. □

Comparing this to the calculation (8.3.1) of $j_*j^*(\underline{IC}^{\bar{p}})$ where $j : U = W - \overline{S_c} \to W$ is the inclusion of the open complement of the closure of S_c we see that the intersection homology sheaf on S_c is the *truncation* of the sheaf $j_*(\underline{IC}^{\bar{p}}|U)$. For example, suppose $\dim(W) = 8$ has strata of dimension $0, 2, 4, 6, 8$ and the perversity is the middle one, $p(c) = (c-2)/2$. Then the stalk cohomology $H^i(\underline{C})_x$ of the sheaf $\underline{IC}^{\bar{p}}$ looks as follows, where L^r means the r-dimensional link of the codimension $r+1$ stratum and the red zeroes represent homology groups that have been killed by the perversity condition:

i	cod0	cod2	cod4	cod6	cod8
0					
-1					0
-2					0
-3				0	0
-4				0	0
-5			0	0	$IH_4(L^7)$
-6			0	$IH_3(L^5)$	$IH_5(L^7)$
-7		0	$IH_2(L^3)$	$IH_4(L^5)$	$IH_6(L^7)$
-8	$\mathbb{Z}$	$IH_1(L^1)$	$IH_3(L^3)$	$IH_5(L^5)$	$IH_7(L^7)$

cod0	cod2	cod4	cod6	cod8
$\mathbb{Z}$	$IH_0(L^1)$	$IH_0(L^3)$	$IH_0(L^5)$	$IH_0(L^7)$
	0	$IH_1(L^3)$	$IH_1(L^5)$	$IH_1(L^7)$
		0	$IH_2(L^5)$	$IH_2(L^7)$
		0	0	$IH_3(L^7)$
			0	0
			0	0
				0
				0

FIGURE 4. Stalk cohomology and compact support cohomology of $\underline{IC}^\bullet$

This gives an inductive way to construct intersection homology using purely sheaf-theoretic operations, to be described in the next lecture.

10. Lecture 10: Truncation

10.1. **Truncation.** If $A^\bullet$ is a complex of sheaves define

$$(\tau_{\le r}A^\bullet)^i = \begin{cases} 0 & \text{if } i > r \\ \ker(d^r) & \text{if } i = r \\ A^i & \text{if } i < r \end{cases}$$

Then $\underline{H}^i(\tau_{\le r}A^\bullet) = \underline{H}^i(A^\bullet)$ for $i \le r$ and is zero for $i > r$.

During a conversation in October 1976 Pierre Deligne suggested that the following construction might generate the intersection homology sheaf.

10.2. **Definition.** Let W be a purely n dimensional oriented Whitney stratified pseudomanifold and let W_r denote the union of the strata of dimension $\le r$. Set $U_k = W - W_{n-k}$ with inclusions

$$U_2 \xrightarrow[j_2]{} U_3 \xrightarrow[j_3]{} \cdots \xrightarrow[j_{n-1}]{} U_n \xrightarrow[j_n]{} U_{n+1} = W$$

Let $\mathcal{L}$ be a local coefficient system defined on the top stratum $U = U_2$. Set

$$\underline{P}^\bullet_{\bar{p}}(\mathcal{L}) = \tau_{\le p(n)} Rj_{n*} \cdots \tau_{\le p(3)} Rj_{3*} \tau_{\le p(2)} Rj_2 * \mathcal{L}.$$

The resulting complex of sheaves will have stalk cohomology that is illustrated as follows (in the case of middle perversity, with $\mathcal{L} = \mathbb{Z}$, for a stratified psuedomanifold W with $\dim(W) = 8$ that is stratified with strata of codimension 0, 2, 4, 6, 8). In this figure we suppress the $\bar{p}$ on $\underline{P}^\bullet_{\bar{p}}$.

i	cod0	cod2	cod4	cod6	cod8
8					
7					0
6					0
5				0	0
4				0	0
3			0	0	$H^3(L^7, \underline{P}^\bullet)$
2			0	$H^2(L^5, \underline{P}^\bullet)$	$H^2(L^7, \underline{P}^\bullet)$
1		0	$H^1(L^3, \underline{P}^\bullet)$	$H^1(L^5, \underline{P}^\bullet)$	$H^1(L^7, \underline{P}^\bullet)$
0	$\mathbb{Z}$	$H^0(L^1, \underline{P}^\bullet)$	$H^0(L^3, \underline{P}^\bullet)$	$H^0(L^5, \underline{P}^\bullet)$	$H^0(L^7, \underline{P}^\bullet)$

We remark, for example, at a point $x \in W$ that lies in a stratum X of codimension 6, the stalk cohomology at x equals the cohomology of the link L^5 of X with coefficients in the part of the sheaf $\underline{P}^\bullet|U_6$ that has been previously constructed over the strictly *larger* strata $Y > X$.

10.3. **Theorem.** *Let W be an oriented n-dimensional stratified pseudomanifold and let $\mathcal{L}$ be a local coefficient system on the top stratum. The orientation map $\mathcal{L}[n] \to \underline{C}_U^{-n}(\mathcal{L})$ induces an isomorphism $\underline{P}^\bullet_{\bar{p}}(\mathcal{L})[n] \cong \underline{IC}^\bullet_{\bar{p}}(\mathcal{L})$ where $U = U_2$ denotes the largest stratum.*

If the link L_x of every stratum is connected then $\underline{P}^\bullet_{\bar{t}}(\mathbb{Z})[n] \to \underline{C}^\bullet(\underline{\mathbb{Z}}_U)$ is a quasi-isomorphism (so that $IH^i_{\bar{t}}(W) = H_{n-i}(W)$) and $P^\bullet_{\bar{0}}(\mathbb{Z}) \to \underline{\mathbb{Z}}$ is a quasi-isomorphism (so that $IH^i_{\bar{0}}(W) = H^i(W)$), where $\bar{0}_c = 0$ and where $\bar{t}_c = c-2$ are the "bottom" and "top" perversities respectively. If $\bar{p}+\bar{q} \leq \bar{t}$ (where $t_c = c-2$) and if $\mathcal{L}_1 \otimes \mathcal{L}_2 \to \mathcal{L}_3$ is a morphism of local systems on U then it extends canonically to a product

$$\underline{P}^\bullet_{\bar{p}}(\mathcal{L}_1) \otimes \underline{P}^\bullet_{\bar{q}}(\mathcal{L}_2) \to \underline{P}^\bullet_{\bar{p}+\bar{q}}(\mathcal{L}_3).$$

Proof. For simplicity we discuss the case of constant coefficients. There are two problems (a) to show that the orientation map $\underline{\mathbb{Z}}_U \to \underline{C}_U^{-n}$ extends to a (uniquely defined) map in the derived category $\underline{P}^\bullet[n] \to \underline{IC}^\bullet$ (for a fixed perversity, which we suppress in the notation) and (b) to show that this map is a quasi-isomorphism. These are proven by induction, adding one stratum at a time. Consider the diagram

$$U_k \xrightarrow[j_k]{} U_{k+1} \xleftarrow[i_k]{} X^{n-k}$$

where X^{n-k} is the union of the codimension k strata. Suppose by induction that we have constructed a quasi-isomorphism $\underline{P}^\bullet_k \to \underline{IC}^\bullet_k$ of sheaves over U_k (where the subscript k denotes the restriction to U_k). Now compare the two distinguished triangles (writing $i = i_k$ and $j = j_k$ to simplify notation),

$$\begin{tikzcd}[column sep=tiny] Ri_*i^!(\underline{P}^\bullet_{k+1}) \arrow[rr] & & \underline{P}^\bullet_{k+1} \arrow[dl] \\ & Rj_*j^*(\underline{P}^\bullet_{k+1})=Rj_*(\underline{P}^\bullet_k) \arrow[ul] & \end{tikzcd} \quad \text{and} \quad \begin{tikzcd}[column sep=tiny] Ri_*i^!(\underline{IC}^\bullet_{k+1}) \arrow[rr] & & \underline{IC}^\bullet_{k+1} \arrow[dl] \\ & Rj_*j^*(\underline{IC}^\bullet_{k+1})=Rj_*(\underline{IC}^\bullet_k) \arrow[ul] & \end{tikzcd}$$

We are actually concerned with the right side of these triangles. By induction we have an isomorphism on the bottom row, so we get an isomorphism of the truncations:

$$\underline{P}^\bullet_{k+1} = \tau_{\leq p(k)} Rj_* P_k \to \tau_{\leq p(k)} Rj_*(\underline{IC}^\bullet_k)$$

This is the upper right corner of the first triangle and we wish to identify it with the upper right corner of the second triangle. So it suffices to show that we have an isomorphism (in the derived category),

$$\underline{IC}^\bullet_{k+1} \cong \tau_{\leq p(k)} Rj_*(\underline{IC}^\bullet_k).$$

But this is exactly what the local calculation says: the stalk of the intersection cohomology is the truncation of the intersection cohomology of the link.

In fact, the formula $\underline{P}^\bullet_{k+1} = \tau_{\leq p(k)} Rj_* \underline{P}^\bullet_k$ implies that the attaching morphism $\underline{P}^\bullet_{k+1} \to Rj_*j^*\underline{P}^\bullet_{k+1}$ is an isomorphism in degrees $r \leq p(k)$, or equivalently, that $H^r(i^!\underline{P}^\bullet) = 0$ for $r \leq p(k)+1$. This is the same as saying that for any $x \in X^{n-k}$,

$$H^r(U_x; \underline{P}^\bullet) = H^r(i_x^! \underline{P}^\bullet) = 0 \text{ for } r < p(k) + 2 + (n-k) = n - q(k)$$

where $q(k) = k - 2 - p(k)$ is the complementary perversity, $i_x : \{x\} \to W$ is the inclusion and U_x is a basic open neighborhood of x in W.

The construction of the pairing is similar. Start with the multiplication

$$\underline{\mathbb{Z}}_{U_2} \otimes \underline{\mathbb{Z}}_{U_2} \to \underline{\mathbb{Z}}_{U-2}.$$

Now apply $\tau_{\leq p(2)} Rj_*$. The truncation of a tensor product is not simply the tensor product of the truncations, there are a lot of cross terms. By examining the effect on the stalk cohomology one eventually arrives at a pairing

$$\underline{P}^\bullet_{\bar{p}} \otimes \underline{P}^\bullet_{\bar{q}} \to \underline{P}^\bullet_{\bar{p}+\bar{q}}.$$

For more details see:
`http://www.math.ias.edu/~goresky/math2710/IH2.pdf` □

10.4. **Duality.** But what is the sheaf-theoretic statement for the Poincaré duality theorem for intersection cohomology? Translating into cohomology indexing this says: if $\bar{p} + \bar{q} = \bar{t}$ then the resulting pairing

$$IH^i_{\bar{p}}(W) \times IH^{n-i}_{\bar{q}}(W) \to IH^{\bar{t}}_n(W) \to \mathbb{Z}$$

is nondegenerate when tensored with any field.

If $S^\bullet$ is a complex of sheaves of k-vector spaces on a topological space, Borel and Moore defined its dual $\mathbf{D}(S^\bullet)$ to be the complex of sheaves associated to the complex of presheaves

$$\mathbf{D}(S)^{-j} = \mathrm{Hom}(\Gamma_c(U, S^j), k).$$

For example the sheaf of *currents* on a smooth manifold is the dual of the sheaf of smooth differential forms. For sheaves of abelian groups (or sheaves of R modules), Hom must be replaced by RHom in the category of groups. In other words, choose an injective resolution $\mathbb{Z} \to I^\bullet$ (or $R \to I^\bullet$), and take the total complex associated to the double complex $\mathrm{Hom}(\Gamma_c(U, S^\bullet), I^\bullet)$. Borel and Moore proved that there are exact cohomology sequences

$$0 \to \mathrm{Ext}(H^{n+1}_c(X, S^\bullet), \mathbb{Z}) \to H^{-n}(X, \mathbf{D}(S^\bullet)) \to \mathrm{Hom}(H^n_c(X, S^\bullet), \mathbb{Z}) \to 0$$

in analogy with the universal coefficient theorem for cohomology:

$$0 \to \mathrm{Ext}(H_{n-1}(X, \mathbb{Z}), \mathbb{Z}) \to H^n(X, \mathbb{Z}) \to \mathrm{Hom}(H_n(X, \mathbb{Z}), \mathbb{Z}) \to 0.$$

Then they defined the Borel-Moore homology to be the homology of the complex $\mathbf{D}(\underline{\mathbb{Z}})$. We have previously seen that this is the homology theory of locally finite chains. The problem with the Borel-Moore theory is that the double dual of $S^\bullet$ did not equal $S^\bullet$.

Then Verdier discovered that their construction could be interpreted sheaf theoretically:

$$\mathbf{D}(S^\bullet) = \underline{\mathrm{RHom}}^\bullet(S^\bullet, \mathbf{D}^\bullet)$$

where $\mathbf{D}^\bullet$ is a particular, universal sheaf called the *dualizing complex*. He then showed that there is a canonical quasi-isomorphism in the derived category,

$$\mathbf{DD}(S^\bullet) \cong S^\bullet.$$

So double duality is restored. And what is this magic dualizing complex? For sheaves of abelian groups, we have no choice:

$$\mathbf{D}^\bullet = \underline{\mathrm{RHom}}^\bullet(\underline{\mathbb{Z}}, \mathbf{D}^\bullet) = \mathbf{D}(\underline{\mathbb{Z}})$$

is the Borel-Moore sheaf of chains! More precisely, it is the quasi-isomorphism class of the Borel-Moore chains. The sheaf of locally finite chains with integer coefficients is not injective, because $\mathbb{Z}$ is not injective. In order to obtain an injective model for the dualizing sheaf it is necessary to injectively resolve $\mathbb{Z}$. Any reasonable injective model of the dualizing complex is a mess.

10.5. **Definition.** A pairing $S^\bullet \otimes T^\bullet \to \mathbf{D}^\bullet$ of sheaves on a Whitney stratified space X is said to be a *Verdier dual pairing* if the resulting morphism $S^\bullet \to \underline{\mathrm{RHom}}^\bullet(T^\bullet, \mathbf{D}^\bullet) = \mathbf{D}(T^\bullet)$ is an isomorphism in $D^b_c(X)$.

In particular, this means that if K is a field then for any open set $U \subset X$,

$$H^i(U, \mathbf{D}(S^\bullet)) \cong \mathrm{Hom}(H_c^{-i}(U, S^\bullet), K).$$

10.6. **Definition.** If $f : X \to Y$ is a continuous map and $S^\bullet$ is a complex of sheaves on Y define $f^!(S^\bullet) = \mathbf{D}_X f^* \mathbf{D}_Y(S^\bullet)$.

10.7. **Theorem** (Verdier duality)**.** *Let $f : X \to Y$ be a stratified mapping between Whitney stratified spaces. Let $A^\bullet$ and $B^\bullet$ be constructible sheaves on X and Y respectively. Then f^*, $f^!$, Rf_* and $Rf_!$ take distinguished triangles to distinguished triangles. There are canonical isomorphisms in $D^b_c(X)$ as follows:*

(1) $\mathbf{DD}(A^\bullet) \cong A^\bullet$
(2) $\mathbf{D}^\bullet_X \cong f^! \mathbf{D}^\bullet_Y$
In particular $\mathbf{D}^\bullet_X = f^!(\mathbb{Z})$ when Y is a point.
(3) $f^!(B^\bullet) = \mathbf{D}_X f^* \mathbf{D}_Y(B^\bullet)$
(4) $Rf_!(A^\bullet) = \mathbf{D}_Y Rf_* \mathbf{D}_X(A^\bullet)$
So $f^!$ is the dual of f^* and $Rf_!$ is the dual of Rf_*.
(5) $f^! \underline{\mathrm{RHom}}^\bullet(B^\bullet, C^\bullet) \cong \underline{\mathrm{RHom}}^\bullet(f^*(B^\bullet), f^!(C^\bullet))$
(6) $Rf_*(\underline{\mathrm{RHom}}^\bullet(A^\bullet, f^! B^\bullet)) \cong \mathrm{RHom}^\bullet(Rf_! A^\bullet, B^\bullet)$ *[Verdier duality theorem]*
This says that $Rf_!$ and $f^!$ are adjoint, just as Rf_* and f^* are.
(7) $Rf_* \underline{\mathrm{RHom}}^\bullet(f^* B^\bullet, A^\bullet) \cong \underline{\mathrm{RHom}}^\bullet(B^\bullet, Rf_* A^\bullet)$
(8) $Rf_! \underline{\mathrm{RHom}}^\bullet(A^\bullet, f^! B^\bullet) \cong \underline{\mathrm{RHom}}^\bullet(Rf_! A^\bullet, B^\bullet)$
(9) If $f : X \to Y$ is the inclusion of an open subset then $f^!(B^\bullet) \cong f^*(B^\bullet)$.
(10) If $f : X \to Y$ is the inclusion of a closed subset then $Rf_!(A^\bullet) \cong Rf_*(A^\bullet)$.
(11) If $f : X \to Y$ is the inclusion of an oriented submanifold in another, and if $B^\bullet$ is cohomologically locally constant on Y then $f^!(B^\bullet) \cong f^*(B^\bullet)[\dim(Y) - \dim(X)]$.

Exercise. Verify (1) for the category of simplicial sheaves using the canonical model for the (simplicial) sheaf of chains. It comes down to a statement about the second barycentric subdivision.

10.8. **Theorem.** *Let X be a Whitney stratified space and let $\bar{p}+\bar{q}=\bar{t}$ be complementary perversities. Let $K=\mathbb{Q}$, $\mathbb{R}$, or $\mathbb{C}$. Let $\mathcal{L}_2 \otimes \mathcal{L}_2 \to \underline{K}$ be a dual pairing of local systems (of K-vector spaces) defined over the top stratum of X. Then the resulting pairing*

$$\underline{IC}^{\bullet}_{\bar{p}}(\mathcal{L}_1)[n] \otimes \underline{IC}^{\bullet}_{\bar{q}}(\mathcal{L}_2) \to \underline{IC}^{\bullet}_{\bar{t}}(K) = \mathbf{D}^{\bullet}(K)$$

is a Verdier dual pairing. For any open set $U \subset X$, the pairing

$$IH^{n-i}_{\bar{p}}(U) \times IH^{i}_{\bar{q},c}(U) \to K$$

is nondegenerate.

The proof is by induction on the strata, as before, adding one stratum at a time, using the long exact sequences, duality, and the above formal properties. It is not difficult.

10.9. **Remark.** Dualizing complexes exist in other derived categories as well. For example, in the category of abelian groups the complex $\mathbb{Q} \to \mathbb{Q}/\mathbb{Z}$ is a dualizing complex.

11. Perverse sheaves

In the next lecture we will prove the following result. It is actually the same as the proof that $\underline{P}^\bullet_{\bar{p}}[n] \cong \underline{IC}^{\bar{p}}$.

11.1. **Theorem.** *Let W be an n-dimensional Whitney stratified set with biggest stratum U and let $\bar{p}$ be a pervesity. Then Deligne's construction*

$$\mathcal{L} \mapsto \underline{P}^\bullet(\mathcal{L}) = \tau_{\le p(n)} Rj_{n*} \cdots \tau_{\le p(3)} Rj_{3*} \tau_{\le p(2)} Rj_2 * \mathcal{L}$$

defines an equivalence of categories between the category of local systems of K-vector spaces ($K = \mathbb{Q}, \mathbb{R}, \mathbb{C}$) on the nonsingular part U, and the full subcategory of $D^b_c(W)$ consisting of "IC sheaves", that is, complexes of sheaves $A^\bullet$, constructible with respect to the given stratification, such that the following conditions hold

1. *$A^\bullet|U_2 \cong \mathcal{L}$ is isomorphic to a local coefficient system*
2. *$\underline{H}^r(A^\bullet) = 0$ for $r < 0$*
3. *$H^r(i_x^* A^\bullet) = 0$ for $r > p(k)$*
4. *$H^r(i_x^! A^\bullet) = 0$ for $r < n - q(k)$*

for all points $x \in W$, where $i_x : \{x\} \to W$ is the inclusion of the point and k denotes the codimension of the stratum containing x and where $q(k) = k - 2 - p(k)$ is the complementary perversity.

This says, in particular, that if $\mathcal{L}$ is a local system on U_2 and if $A^\bullet$ is a constructible complex of sheaves that satisfies the above conditions, then there is a canonical isomorphism $A^\bullet \cong \underline{IC}^\bullet_{\bar{p}}(\mathcal{L})$. Moreover, it says that if $\mathcal{L}_1, \mathcal{L}_2$ are local systems on U_2 then

$$\mathrm{RHom}(\underline{IC}^\bullet(\mathcal{L}_1), \underline{IC}^\bullet(\mathcal{L}_2)) \cong \mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2).$$

If $\mathcal{L}$ is an indecomposable local system (which is not isomorphic to a direct sum of two nontrivial local systems) then $\underline{IC}^\bullet_{\bar{p}}(\mathcal{L})$ is an indecomposable complex of sheaves (and is not isomorphic to a direct sum of two nontrivial complexes of sheaves).

For a perversity $\bar{p}$ let $p^{-1}(j) = \min\{c|\ p(c) \ge j\}$ and $p^{-1}(j) = \infty$ if $j > p(n)$. We can reformulate these conditions (2,3,4) in a way that does not refer to a particular stratification as follows:

(S1) $\dim\{x \in W|\ H^r(i_x^* A^\bullet) \ne 0\} \le n - p^{-1}(j)$ for all $j > 0$

(S2) $\dim\{x \in W|\ H^r(i_x^! A^\bullet) \ne 0\} \le n - q^{-1}(n-j)$ for all $j < n$.

As above, the condition (S2) is the Verdier dual of condition (S1) and may be expressed as

(S2') $\dim\{x \in W|\ H^r(i_x^* \mathbf{D}(A^\bullet)) \ne 0\} \le n - q^{-1}(j)$ for all $j > 0$.

11.2. Let W be a Whitney stratified space with a given stratification. We have two notions of the constructible derived category,

1. As complexes of sheaves that are cohomologically constructible with respect to the given stratification

(2) As complexes of sheaves that are cohomologically constructible with respect to some stratification

In order to reduce the total number of words in these notes, we shall simply refer to "the constructible derived category", meaning either one of these two possibilities.

Perverse sheaves are defined for any perversity but the indexing conventions are messy in general. In the following we will give the definition for the middle perversity, which is the only case of real importance. In the examples we will discuss a few others as well.

11.3. **Definition.** Let W be a n-dimensional Whitney stratified (or stratifiable) space that can be stratified with strata of even dimension. Let $K = \mathbb{Q},$, $\mathbb{R}$, or $\mathbb{C}$. Fix a perversity $\bar{p}$. A (middle perversity) perverse sheaf on W is a complex of sheaves $A^\bullet$ in the bounded constructible derived category $D_c^b(W)$ of K-vector spaces, such that (see Figure 8):

(P1) $\dim\{x \in W |\ H^r(j_x^* A^\bullet) \neq 0\} \leq n - 2r$

(P2) $\dim\{x \in W |\ H^r(j_x^! A^\bullet) \neq 0\} \leq 2r - n$

i	cod0	cod2	cod4	cod6	cod8
8	c	c	c	c	c
7			c	c	c
6				c	c
5					c
4					0
3					x
2				x	x
1			x	x	x
0	x	x	x	x	x
	$\underline{IC}^\bullet$ support				

i	cod0	cod2	cod4	cod6	cod8
8	c	c	c	c	c
7		c	c	c	c
6			c	c	c
5				c	c
4					cx
3				x	x
2			x	x	x
1		x	x	x	x
0	x	x	x	x	x
	Perverse sheaf support				

FIGURE 5. Stalk and co-stalk cohomology of $\underline{IC}^\bullet$ and perverse sheaves

In these figures, "x" denotes regions of possibly nontrivial stalk cohomology and "c" denotes regions of possibly nontrivial stalk cohomology with compact support, $H^r(j_x^! A^\bullet)$.

11.4. **Definition.** Fix a Whitney stratified (or stratifiable) space. Fix a perversity $\bar{p}$. The *category of perverse sheaves* (with perversity p) is the full subcategory of $D_c^b(W)$ whose objects are perverse sheaves with perversity $\bar{p}$.

11.5. **Theorem** (Beilinson, Bernstein, Deligne)**.** *The category of perverse sheaves with perversity $\bar{p}$ forms an abelian subcategory of the derived category $D_c^b(W)$. If $A^\bullet$ is $\bar{p}$-perverse then its Verdier dual $\mathbf{D}(A^\bullet)$ is $\bar{q}$-perverse, where $\bar{q}$ is the perversity complementary to p.*

There is something very mysterious about this theorem. If $\phi : A^\bullet \to B^\bullet$ is a morphism (in $D^b_c(W)$) between two perverse sheaves, it can be lifted to an honest morphism of complexes $A^\bullet \to B^\bullet$. As such, it has a kernel and a cokernel. These are unlikely to be perverse, and moreover, they may change if we choose different (but quasi-isomorphic) representative complexes for $A^\bullet, B^\bullet$. However, the kernel and cokernel of ϕ *in the category of perverse sheaves* are again perverse sheaves, and are well defined as elements of the derived category. Moreover, various constructions from the theory of abelian cartegories can be implemented. For example, suppose $A^\bullet_0 \xrightarrow{d} A^\bullet_1 \xrightarrow{d} A^\bullet_2 \xrightarrow{d} \cdots$ is a complex of perverse sheaves, that is a complex such that $d \circ d = 0$ in the derived category. Then $\ker(d)/\mathsf{Im}(d)$ makes sense as a perverse sheaf, so we obtain the *perverse cohomology* ${}^pH^r(A^\bullet_\bullet)$ of such a complex. (Clearly these bullets will start to get in the way so it is customary to drop them at this point.)

11.6. **Historical comment.** Around 1980 Kazhdan and Lusztig realized that certain questions involving representations of Hecke algebras, Verma modules, and Kazhdan Lusztig polynomials were related to the failure of Poincaré duality for Schubert varieties. On the advice of Raoul Bott they spoke at length with MacPherson, who replied with a long letter which, at the end, suggested the use of intersection cohomology as a solution to their problem. Consequently the Kazhdan Lusztig polynomials were shown to coincide with the intersection cohomology local Poincaré polynomial of one Schubert variety at a point in another Schubert variety. This resulted in a further series of conjectures, by Kazhdan and Lusztig concerning representations of Verma modules and their relation to Kazhdan-Lusztig polynomials. These conjectures were eventually proven indepedently by Beilinson-Bernstein and by Brylinski-Kashiwara. On an algebraic manifold (such as the flag manifold) there is a ring $\mathcal{D}$ (or rather, a sheaf of rings) of differential operators. To each $\mathcal{D}$-module there corresponds a sheaf of solutions, which is a constritible sheaf. B-B and K-L showed that each Verma module can be associated to a certain $\mathcal{D}$-module whose sheaf of solutions turns out to be the IC sheaf. This provided the link between the Kazhdan-Lusztig polynomials and Verma modules. However, the category of $\mathcal{D}$-modules is an abelian category, whereas the (derived) category of constrictible sheaves is not abelian, so it was conjectured that there might correspond an abelian subcategory of the derived category that "receives" the solutions of $\mathcal{D}$-modules. This turned out to be the category of perverse sheaves, with middle perversity. On the other hand, intersection homology is a topological invariant, so then the question arose as to whether this category of perverse sheaves could be constructed purely topologically, and for other perversities as well. The book of BBD completely answers this question, giving a very general setting in which the category of perverse sheaves, an abelian subategory of the derived category, could be constructed.

12. Lecture 12: Examples of perverse sheaves

12.1. **IC of subvarieties.** As above we consider the middle perversity $\bar{m}$ and a Whitney stratified space of dimension n with even dimensional strata. Let Y denote the closure of a single stratum, Y^o. Let $\mathcal{L}_Y$ be a local system on the stratum Y^o. Then the intersection complex $\underline{IC}_Y^{\bar{m}}(\mathcal{L}_Y)[-\mathrm{cod(Y)}/2]$ is $\bar{m}$-perverse. Here are the support diagrams for an 8 dimensional stratified space with strata of dimension 0, 2, 4, 6, 8 where, as above, "x" denotes possibly nonzero stalk cohomology and "c" denotes possibly nonzero stalk cohomology with compact support.

i\k	0	2	4	6	8
8	c	c	c	c	c
7			c	c	c
6				c	c
5					c
4					0
3					x
2				x	x
1			x	x	x
0	x	x	x	x	x
$\dim(Y) = 8$					

i\k	0	2	4	6
6	c	c	c	c
5			c	c
4				c
3				0
2				x
1			x	x
0	x	x	x	x
$\dim(Y) = 6$				

i\k	0	2	4
4	c	c	c
3			c
2			0
1			x
0	x	x	x
$\dim(Y) = 4$			

i\k	0	2
2	c	c
1		0
0	x	x
$\dim(Y) = 2$		

i\k	0
0	cx
$d = 0$	

Figure 6. Shifted IC of subvarieties

Adding these up gives the support diagram (Figure 8) for a perverse sheaf. (It is hoped that the Reader will appreciate the manner in which the Author coaxed latex into lining these up correctly.)

12.2. **Logarithmic perversity.** Because the support conditions for (middle) perverse sheaves are relaxed slightly from those for $\underline{IC}^\bullet$, there are several other perversities for which intersection cohomology forms a (middle) perverse sheaf. These include the *logarithmic* perversity $\bar{\ell}$, given by $\bar{\ell}(k) = k/2 = \bar{m}(k) + 1$ and its Verdier dual, the *sublogarithmic* perversity, $\bar{s}$ given by $\bar{s}(k) = \bar{m}(k) - 1$.

12.3. Let Y^o be a stratum of W (which is stratified by even dimensional strata). Let Y be its closure with inclusion $j_Y : Y \to W$. It is stratified by even dimensional strata. Let $A^\bullet$ be a perverse sheaf on Y. Then $Rj_*(A^\bullet)[-\mathrm{cod(Y)}/2]$ is a perverse sheaf on W.

12.4. **Hyperplane complements.** Let $\{H_1, H_2, \cdots, H_r\}$ be a collection of complex affine hyperplanes in $W = \mathbb{C}^n$. Stratify W according to the multi-intersections of the hyperplanes. The largest stratum is

$$W^o = W - \bigcup_{j=1}^{r} H_j$$

i	cod0	cod2	cod4	cod6	cod8
8	c	c	c	c	c
7				c	c
6					c
5					0
4					x
3				x	x
2			x	x	x
1			x	x	x
0	x	x	x	x	x
	$\underline{IC}^\bullet_{\bar{\ell}}$ support				

i	cod0	cod2	cod4	cod6	cod8
8	c	c	c	c	c
7		c	c	c	c
6			c	c	c
5				c	c
4					c
3					0
2					x
1				x	x
0	x	x	x	x	x
	$\underline{IC}^\bullet_{\bar{s}}$ support				

FIGURE 7. Support/cosupport of logarithmic and sublogarithmic IC sheaf

and it has a highly nontrivial fundamental group. let $\mathcal{L}$ be a local coefficient system on this hyperplane complement. Then $\underline{IC}^\bullet_{\bar{s}}(\mathcal{L})$, $\underline{IC}^\bullet_{\bar{m}}(\mathcal{L})$ and $\underline{IC}^\bullet_{\bar{\ell}}(\mathcal{L})$ are perverse sheaves on $W = \mathbb{C}^n$. These are surprisingly complicated objects, and even the case of middle perversity, when the hyperplanes are the coordinate hyperplanes, has been extensively studied. Notice, in this case, that the space $W = \mathbb{C}^n$ is nonsingular, the hyperplane complement W^o is nonsingular, and the sheaf $\underline{IC}(\mathcal{L})$ is constructible (with respect to this chosen stratification) but to analyze this sheaf we are forced to consider the singularities of the multi-intersections of the hyperplanes.

In the simplest case, $(\mathbb{C}, \{0\})$ the category of perverse sheaves is equivalent to the category of representations of the following quiver

$$\begin{tikzcd} \bullet \arrow[r, shift left, "\alpha"] & \bullet \arrow[l, shift left, "\beta"] \end{tikzcd}$$

where $I - \alpha\beta$ and $I - \beta\alpha$ are invertible.

For $\mathbb{C}^2$, $xy = 0$ (the coordinate axes) the perverse category is equivalent to the category of representations of the quiver

$$\begin{tikzcd} \bullet \arrow[r, shift left] \arrow[d, shift left] & \bullet \arrow[l, shift left] \arrow[d, shift left] \\ \bullet \arrow[r, shift left] \arrow[u, shift left] & \bullet \arrow[l, shift left] \arrow[u, shift left] \end{tikzcd}$$

with the same conditions on each of the horizontal and vertical pairs, such that all possible ways around the outside of the square commute.

12.5. **Small and semismall maps.** Let M be a compact complex algebraic manifold and let $\pi : M \to W$ be an algebraic mapping. Then π is said to be *semismall* if

$$\mathrm{cod_W}(\{\mathrm{x} \in \mathrm{X}|\ \dim \pi^{-1}(\mathrm{x}) \geq \mathrm{k}\}) \geq 2\mathrm{k}.$$

In other words, if the map has been stratified then for each stratum $S \subset W$ the dimension of the fiber over S is $\leq \frac{1}{2}$ the codimension of S. The map is *small* if, for each singular stratum S, $\dim \pi^{-1}(x) < \frac{1}{2}\mathrm{cod(S)}$ (for all $x \in S$).

If π is small then $R\pi_*(\underline{\mathbb{Q}})$ is a self dual sheaf on W whose support satisfies the support conditions of (middle) intersection cohomology. It follows from the axiomatic characterization that there is a canonical isomorphism (in $D^b_c(W)$), $R\pi_*(\underline{QQ}) \cong \underline{IC}_{\bar{m}}(W)$. In other words, the intersection cohomology of W is canonically isomorphic to the ordinary cohomology of M.

If π is semi-small then $R\pi_*(\underline{\mathbb{Q}})$ is (middle) perverse.

Let $W = \{P \subset \mathbb{C}^4|\ \dim(P) = 2, \dim(P \cap \mathbb{C}^2) \geq 1\}$ be the singular Schubert variety in the Grassmannian of 2-planes in 4-space. It has a singularity when $P = \mathbb{C}^2$. A resolution of singularities is $\widetilde{W} = \ \{(P, L)|\ P \in W, \ \text{ and } L \subset P \cap \mathbb{C}^2 \subset \mathbb{C}^4\}$. Then $\pi : \widetilde{W} \to W$ is a small map so $R\pi_*(\mathbb{Q}) \cong \underline{IC}^\bullet_W$ hence $IH^*(W) \cong H^*(\widetilde{W})$.

12.6. **Sheaves on $\mathbb{P}^1$.** Let us stratify $\mathbb{P}^1$ with a single zero dimensional stratum, N (the north pole, say). The support diagram is the following:

i\cod	0	2
2	c	c
1		cx
0	x	x

So the skyscraper sheaf supported at the point, $\underline{\mathbb{Q}}_N[-1]$ is perverse. We also have the following:

$$\underline{\mathbb{Q}}_N[-1] : \begin{array}{|c||c|c|} 2 & & \\ 1 & & \mathrm{cx} \\ 0 & & \end{array} \qquad \mathbb{Q}_{\mathbb{P}^1} : \begin{array}{|c||c|c|} 2 & \mathrm{c} & \mathrm{c} \\ 1 & & \\ 0 & \mathrm{x} & \mathrm{x} \end{array} \qquad j_!(\mathbb{Q}_U) : \begin{array}{|c||c|c|} 2 & \mathrm{c} & \mathrm{c} \\ 1 & & \mathrm{c} \\ 0 & \mathrm{x} & \end{array} \qquad j_*(\underline{\mathbb{Q}}_U) : \begin{array}{|c||c|c|} 2 & \mathrm{c} & \\ 1 & & \mathrm{x} \\ 0 & \mathrm{x} & \mathrm{x} \end{array}$$

The first sheaf is self dual. The second sheaf is self dual. The third and fourth sheaves are dual to each other. It turns out that there is one more indecomposable perverse sheaf on this space, which is not an IC sheaf, and its support diagram is the full diagram. It is self dual. Here is how to construct it. Take a closed disk and put the constant sheaf on the interior, zero on the boundary, except for one point (or even one segment). Then map this disk to the 2-sphere, collapsing the boundary to the N pole, and push this sheaf forward.

If we started with zero on the boundary and pushed forward we would gert the sheaf $Rj_!(\mathbb{Q}_U)$. If we started with the full constant sheaf on the disk and pushed forward we would get the sheaf $Rj_*(\mathbb{Q})$. This new sheaf has both stalk cohomology and compact support stalk cohomology in degree 1, at the singular point. Verdier duality switches these two types of boundary conditions, so when we have a mixed boundary condition as in this case, we obtain a self dual sheaf.

In this case the category of perverse sheaves is equivalent to the category of representations of the quiver

$$\begin{tikzcd} \bullet \arrow[r, shift left, "\alpha"] & \bullet \arrow[l, shift left, "\beta"] \end{tikzcd}$$

where $\alpha\beta = \beta\alpha = I$. There are five indecomposable objects, one of which is has $\mathbb{Q} \oplus \mathbb{Q}$ on one of the vertices of the graph.

12.7. **Deligne's numbering system.** In their book [**?**] Beilinson, Bernstein and Deligne modified the indexing system for cohomology in a way that vastly reduces the amount of notation and arithmetic involving indices. Although the new system is extremely simple, it is deceptively so, because it takes us one step further away from any intuition concerning perverse sheaves. The new system works best in the case of a complex algebraic (or analytic) variety W, stratified with complex algebraic (or analytic) strata, and counted according to their complex dimensions. The idea is simply to shift all degrees by $\dim_{\mathbb{C}}(W) = \dim(W)/2$. So the support conditions look like this:

new	old	cod0	cod2	cod4	cod6	cod8
4	8	c	c	c	c	c
3	7		c	c	c	c
2	6			c	c	c
1	5				c	c
0	4					cx
-1	3				x	x
-2	2			x	x	x
-3	1		x	x	x	x
-4	0	x	x	x	x	x
		Perverse sheaf support				

In symbols,

$$\dim \operatorname{spt} H^{-r}(A^\bullet) \le r \text{ and } \dim \operatorname{spt} H^{-r}(\mathbf{D}(A^\bullet)) \le r$$

or equivalently,

$$\dim\{x \in W|\ H^i_x(A^\bullet) \neq 0\} \le -i \text{ for all } i \in \mathbb{Z}$$
$$\dim\{x \in W|\ H^i_c(A^\bullet) \neq 0\} \le i \text{ for all } i \in \mathbb{Z}.$$

12.8. **Perversity zero.** Let W be a stratified pseudomanifold of dimension n (with a fixed stratification). The category of Perverse sheaves on W with perversity zero, constructible with respect to this stratification, is equivalent to the category of sheaves on W (nb: this means *sheaves*, rather than complexes of sheaves) that are constructible with respect to this stratification, that is, sheaves whose restriction to each stratum is locally trivial. In this case, the "abelian subcategory" defined by the perversity condition simply coincides with the abelian category structure of the category of sheaves. Here are support diagrams for intersection cohomology with perversity zero, and for perverse sheaves with perversity zero.

i	cod0	cod1	cod2	cod3	cod4	cod5	cod6
6	c	c	c	c	c	c	c
5				c	c	c	c
4					c	c	c
3						c	c
2							c
1							0
0	x	x	x	x	x	x	x
	$\underline{IC}^\bullet_{\bar\ell}$ support						

i	cod0	cod1	cod2	cod3	cod4	cod5	cod6
6	c	c	c	c	c	c	c
5			c	c	c	c	c
4				c	c	c	c
3					c	c	c
2						c	c
1							c
0	x	x	x	x	x	x	x
	perverse sheaf support						

FIGURE 8. Support/cosupport for $\bar{p} = 0$ IC sheaf and perverse sheaf

TECHNICAL INTERLUDE

The following lemma that provides lifts of morphisms in the derived category, see [GMII] §?¿

12.9. **Lemma.** *Let $A^\bullet, B^\bullet$ be a objects in the derived category. Suppose $\underline{H}^r(A^\bullet) = 0$ for all $r > p$ and suppose that $\underline{H}^r(B^\bullet) = 0$ for all $r < p$. Then the natural map*

$$\mathrm{Hom}_{D^b_c(X)}(A^\bullet, B^\bullet) \to \mathrm{Hom}_{Sh(X)}(\underline{H}^p(A^\bullet), \underline{H}^p(B^\bullet))$$

is an isomorphism.

Proof. When we wrote IH II, Verdier (who was one of the referees) showed us how to replace our 4 page proof with the following simple proof. Up to quasi-isomorphism it is possible to replace the complexes $A^\bullet$, B/b with complexes

$$\begin{array}{ccccccccccc} \cdots & \longrightarrow & A^{p-1} & \xrightarrow{d_A} & A^p & \longrightarrow & 0 & \longrightarrow & 0 & \longrightarrow & \cdots \\ \cdots & \longrightarrow & 0 & \longrightarrow & I^p & \xrightarrow[d_B]{} & I^{p+1} & \longrightarrow & I^{p+2} & \longrightarrow & \cdots \end{array}$$

where I^r are injective. This means that a morphism in the derived category is represented by an honest morphism between these complexes, that is, a mapping

$$\phi : \underline{H}^p(A^\bullet) = \mathrm{coker}(d_A) \to \ker(d_b) = \underline{H}^p(B^\bullet). \qquad \square$$

12.10. **Proof of Theorem 11.1.** We have a Whitney stratification of W and inclusions

$$U_2 \xrightarrow[j_2]{} U_3 \xrightarrow[j_3]{} \cdots \xrightarrow[j_{n-1}]{} U_n \xrightarrow[j_n]{} U_{n+1} = W$$

Let us suppose that $A^\bullet$ is constructible with respect to this stratification and that it satisfies the support (but not necessarily the co-support) conditions, that is

$$H^r(A^\bullet)_x = 0 \text{ for } r \geq p(c) + 1$$

whenever $x \in X^{n-c}$ lies in a stratum of codimension c. Fix $k \geq 2$ and consider the situation

$$U_k \xrightarrow[j_k]{} U_{k+1} \xleftarrow[i_k]{} X^{n-k}$$

where X^{n-k} is the union of the codimension k strata. Let $A^\bullet_k = A^\bullet | U_k$. Let $\bar{q}$ be the complementary perversity, $q(c) = c - 2 - p(c)$. The following proposition says that the vanishing of the stalk cohomology with compact supports $H^r(i_x^! A]b)$ is equivalent to the condition that the attaching map is an isomorphism:

12.11. **Proposition.** *The following statements are equivalent.*

(1) $A^\bullet_{k+1} \cong \tau_{\leq p(k)} Rj_{k*} A^\bullet_k$
(2) $\underline{H}^r(A^\bullet_{k+1})_x \to \underline{H}^r(Rj_{k*}A^\bullet_k)_x$ *is an isomorphism for all* $x \in X^{n-k}$
(3) $H^r(i_k^! A^\bullet_{k+1}) = 0$ *for all* $r \leq p(k) + 1$
(4) $H^r(i_x^! A^\bullet_{k+1}) = 0$ *for all* $r < n - q(k)$ *for all* $x \in X^{n-k}$

Proof. Items (1) and (2) are equivalent because there is a canonical morphism

$$A^\bullet_{k+1} \to Rj_{k*} j_k^* A^\bullet = Rj_{k*} A_k$$

truncation $\tau_{\leq p(k)}$ leaves an isomorphism in degrees $\leq p(k)$. Items (3) and (4) are equivalent because $i_x : \{x\} \to X^{n-k}$ is the inclusion into a manifold so $i_x^! = i_k^![n-k]$, and because $r < p(k) + 2 + (n-k) = n - (k - 2 - p(k)) = n - q(k)$. Items (2) and (3) are equivalent because there is a distinguished triangle,

$$\begin{tikzcd} Ri_{k*} i_k^! (\underline{A}^\bullet_{k+1}) \arrow[rr] & & \underline{A}^\bullet_{k+1} \arrow[dl, "\alpha"] \\ & Rj_{k*} j_k^* (\underline{A}^\bullet_{k+1}) \arrow[ul] & \end{tikzcd}$$

and therefore an exact sequence on stalk cohomology as follows:

$$\begin{tikzcd}
\underline{H}^{p+2}(i_k^! A^\bullet_{k+1})_x \arrow[r] & \underline{H}^{p+2}(A^\bullet_{k+1})_x \arrow[r] & \underline{H}^{p+2}(Rj_{k*}A^\bullet_k)_x \\
\underline{H}^{p+1}(i_k^! A^\bullet_{k+1})_x \arrow[r] & \underline{H}^{p+1}(A^\bullet_{k+1})_x \arrow[r] & \underline{H}^{p+1}(Rj_{k*}A^\bullet_k)_x \arrow[ull] \\
\underline{H}^{p}(i_k^! A^\bullet_{k+1})_x \arrow[r] & \underline{H}^{p}(A^\bullet_{k+1})_x \arrow[r, "\alpha"'] & \underline{H}^{p}(Rj_{k*}A^\bullet_k)_x \arrow[ull] \\
\underline{H}^{p-1}(i_k^! A^\bullet_{k+1})_x \arrow[r] & \underline{H}^{p-1}(A^\bullet_{k+1})_x \arrow[r, "\alpha"'] & \underline{H}^{p-1}(Rj_{k*}A^\bullet_k)_x \arrow[ull]
\end{tikzcd}$$

Now use the fact that the yellow highlighted terms are zero and the green highlighted morphisms are isomorphisms to conclude the proof of the Proposition. □

Since the sheaf $\underline{IC}^\bullet_{\bar{p}}[-n]$ satisfies these conditions, this proposition proves (by induction) that it is isomorphic (in the derived category) to the sheaf $\underline{P}^\bullet_{\bar{p}}$ that is defined by Deligne's construction. We gave an intuitive argument for this statement a few lectures ago, but the above constitutes a proof.

12.12. **Continuation of the proof of Theorem 11.1.** Now let us show that if $\mathcal{L}_1, \mathcal{L}_2$ are local systems on U_2 and if $A^\bullet = \underline{P}^\bullet_{\bar{p}}(\mathcal{L}_1)$ and if $B^\bullet = \underline{P}^\bullet_{\bar{p}}(\mathcal{L}_2)$ then we have an isomorphism

$$\mathrm{Hom}_{Sh}(\mathcal{L}_1, \mathcal{L}_2) \cong \mathrm{Hom}_{D^b_c(X)}(A^\bullet, B^\bullet).$$

As before, let $A^\bullet_{k+1} = A^\bullet | U_{k+1} = \tau_{\le p(k)} Rj_{k*} A^\bullet_k$. Assume by induction that we have established an isomorphism

$$\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2) \cong \mathrm{Hom}_{D^b_c(U_k)}(A^\bullet_k, B^\bullet_k).$$

Using the above triangle for $B^\bullet$ we get an exact triangle of $\underline{\mathrm{RHom}}$ sheaves,

$$\begin{tikzcd}
\underline{\mathrm{RHom}}^\bullet(A^\bullet_{k+1}, Ri_{k*}i_k^! B^\bullet_{k+1}) \arrow[rr] & & \underline{\mathrm{RHom}}^\bullet(A^\bullet_{k+1}, B^\bullet_{k+1}) \arrow[dl, "\alpha"] \\
& \underline{\mathrm{RHom}}^\bullet(A^\bullet_{k+1}, Rj_{k*} B^\bullet_k) \arrow[ul, "{[1]}"] &
\end{tikzcd}$$

By Lemma 12.9 and the support conditions, we see that α is an isomorphism in degree zero,

$$\mathrm{Hom}_{D^b_c X}(A^\bullet_{k+1}, B^\bullet_{k+1}) = H^0(U_{k+1}; \underline{\mathrm{RHom}}^\bullet(A^\bullet_{k+1}, B^\bullet_{k+1}) \cong H^0(U_{k+1}; \underline{\mathrm{RHom}}^\bullet(A^\bullet_{k+1}, Rj_{k*}B^\bullet_k)).$$

Moreover,

$$Rj_{k*}\underline{\mathrm{RHom}}^\bullet(A^\bullet_k, B^\bullet_k) \cong Rj_{k*}\underline{\mathrm{RHom}}^\bullet(j_k^* A^\bullet_{k+1}, B^\bullet_k) \cong \underline{\mathrm{RHom}}^\bullet(A^\bullet_{k+1}, Rj_{k*}B^\bullet_k)$$

by the standard identites (above), whose cohomology is

$$H^0(U_{k+1}; Rj_{k*}\underline{\mathrm{RHom}}^\bullet(A_k^\bullet, B_k^\bullet)) \cong H^0(U_k; \underline{\mathrm{RHom}}^\bullet(A_k^\bullet, B_k^\bullet)) \cong \mathrm{Hom}_{D_c^b(U_k)}(A_k^\bullet, B_k^\bullet).$$

So, putting these together we have a canonical isomorphism

$$\mathrm{Hom}_{D_c^b(U_k)}(A_k^\bullet, B_k^\bullet) \cong \mathrm{Hom}_{D_c^b(U_{k+1}}(A_{k+1}^\bullet, B_{k+1}^\bullet)$$

which was canonically isomorphic to $\mathrm{Hom}_{Sh}(\mathcal{L}_1, \mathcal{L}_2)$ by induction. This completes the proof of the theorem, but the main point is that the depth of the argument is the moment in which Lemma 12.9 was used in order to lift a morphism $A_{k+1}^\bullet \to Rj_{k*}B_k^\bullet$ to a morphism $A_{k+1}^\bullet \to B_{k+1}^\bullet$. $\square$

13. Lecture 13: t-structures and Perverse cohomology

13.1. **Definition.** An indecomposable object A in an abelian category is one that cannot be expressed nontrivially as a direct sum $A = B \oplus C$. A *simple* object A is one that has no nontrivial subobjects $B \to A$ (where the morphism is a monomorphism). An object is *semisimple* if it is a direct sum of simple objects. An *Artinian* category is one in which descending chains stabilize, which implies that every object can be expressed as a finite iterated sequence of extensions of simple objects.

13.2. **Perversity zero.** Let W be a stratified space. The category of (ordinary) sheaves on W that are constructible with respect to this stratification is Artinian. If $j : X \to W$ is the inclusion of a single stratum and if $\mathcal{L}$ is a local system on X then its extension by zero, $j_!(\mathcal{L})$ is a simple object in this category. Every object S in this category is an interated extension of such sheaves, for if S is such a sheaf then there is a smallest stratum $j : X \to W$ such that $S|X$ is nonzero. Let $j : X \to W$ be the inclusion. The sheaf $j^*(S)$ is a local system on S so it decomposes as a direct sum of local systems. We get an exact sequence

$$0 \to j_! j^* S \to S \to \underline{\text{coker}} \to 0.$$

and the cokernel is supported on fewer strata so we may proceed by induction.

Now let $A^\bullet$ be a complex of sheaves. We have truncation functors

$$\begin{aligned}
A^\bullet &= (\cdots \xrightarrow{d^{r-2}} A^{r-1} \xrightarrow{d^{r-1}} A^r \xrightarrow{d^r} A^{r+1} \xrightarrow{d^{r+1}} \cdots) \\
\tau_{\leq r} A^\bullet &= (\cdots \longrightarrow A^{r-1} \longrightarrow \underline{\ker}(d^r) \longrightarrow 0 \longrightarrow \cdots) \\
\tau^{\geq r} A^\bullet &= (\cdots \longrightarrow 0 \longrightarrow \underline{\text{coker}}(d^{r-1}) \longrightarrow A^{r+1} \longrightarrow \cdots)
\end{aligned}$$

Then there is a short exact sequence $0 \to \tau_{\leq 0} A^\bullet \to A^\bullet \to \tau^{\geq 1} A^\bullet \to 0$, and the cohomology sheaf of $A^\bullet$ is given by

$$\underline{H}^r(A^\bullet) = \tau_{\leq r}\left(\tau^{\geq r} A^\bullet\right) = \tau^{\geq r}\left(\tau_{\leq r} A^\bullet\right)$$

To summarize, let $Sh_c(W)$ be the category of (ordinary) sheaves on W that are constructible with respect to this stratification. Then the the following holds:

13.3. **Theorem.** *The cohomology functor $\underline{H}^r : D^b_c(W) \to Sh_c(W)$ is given by $\tau_{\leq r} \circ \tau^{\geq r}$ and also by $\tau^{\geq r} \circ \tau_{\leq r}$. The functor $\underline{H}^0$ restricts to an equivalence of categories between $Sh_c(W)$ and the subcategory of $D^b_c(W)$ whose objects are complex $A^\bullet$ such that $\underline{H}^r(A^\bullet) = 0$ for $r \neq 0$. This category is Artinian and its simple objects are the sheaves $j_!(\mathcal{L})$ where $\mathcal{L}$ is a simple local system on a single stratum $j : X \to W$.*

13.4. Fix a perversity $\bar{p}$. Let $\mathcal{P}(W)$ denote the category of perverse sheaves on W, that are constructible with respect to a given stratification. There are truncation functors

$$^{\bar{p}}\tau_{\leq r} \text{ and } {}^{\bar{p}}\tau^{\geq r} : D^b_c(W) \to D^b_c(W)$$

which are cohomological, that is, they take distinguished triangles to exact sequences, and satisfy

(T1) $^{\bar{p}}\tau_{\leq r}(A^\bullet) = ({}^{\bar{p}}\tau_{\leq 0}(A^\bullet[r]))[-r]$.

From this, define the *perverse cohomology*

$$^{\bar{p}}H^r(A^\bullet) = {}^{\bar{p}}\tau_{\leq r}\left({}^{\bar{p}}\tau^{\geq r}A^\bullet\right).$$

Then $^{\bar{p}}H^r : D^b_c(W) \to \mathcal{P}(W)$ and $A^\bullet \in \mathcal{P}(W)$ if and only if $^{\bar{p}}H^r(A^\bullet) = 0$ for all $r \neq 0$. In this case $^{\bar{p}}H^0(A^\bullet) = A^\bullet$.

13.5. **Theorem.** *The cohomology functor* $^{\bar{p}}H^r : D^b_c(W) \to \mathcal{P}(W)$ *is given by* $^{\bar{p}}\tau_{\leq r} \circ {}^{\bar{p}}\tau^{\geq r}$ *and also by* $^{\bar{p}}\tau^{\geq r} \circ {}^{\bar{p}}\tau_{\leq r}$. *The functor* $^{\bar{p}}H^0$ *restricts to an equivalence of categories between* $\mathcal{P}(W)$ *and the subcategory of* $D^b_c(W)$ *whose objects are complex* $A^\bullet$ *such that* $^{\bar{p}}H^r(A^\bullet) = 0$ *for* $r \neq 0$. *This category is Artinian and its simple objects are the sheaves* $Rj_*(\underline{IC}^\bullet_{\bar{p}}(\mathcal{L}))$ *where* $\mathcal{L}$ *is a simple local system on a single stratum* X *and where* $j : \bar{X} \to W$ *is the inclusion.*

In particular, a *semisimple* perverse sheaf is one which is a direct sum of (appropriately shifted) intersection cohomology sheaves of (closures of) strata.

13.6. **t structures.** But how to define these truncation functors? At this point it becames almost essential to shift to Deligne's numbering scheme, otherwise we will have shifts depending on the strata that will drive you crazy. Define the subcategory $D^b_c(W)_{\leq 0}$ to be the category of constructible complexes $A^\bullet$ that satisfy the support condition, that is,

$$\dim_{\mathbb{C}}\{x \in W|\ \mathrm{spt}\underline{\mathrm{H}}^{\mathrm{i}}(\mathrm{A}^\bullet)_{\mathrm{x}} \neq 0\} \leq -\mathrm{i}$$

and define $D^b_c(W)^{\geq 0}$ to be the category of complexes $A^\bullet$ that satisfy the cosupport condition,

$$\dim_{\mathbb{C}}\{x \in W|\ H^i(i^!_x A^\bullet) \neq 0\} \leq i.$$

Note that

(T2) $D^b_c(W)_{\leq 0} \subset D^b_c(W)_{\leq 1}$ and $D^b_c(W)^{\geq 0} \supset D^b_c(W)^{\geq 1}$.

(T3) $\mathrm{Hom}_{D^b_c(W)}(A^\bullet, B^\bullet) = 0$ for all $A^\bullet \in D^b_c(W)_{\leq 0}$ and $B \in D^b_c(W)^{\geq 1}$

by the basic lemma proven earlier. Moreover, $\mathcal{P}(W) = D^b_c(W)_{\leq 0} \cap D^b_c(W)^{\geq 0}$.

(T4) For any complex $X^\bullet$ there is a distinguished triangle

$$\begin{tikzcd} A^\bullet \arrow[rr] & & X^\bullet \arrow[dl] \\ & B^\bullet \arrow[ul, "{[1]}"] & \end{tikzcd}$$

where $A^\bullet \in D^b_c(W)^{\leq 0}$ and where $B^\bullet \in D^b_c(W)^{\geq 1}$.

In this case we define

$$\boxed{{}^{\bar{p}}\tau_{\leq 0}X^\bullet = A^\bullet \qquad \text{and} \qquad {}^{\bar{p}}\tau^{\geq 1}X^\bullet = B^\bullet.}$$

Then, the perverse cohomology is ${}^{\bar{p}}\underline{H}^0(X^\bullet) = {}^{\bar{p}}\tau^{\geq 0}(A^\bullet)$, so everything depends on the definition of the truncation functors.

13.7. **Further properties.** There is a beautiful way to see the kernel and cokernel of a morphism $\Phi : A^\bullet \to B^\bullet$ of perverse sheaves. Let $M^\bullet$ be the mapping cone of the morphism. Then the long exact sequence on perverse cohomology reads:

$$\begin{tikzcd}
\cdots \arrow[r] & {}^{\bar{p}}H^{-1}(B^\bullet) \arrow[r] \arrow[d, equal] & {}^{\bar{p}}H^{-1}(M^\bullet) \arrow[r] \arrow[d, equal] & {}^{\bar{p}}H^0(A^\bullet) \arrow[r] \arrow[d, equal] & {}^{\bar{p}}H^0(B^\bullet) \arrow[r] \arrow[d, equal] & {}^{\bar{p}}H^0(M^\bullet) \arrow[r] \arrow[d, equal] & {}^{\bar{p}}H^1(A^\bullet) \arrow[r] \arrow[d, equal] & \cdots \\
& 0 \arrow[r] & \ker(\Phi) \arrow[r] & A^\bullet \arrow[r, "\Phi"] & B^\bullet \arrow[r] & \operatorname{coker}(\Phi) \arrow[r] & 0 &
\end{tikzcd}$$

Perverse versions of other functors can be defined by using ${}^{\bar{p}}\underline{H}^0$ to "project" the result into the category $\mathcal{P}(W)$. For example, if $j : U \to X$ and if $A^\bullet$ is perverse on U then ${}^p j_* A^\bullet = {}^p H^0(Rj_* A^\bullet)$ and ${}^p j_! A^\bullet = {}^p H^0(Rj_! A^\bullet)$. In particular, if we start with a local system $\mathcal{L}$ on the nonsingular part $U = U_2$ of a stratified space W then

$${}^p j_!(\mathcal{L}) = \underline{IC}^\bullet_{\bar{s}}(\mathcal{L}) \quad \text{and} \quad {}^p j_*(\mathcal{L}) = \underline{IC}^\bullet_{\bar{\ell}}(\mathcal{L}).$$

Moreover, the (perverse) image of the first in the second is the middle IC sheaf, and it is sometimes referred to as

$${}^p j_{!*}(\mathcal{L}) = \underline{IC}^\bullet_{\bar{m}}(\mathcal{L}).$$

13.8. The truncation functors are constructed by induction. If W is a manifold and if $X^\bullet$ is cohomologically locally constant on W then take $A^\bullet = \tau_{\leq 0}X^\bullet$ and $B^\bullet = \tau^{\geq 1}X^\bullet$ as above, that is, ${}^p\tau = \tau$ is the usual truncation. Otherwise, assume (as usual) by induction that we have defined these truncation functors on an open union U of strata, and consider the addition of a single stratum S. We may assume that $W = U \cup S$.

$$U \xrightarrow[j]{} W \xleftarrow[i]{} S$$

Given a complex $X^\bullet$ on W let $Y^\bullet[1]$ be the mapping cone of $X^\bullet \to Rj_*{}^p\tau^{\geq 1}j^*X^\bullet$ and let $A^\bullet[1]$ be the mapping cone of $Y^\bullet \to i_*{}^p\tau^{\geq 1}i^*Y^\bullet$.

$$\begin{tikzcd}
& X^\bullet \arrow[rr] & & Rj_*{}^p\tau^{\geq 1}j^*X^\bullet \arrow[dl] \\
A^\bullet \arrow[ur, dotted] \arrow[rr] & & Y^\bullet \arrow[ul, "{[1]}"'] \arrow[dl] & \\
& i_*{}^p\tau^{\geq 1}i^*Y^\bullet \arrow[ul, "{[1]}"] & &
\end{tikzcd}$$

This gives the desired morphism $A^\bullet \to X^\bullet$ and then $B^\bullet$ is the third term.

For example, suppose there are two strata and we let $X = Rj_*(\mathbb{Q})$. Then ${}^p\tau^{\geq 1}j^*X = 0$ so $Y = X$. Now the stalk cohomology of $i^*(Y)$ equals the cohomology of the link. $\tau^{\geq 1}i^*(Y)$ is the cohomology of the link in degrees above the middle. So the mapping cone, which is A, keeps the cohomology of the link in degrees $\leq$ the middle. We end up with $0 \to A^\bullet \to Y^\bullet \to B^\bullet \to 0$ where $A^\bullet$ is perverse and where $B^\bullet$ is in $D^b_c(W)^{\geq 1}$. If we start with $X = Rj_!(\mathbb{Q})$ then this is already in $D^b_c(W)^{\leq 0}$. The relevant section is [BBD] p. 48.

13.9. More generally, a t-structure on a category $\mathcal{D}$ is defined to be a pair of subcategories $\mathcal{D}_{\leq 0}$ and $\mathcal{D}^{\geq 0}$ satisfying (T1), (T2), (T3), (T4) above. It is proven in [BBD] that under these hypotheses the *heart* $P = \mathcal{D}_{\leq 0} \cap \mathcal{D}^{\geq 0}$ is abelian.

14. Lecture 14: Algebraic varieties and the decomposition theorem

14.1. **Lefschetz theorems.** Suppose $W \subset \mathbb{CP}^N$ is a complex projective algebraic variety of complex dimension n. Let $L^j \subset \mathbb{CP}^n$ be a codimension j linear subspace. Let $Y^j = L^j \cap W$. If L^j is transverse to each stratum of a Whitney stratification of W then there are natural morphisms $\alpha : IH^r(W;\mathbb{Q}) \to IH^r(Y^j;\mathbb{Q})$ and $\beta : IH^s(Y^j;\mathbb{Q}) \to IH^{s+2j}(W;\mathbb{Q})$.

14.2. **Theorem.** *If H^1 is transverse to W then the restriction mapping $IH^r(W;\mathbb{Z}) \to IH^r(Y^1;\mathbb{Z})$ is an isomorphism for $r \le n-2$ and is an injection for $r = n-1$. If $j \ge 1$ and L^j is transverse to W then the composition $L^j : IH^{n-j}(W;\mathbb{Q}) \to IH^{n-j}(Y^j;\mathbb{Q}) \to IH^{n+j}(W;\mathbb{Q})$ is an isomorphism.*

These maps are illustrated in the following diagram.

$$\begin{tikzcd}
IH^0 & IH^1 & IH^2 & IH^3 & IH^4 & IH^5 & IH^6 & IH^7 & IH^8 & IH^9 & IH^{10} \\
IP^0 \arrow[rr] & & L(IP^0) \arrow[rr] & & L^2(IP^0) \arrow[rr] & & L^3(IP^0) \arrow[rr] & & L^4(IP^0) \arrow[rr] & & L^5(IP^0) \\
& IP^1 \arrow[rr] & & L(IP^1) \arrow[rr] & & L^2(IP^1) \arrow[rr] & & L^3(IP^1) \arrow[rr] & & L^4(IP^1) & \\
& & IP^2 \arrow[rr] & & L(IP^2) \arrow[rr] & & L^2(IP^2) \arrow[rr] & & L^3(IP^2) & & \\
& & & IP^3 \arrow[rr] & & L(IP^3) \arrow[rr] & & L^2(IP^3) & & & \\
& & & & IP^4 \arrow[rr] & & L(IP^4) & & & & \\
& & & & & IP^5 & & & & &
\end{tikzcd}$$

where (for $j \le n$) the *primitive intersection cohomology* $IP^j \subset IH^j$ is the kernel of $\cdot L^{n-j+1}$. (It may also be identified with the cokernel of $\cdot L : IH^{j-2} \to IH^j$.) The resulting decomposition is called the *Lefschetz decomposition* $IH^r \cong \oplus_{j=0}^{[r/2]} L^j \cdot IP^{r-2j}$. The combination of Poincaré duality

$$IH^{n+r}(W;\mathbb{Q}) \cong \mathrm{Hom}(IH^{n-r}(W;\mathbb{Q}),\mathbb{Q})$$

and the Lefschetz isomorphism $L^r : IH^{n-r}(W;\mathbb{Q}) \cong IH^{n+r}(W;\mathbb{Q})$ induces a nondegenerate bilinear pairing on $IH^{n-r}(W;\mathbb{Q})$. With respect to this pairing the Lefschetz decomposition is orthogonal and its signature is described by the Hodge Riemann bilinear relations.

14.3. **Hodge theory and purity.** Let W be a complex projective algebraic variety. Then there is a natural decomposition $IH^r(W;\mathbb{C}) \cong \oplus_{a+b=r} IH^{a,b}(W)$ with $\overline{IH^{a,b}} \cong H^{b,a}$ and the Lefschetz operator $\cap H^1$ induces $IH^{a,b} \to IH^{a+1,b+1}$.

Let X be a projective algebraic variety defined over a finite field k with $q = p^t$ elements. Then the étale intersection cohomology $IH^s_{et}(X(\bar{k});\mathbb{Q}_\ell)$ carries an action of $\mathrm{Gal}(\bar{k}/k)$ which is topologically generated by the Frobenius Fr. The eigenvalues of Fr on $IH^s_{et}(X)$ have absolute value equal to $\sqrt{q}^s$.

14.4. **Modular varieties.** Let G be a semisimple algebraic group defined over $\mathbb{Q}$ of Hermitian type, with associated symmetric space $D = G(\mathbb{R})/K$ where $K \subset G(\mathbb{R})$ is a maximal compact subgroup. Let $\gamma \subset G(\mathbb{Q})$ be a torsion free arithmetic group. The space $X = \Gamma\backslash D$ is a Hermitian locally symmetric space and it admits a natural "invariant" Riemannian metric. Let $\mathcal{L}$ be a finite dimensional metrized local system on X. The L^2 cohomology $H^r_{(2)}(X; \mathcal{L})$ is defined to be the cohomology of the complex of differential forms

$$\Omega^j_{(2)}(X) = \left\{ \omega \in \Omega^j(X;\mathbb{C}) | \int_X \omega \wedge *\omega < \infty, \int_X (d\omega) \wedge *(d\omega) < \infty \right\}.$$

This may be interpreted as the global sections of a complex of sheaves $\underline{\Omega}^r$ on the Baily Borel compactification $\overline{X}$ whose sections over an open set $U \subset \bar{X}$ consist of differetial forms $\omega \in \Omega^r(U \cap X)$ such that $\omega, d\omega$ are square integrable. The following conjecture of S. Zucker was proven by E. Looijenga and independently by L. Saper and M. Stern.

14.5. **Theorem.** *There is a natural quasi-isomorphism of sheaves* $\underline{\Omega}^{\bullet}_{(2)}(X) \cong \underline{IC}^{\bullet}_{\overline{X}}$ *which induces an isomorphism* $H^r_{(2)}(X) \cong IH^r(\overline{X}; \mathbb{C})$.

14.6. **Morse theory.** Let W be a Whitney stratified (with a fixed stratification) closed subset of some smooth manifold M. Let $f : M \to \mathbb{R}$ be a smooth function. For any $t \in \mathbb{R}$ let $W_{\leq t} = W \cap f^{-1}(-\infty, t]$. We say that f is a Morse function on W if the following holds

(1) For each stratum $W \subset W$ the restriction $f|W$ is a Morse function, that is, its critical points are isolated and nondegenerate.
(2) For each stratum $X \subset W$ and for each critical point $x \in X$ the following further nondegeneracy condition is required: suppose that $X < Y$ are strata of W, that $y_i \in Y$ is a sequence converging to $x \in X$ and suppose that the tangent planes $T_{y_i}Y$ converge to some limiting plane $\tau \subset T_xM$. Then $df(x)(\tau) \neq 0$.

In other words, $df(x)$ does not kill any limit of tangent planes from larger strata. It is a theorem from [SMT] that if $f : M \to \mathbb{R}$ is a smooth function that is Morse on W and if $x \in S \subset W$ is a critical point with critical value $v = f(x)$ then the group

$$M^i_x := H^i(W_{\leq v+\epsilon}, W_{\leq v-\epsilon})$$

is independent of ϵ, provided ϵ is chosen sufficiently small. It is called the $i-th$ Morse group at the critical point x. If x lies in the top stratum of W then $M^i_x = 0$ for all values of i except for one, the Morse index, and it equals the dimension of the negative eigenspace of the Hessian matrix $d^2f(x)$. However at a general singular point x the Morse groups M^i_x may be nonzero in many different degrees i.

14.7. **Theorem.** *Suppose $W \subset M$ is a Whitney stratified complex algebraic or complex analytic subvariety of a complex manifold and that the strata closures are also complex analytic. Let $f : M \to \mathbb{R}$ be a Morse function in the above sense. Let $A^\bullet$ be a perverse sheaf on W (for example, the $\underline{IC}^\bullet$ sheaf). Let $x \in S$ be a critical point of f. Then the Morse group*

$$M_x^i(A^\bullet) = H^i(W_{\le v+\epsilon}, W_{\le v-\epsilon}; A^\bullet)$$

is nonzero in at most a single degree, $i = \mathrm{cod}_{\mathbb{C}}(\mathrm{S}) + \lambda$ where λ is the Morse index of $f|S$.

This allows you to use Morse theory to analyze the cohomology of a perverse sheaf but it only works in the complex analytic setting.

14.8. **Decomposition theorem.** Let $f : X \to Y$ be a proper complex projective algebraic map with X nonsingular. The decomposition theorem says that $Rf_*(\underline{IC}_X^\bullet)$ breaks into a direct sum of intersection complexes of subvarieties of Y, with coefficients in various local systems, and with shifts. In many cases this statement is already enough to determine the constituent sum. More precisely,

(1) $Rf_*(\underline{IC}_X^\bullet) \cong \bigoplus_i {}^{\bar{p}}\underline{H}^i(Rf_*(\underline{IC}_X^\bullet))[-i]$ (This says that the push forward sheaf is a direct sum of perverse sheaves; these are its own perverse cohomology sheaves, shifted.)
(2) Each ${}^{\bar{p}}\underline{H}^i(Rf_*(\underline{IC}_X^\bullet))$ is a semisimple perverse sheaf. (This says that it is a direct sum of $\underline{IC}^\bullet$ sheaves of strata. In particular, each summand enjoys all the remarkable properties of intersection cohomology that were described in the previous lecture.)
(3) The big summands come in pairs, ${}^{\bar{p}}\underline{H}^i(Rf_*(\underline{IC}_X^\bullet)) \cong {}^{\bar{p}}\underline{H}^{-i}(\mathbf{D}(Rf_*(\underline{IC}_X^\bullet)))$ (This is because $\underline{IC}_X^\bullet$ is self-dual, hence so is its pushforward.)
(4) If η is the class of a hyperplane on X then for all r,
$$\cdot\eta^r : {}^{\bar{p}}\underline{H}^{-r}(Rf_*(\underline{IC}_X^\bullet)) \to {}^{\bar{p}}\underline{H}^r(Rf_*(\underline{IC}_X^\bullet))$$
is an isomorphism. (This is the relative hard Lefschetz theorem.)
(5) If L is the class of a hyperplane on Y then for all s and all r,
$$\cdot L^s : H^{-s}(Y, {}^{\bar{p}}\underline{H}^r(Rf_*(\underline{IC}_X^\bullet))) \to H^s(Y, {}^{\bar{p}}\underline{H}^r(Rf_*(\underline{IC}_X^\bullet))$$
is an isomorphism. (This is just the statement that each summand satisfies hard Lefschetz.)

Moreover, if the mapping f can be stratified then the resulting perverse sheaves are constructible with respect to the resulting stratification of Y.

14.9. The hard Lefschetz isomorphisms give rise to the Lefschetz decomposition into primitive pieces as above. The combination of the Lefschetz isomorphism and the Poincaré duality isomorphism gives a nondegenerate bilinear form on each $H^r(Rf_*(\underline{IC}^\bullet)$. The Lefschetz decomposition is orthogonal with respect to this pairing, and the signature of the components is given by the Hodge Rieman bilinear relations.

14.10. If $f: X \to Y$ is a proper projective algebraic map, recall that the i-th cohomology sheaf of $Rf_*(\underline{IC}^\bullet_X)$ is the constructible sheaf

$$R^i f_*(\underline{IC}^\bullet_X) = \underline{H}^i(Rf_*(\underline{IC}^\bullet_X)$$

whose stalk at a point $y \in Y$ is the cohomology $H^i(f^{-1}(y); \underline{IC}^\bullet_X | f^{-1}(y))$. Let $U \subset Y$ be the nonsingular part. Then the *invariant cycle theorem* says that

(6) The restriction map

$$IH^i(X) \to H^0(U, R^i f_*(\underline{IC}^\bullet_X)) = \Gamma(U, R^i f_*(\underline{IC}^\bullet_X))$$

is surjective.

Let $y \in Y$ and let B_y be the intersection of Y with a small ball around y. Let $y_0 \in B_y \cap U$. For simplicity assume X is nonsingular so that $\underline{IC}^\bullet_X = \mathbb{Q}_x[n]$. Then the *local invariant cycle theorem* says

(7) restriction mapping

$$H^i(f^{-1}(B_y, \mathbb{Q}) = H^i(f^{-1}(y), \mathbb{Q}) \to H^0(B_y, R^i f_*(\mathbb{Q}_X[n])) \cong H^i(f^{-1}(y_0))^{\pi_1(U \cap B_y)}$$

is surjective.

14.11. Let us suppose that X is nonsingular so that (in Deligne's numbering system) $\underline{IC}^\bullet_X \cong \mathbb{Q}_X[n]$. The decomposition theorem contains two hard Lefschetz theorems and they work against each other to limit the types of terms that can appear in this decomposition. Let $[\eta] \in H^2(X)$ denote a hyperplane class and let $[L] \in H^2(Y)$ denote a hyperplane class. Statement (4) says that, for each $j \geq 0$ the cup product with $[\eta]^j$ induces an isomorphism

$$H^r(Y; {}^p H^{-j}(Rf_*(\mathbb{Q}[n]))) \cong H^{r+2j}(Y; {}^p H^j(Rf_*(\mathbb{Q}[n]))) \text{ for all } r.$$

Statement (2) says that ${}^p H^j(Rf_*(\mathbb{Q}[n]))$ is a direct sum of intersection cohomology sheaves, each of which satisfies hard Lefschetz (with respect to L) so that, for any $t \geq 0$ and for all j, the cup product with L^t induces an isomorphism

$$H^{r-t}(Y; {}^p H^j(Rf_*(\mathbb{Q}[n])) \cong H^{r+t}(Y; {}^p H^j(Rf_*(\mathbb{Q}[n])).$$

14.12. Suppose $\pi: X \to Y$ is a resolution of singularities. The decomposition theorem says that $R\pi_*(\mathbb{Q}[n])$ is a direct sum of intersection cohomology sheaves of subvarieties. The stalk cohomology of this sheaf, at any nonsingular point $y \in Y$ is $H^*(\pi^{-1}(y); \mathbb{Q}[n])$ which is $\mathbb{Q}$ in degree $-n$. So the sheaf $\underline{IC}^\bullet_Y$ is one of the summands, that is: the intersection cohomology of Y appears as a summand in the cohomology of any resolution.

14.13. Suppose X, Y are nonsingular and $f : X \to Y$ is an algebraic fiber bundle. Then $Rf_*(\mathbb{Q}_X[n])$ decomposes into a direct sum of perverse sheaves on Y, each of which is therefore a local system on Y, that is,

$$H^r(X;\mathbb{Q}) \cong \oplus_{i+j=r} H^i(Y; H^j(F))$$

where $H^j(F)$ denotes the cohomology of the fiber, thought of as a local system on Y. In other words, the Leray spectral sequence for this map degenerates (an old theorem of Deligne) and hard Lefschetz applies both to $H^j(F)$ and to $H^i(Y)$.

14.14. **Three proofs.** The first and original proof is in [BBD] and uses reduction to varieties in characteristic $p > 0$, purity of Frobenius, and Deligne's proof of the Weil conjectures. The second proof is due to Morihiko Saito, who developed a theory of mixed Hodge modules in order to extend the proof to certain analytic settings. The third proof is due to deCataldo and Migliorini, who used classical Hodge theory. Their proof works in the complex analytic setting and some people feel it is the most accessible of the three.

15. Lecture 15: cohomology of toric varieties

15.1. In 1915 Emmy Noether proved that if a Hamiltonian system is preserved by an 1-parameter infinitesimal symmetry (that is to say, by the action of a Lie group) then a certain corresponding "conjugate" function, or "first integral" is preserved under the time evolution of the system. Time invariance gives rise to conservation of energy. Translation invariance gives rise to conservation of momentum. Rotation invariance gives rise to conservation of angular momentum.

Figure 9. Emmy Noether

Today, this is known as the *moment map*: Suppose (M, ω) is a symplectic manifold, and suppose a compact lie group G acts on M and preserves the symplectic form. The infinitesimal action of G in the direction of V defines a vector field X on M. Contract this with the symplectic form to obtain a 1-form $\theta = \iota_X(\omega)$. It follows that $d\theta = 0$. If the action of G is *Hamiltonian* then in fact, $\theta = df$ for some smooth function $f : M \to \mathbb{R}$ (defined up to a constant). This is the conserved quantity. In summary, if the action is Hamiltonian then there exists a *moment map*, that is, a smooth mapping $\mu : M \to \mathfrak{g}^*$ so that for each $X \in \}$ the differential of the function $p \mapsto \langle \mu(p), X\rangle$ equals $\iota_X(\omega)$.

Now consider the Fubini Study metric $h(z, w) = \sum dz_i \wedge d\bar{z}_i$ on projective space. The real and imaginary part, $h = R + i\omega$ are respectively, positive definite and sympletic. Fix $a_0, a_1, \cdots, a_n \in \mathbb{Z}$.

If $\lambda \in \mathbb{C}^\times$ acts on $\mathbb{C}^{n+1}$ by

$$\lambda \cdot (z_0, z_1, \cdots, z_n) = (\lambda^{a_0} z_0, \lambda^{a_1} z_1, \cdots, \lambda^{a_n} z_n)$$

then, restricting the action to (S^1), the resulting moment map $\mu : \mathbb{CP}^n \to \mathbb{R}$ is

$$\mu([z_0 : z_1 : \cdots : z_n]) = \frac{a_0|z_0|^2 + a_1|z_1|^2 + \cdots + a_n|z_n|^2}{(|z_0|^2 + \cdots + |z_n|^2)}.$$

If $(\lambda_0, \lambda_1, \cdots, \lambda_n) \in (\mathbb{C}^\times)^n$ acts on $\mathbb{C}^n$ by

$$(\lambda_0, \cdots, \lambda_n) \cdot (z_0, \cdots, z_n) = (\lambda_0 z_0, \cdots, \lambda_n z_n)$$

then, restricting the action to $(S^1)^n$ the restulting moment map $\mu : \mathbb{CP}^n \to \mathbb{R}^n$

$$\mu([z_0 : \cdots : z_n]) = \frac{(|z_0|^2, |z_1|^2, \cdots, |z_n|^2)}{(|z_0|^2 + \cdots + |z_n|^2)}$$

and it is the standard simplex contained in the hyperplane $x_0 + \cdots + x_n = 1$. These actions are Hamiltonian and the moment map collapses orbits of $(S^1)^n$.

15.2. Now let $X \subset \mathbb{CP}^N$ be an n-dimensional subvariety on which a torus $T = (\mathbb{C}^\times)^n$ acts with finitely many orbits. In this case the action extends to a linear action on projective space of the sort described above and the moment map image (for the action of $(S^1)^n$), $\mu(X) \subset \mu(\mathbb{CP}^n)$ is convex. In fact, the *convexity theorem* of Atiyah, Kostant, Guillemin, Sternberg states that

15.3. **Theorem.** *The moment map image $\mu(X)$ is the convex hull of the images $\mu(x_i)$ of the T-fixed points in X. The image of each k-dimensional T-orbit is a single k-dimensional face of this polyhedron.*

It turns out, moreover, that the toric variety is a rational homology manifold if and only $\mu(X)$ is a simple polytope, meaning that each vertex is adjacent to exactly n edges.

Algebraic geometers prefer a presentation of a toric variety from a *fan*, a collection of homogeneous cones in Euclidean space. From a fan one constructs a convex polynedron by intersecting the fan with a ball centered at the origin, and then flattening the faces. The resulting convex polyhedron is the dual of the moment map polyhedron. If the moment map polyhedron is *simple* then the fan-polyhedron is *simplicial*, meaning that the faces are simplices.

15.4. **Definition.** If Y is a complex algebraic variety define the intersection cohomology Poincaré polynomial

$$h(Y, t) = h_0 + h_1 t + h_2 t^2 + \cdots + h_n t^n$$

where $h_r = \operatorname{rank} IH^r(Y; \mathbb{Q})$. If $y \in Y$ define the local Poincaré polynomial $h_y(Y, t) = \sum_{r \geq 0} \operatorname{rank} (\underline{H}^r(\underline{IC}^\bullet)_y)\, t^r$.

If Y is defined over $\mathbb{F}_q$ we use the same notation for the Poincaré polynomial of the étale intersection cohomology.

15.5. **Counting points.** There is a very general approach to understanding the cohomology and intersection cohomology of an n-dimensional algebraic variety defined over a finite field $\mathbb{F}_q$, provided its odd degree cohomology groups vanish. The variety $Y/\mathbb{F}_q$ is said to be *pure* if the eigenvalues of Frobenius on $H^r(Y;\mathbb{Q}_\ell)$ have absolute value $\sqrt{q}^r$ with respect to any embedding into the complex numbers. The Weil conjectures (proven by Deligne) say that

$$\sum_{r=0}^{2n}(-1)^r\mathrm{Tr}(Fr_q : H^r(Y) \to H^r(Y)) = |Y(\mathbb{F}_q)|$$

the right hand side being the number of points that are fixed by the Frobenius morphism. The intersection cohomology of any projective algebraic variety is pure. If the variety Y is also nonsingular (so that $IH^*(Y) = Y^*(Y)$ and *Tate* (which means that the eigenvalues on H^r are in fact equal to $(\sqrt{q})^r$) then this gives

$$h(Y,\sqrt{q}) = \sum_{r=0}^{n}\mathrm{rank}H^{2r}(Y)q^r = |Y(\mathbb{F}_q)|.$$

For example, if such a variety Y is defined over the integers, is nonsingular and $Y(\mathbb{C})$ has an algebraic cell decomposition with m_r cells of (complex) dimension r then $h_{2r+1} = 0$ and $h_{2r} = m_r$ accounts for $m_r q^r$ points over $\mathbb{F}_q$. In the case of a nonsingular toric variety whose moment map image is a convex polyhedron with f_r faces of dimension r this gives

$$h(Y,\sqrt{q}) = \sum_{s=0}^{n}\mathrm{rank}H^{2s}(Y;\mathbb{C})q^s = \sum_{r=0}^{n} f_r(q-1)^r$$

since each r-dimensional orbit is itself (isomorphic to) a torus of dimension r. The hard Lefschetz theorem says $h_{2s-2} \le h_{2s}$ for $2s \le n$ which in turn gives inequalities between the face numbers, as observed by Stanley in 1980.

15.6. If we wish to use intersection cohomology rather than ordinary cohomology in the Weil conjectures then the formula becomes

$$\sum_{r=0}^{2n}(-1)^r\mathrm{Tr}(Fr_q : IH^r(Y) \to IH^r(Y)) = |Y(\mathbb{F}_q)|_{\mathrm{mult}}$$

where each point $y \in Y(\mathbb{F}_q)$ is counted with a multiplicity equal to the (alternating sum of) trace of Frobenius acting on the stalk of the intersection cohomology at $y \in Y(\mathbb{F}_q)$. If this is pure and if the stalk cohomology vanishes in odd degrees, then this multiplicity equals the Poincaré polynomial $h_y(Y,\sqrt{q})$ of the stalk of the intersection cohomology. In conclusion, if the intersection cohomology of Y is Tate and vanishes in odd degrees then

$$\text{(15.6.1)} \qquad \sum_{s=0}^{n} \mathrm{Tr}(Fr_q|(IH^{2s}(Y))) = \sum_{s=0}^{n} \mathrm{rank}\ IH^{2s}(Y)q^s = h(Y, \sqrt{q}) = \sum_{y \in Y(\mathbb{F}_q)} h_y(Y, \sqrt{q}).$$

Let us now try to determine these multiplicities h_y. If $f(x) = a_0 + a_1x + \cdots + a_nx^n$ define the truncation $\tau_{\le r} f$ to be the polynomial $a_0 + \cdots + a_rx^r$ consisting of those terms of degree $\le r$.

15.7. **Lemma.** *Let $Z \subset \mathbb{C}P^{N-1}$ be a projective algebraic variety of dimension d, with intersection cohomology Poincaré polynomial*

$$g(t) = g_0 + g_1t + \cdots + g_{2d}t^{2d} = \sum_{r=0}^{2d} \dim(IH^r(Z))t^r.$$

Then the stalk of the intersection cohomology of the complex cone $Y = \mathrm{cone}_{\mathbb{C}}(Z) \subset \mathbb{C}P^N$ at the cone point $y \in Y$ has Poincaré polynomial

$$\text{(15.7.1)} \qquad h_y(Y, t) = \tau_{\le d}\left(g(t)(1 - t^2)\right)$$

Proof. The complex projective space $\mathbb{CP}^N$ is the complex cone over $\mathbb{CP}^{N-1}$. In fact, if we remove the cone point then what remains is a line bundle $\mathcal{L} \to \mathbb{CP}^{N-1}$ whose first Chern class $c^1(\mathcal{L}) \in H^2(\mathbb{CP}^{N-1})$ is the class of a hyperplane section. This is to say that there exists a section of this bundle that vanishes precisely on a hyperplane; it may be taken to be

$$s([z_0 : \ldots z_{N-1}]) = [z_0 : \ldots : z_{N-1}, \Sigma_j a_j z_j] \in \mathbb{CP}^N$$

for any fixed choice (not all zero) of $a_0, a_1, \cdots, a_{N-1} \in \mathbb{C}$. The vanishing of the last coordinate is a hyperplane in $\mathbb{CP}^{N-1}$. So this class may be used as a hard Lefschetz class.

If $Z \subset \mathbb{CP}^{N-1}$ is a projective algebraic variety then $\mathrm{cone}_{\mathbb{C}}(Z) \subset \mathbb{CP}^N$ is a singular variety and the link L of the cone point can be identified with the sphere bundle of this line bundle $\mathcal{L}_Z \to Z$. The Gysin sequence becomes

$$IH^{i-2}(Z) \xrightarrow{\cup c^1} IH^i(Z) \longrightarrow IH^i(L) \longrightarrow IH^{i-1}(Z) \xrightarrow{\cup c^1} IH^{i+1}(Z) \longrightarrow$$

It follows from the hard Lefschetz theorem for Z that $IH^i(L)$ is the primitive part of the intersection cohomology of Z for $i \le d$, that is,

$$IH^i(L) \cong IP^i(Z) = \mathrm{coker}(\cdot c^1(\mathcal{L}) : IH^{i-2}(Z) \to IH^i(Z))$$

for $i \le \dim(Z)$, and hence its Poincaré polynomial is given by

$$g_0 + g_1t + (g_2 - g_0)t^2 + (g_3 - g_1)t^3 + (g_4 - g_2)t^4 + \cdots + (g_d - g_{d-2})t^d = \tau_{\le d}g(t)(1 - t^2). \qquad \square$$

15.8. **Some geometry.** Let $\mu : Y \to P \subset \mathbb{R}^m$ be the moment map corresponding to the action of a torus $T \cong (\mathbb{C}^\times)^m$ on a toric variety Y. If F is a face of P, the link of F can be realized as another conex polyhedron, $L_F = P \cap V$ where $V \subset \mathbb{R}^N$ is a linear subspace such that $\dim(V) + \dim(F) = N - 1$, which passes near F and through P. (For example, V may be taken to lie completely in the plane $F^\perp$.) In fact, L_F is the moment map image of a sub-toric variety Y_F on which a sub-torus T_F acts.

15.9. In the case of a toric variety Y, a given face F corresponds to a stratum S_F of the toric variety. The link of this stratum is therefore isomorphic to a circle bundle over a toric variety whose moment map image is the link L_F of the face F. Let $h(Y_F, t)$ be the intersection cohomology Poincaré polynomial of this "link" toric variety. Then equations (15.7.1) and (15.6.1) give:

15.10. **Theorem.** *The IH Poincaré polynomial of Y is*

$$h(Y,t) = \sum_F (t^2-1)^{\dim(F)}.\tau_{\le n-\dim(F)}\left((1-t^2)h(Y_F,t)\right)$$

15.11. In particular, the intersection cohomology only depends on the combinatorics of the moment map image $P = \mu(Y)$, and moreoer, the functions $h(Y_F, t)$ may be determined (inductively) from the moment map images $L_F = \mu_F(Y_F)$. The hard Lefschetz theorem (which says that $h_{2r} \ge h_{2r-2}$ for all $r \le \dim(Y)$) then implies a collection of inequalities among the numbers of chains of faces.

15.12. **Remarks.** This formula simplifies if P is a simple polyhedron, to:

$$h(Y,t) = \sum_F (t^2-1)^{\dim(F)} = f(t^2-1)$$

where $f(s) = f_0 + f_1 s + \cdots + f_d s^d$ and f_j is the number of faces of dimension j. The polytopes considered here are always rational, meaning that the vertices are rational points in $\mathbb{R}^d$, or equivalently, the faces are the kernels of linear maps $\mathbb{R}^d \to \mathbb{R}$ with rational coefficients. Any simple (or simplicial) polytope can be perturbed by moving the faces (resp. the vertices) so as to make them rational. Therefore the inequalities arising from hard Lefschetz apply to all simple polytopes. However a general polytope cannot necessarily be perturbed into a rational polytope with the same face combinatorics. The Egyptian pyramid, for example, has a square face. Lifting one of the vertices on this face, an arbitrarily small amount, will force the face to "break". In order to prove that the inequalities arising from hard Lefschetz for intersection cohomology can be applied to any polytope it was necessary to construct something like intersection cohomology in the non-rational case. This was accomplished by (Barthel, Brasselet, Fieseler, Kaup) and K. Karu [] (who proved that it satisfies the hard Lefschetsz formula).

15.13. There is another way to prove this result using the decomposition theorem (which does not involve passing to varieties over a finite field). The singularities of the toric variety Y can be resolved by a sequence of steps, each of which is toric with moment maps that correspond to 'cutting off the faces" that are singular. For example, the Egyptian pyramid has a single singular point. The singularity is resolved by a mapping $\pi : \tilde{Y} \to Y$ as illustrated in this diagram:

insert diagram here.

Let us examine the decomposition theorem for this mapping. The mapping is an isomorphism everywhere except over the singular point $y \in Y$ and $\pi^{-1}(y) \cong \mathbb{P}^1 \times \mathbb{P}^1$. The stalk cohomology of the pushforward $R\pi_*(\mathbb{Q}_{\tilde{Y}})$ is $(\mathbb{Q}, 0, \mathbb{Q} \oplus \mathbb{Q}, 0, \mathbb{Q})$. Put this into the support diagram for a 3 dimensional variety:

i	j	cod0	cod2	cod4	cod6	$H^*(\pi^{-1}(y))$
6	3	c	c	c	c	
5	2			c	c	
4	1				c	$\mathbb{Q}$
3	0				0	0
2	-1				x	$\mathbb{Q} \oplus \mathbb{Q}$
1	-2			x	x	0
0	-3	x	x	x	x	$\mathbb{Q}$
		$\underline{IC}^\bullet$ support				

(Here, the degree i is the "usual" cohomology degree notation and the degree j is the "perverse degree" notation.) From the decomposition theorem we know that one term will be $\underline{IC}^\bullet(Y)$ and that there are additional terms supported at the singular point y. From the support condition it is clear that $\mathbb{Q}[3]$ (on the bottom row) is part of the IC sheaf. It is not so clear how much of the $(\mathbb{Q} \oplus \mathbb{Q})[1]$ belongs to $\underline{IC}^\bullet(Y)$ and how much belongs to the other terms. However the $\mathbb{Q}[-1]$ (at the top of the column; in degree $j = 1$) is definitely not part of IC. By Poincaré and especially by Hard Lefschetz, it must be paired with one copy of $\mathbb{Q}$ in degree -1. So this leaves $\mathbb{Q}[3] \oplus \mathbb{Q}[1]$ (in degrees -3 and 1 respectively) for the IC sheaf. A closer inspection of this argument shows that these two terms constitute the primitive cohomology of the fiber, as we saw earlier.

Thus, the decomposition theorem singles out the primitive cohomology of the fiber as belonging to the IC sheaf. Now, assuming by induction that the formula holds for $IH^*(\tilde{Y}) = H^*(\tilde{Y})$ (which is less singular that Y) and knowing how these terms decompose, it is easy to conclude that the formula must also hold for $IH^*(Y)$.

16. Lecture 16: Springer Representations

16.1. **The flag manifold.** Let $G = SL_n(\mathbb{C})$. It acts transitively on the set $\mathcal{F}$ of complete flags $0 \subset F^1 \subset \cdots \subset F^{n-1} \subset \mathbb{C}^n$ and the stabilizer of the standard flag is the "standared" Borel subgroup B of (determinant $= 1$) upper triangular matrices, giving an isomorphism $\mathcal{F} \cong G/B$. The Lie algebras are $\mathfrak{g}$ (matrices with trace $= 0$) and $\mathfrak{b} =$ upper triangular matrices with trace $= 0$. If $x \in G$ and $xBx^{-1} = B$ then $x \in B$. So we may identify $\mathcal{F}$ with the set $\mathcal{B}$ of all subgroups of G that are conjugate to B or equivalently to the set of all subalgebras of $\mathfrak{g}$ that are conjugate to $\mathfrak{b}$, that is, the *variety of Borel subalgebras* of $\mathfrak{g}$.

16.2. **Definition.** Let $\mathcal{N}$ be the set of all nilpotent elements in $\mathfrak{g}$. Define

$$\begin{array}{ccc} \widetilde{\mathcal{N}}=\{(x \in \mathcal{N}, A \in \mathcal{B})|\ x \in \mathrm{Lie}(A)\} & \xrightarrow{\phi} & \mathcal{B} \\ \downarrow{\scriptstyle\pi} & & \\ \mathcal{N} & & \end{array}$$

The mapping π is proper and its fibers $\mathcal{B}_x = \pi^{-1}(x)$ are called *Springer fibers*. In a remarkable series of papers [Springer 1976, 1978], T. A. Springer constructed an action of the symmetric group W on the cohomology of each Springer fiber $\mathcal{B}_x$, even though W does not actually act on $\mathcal{B}_x$.

Let A be the subgroup that preserves a flag $F_A = (0 = A^0 \subset A^1 \subset \cdots \subset A^n = \mathbb{C}^n)$ then the following are equivalent:

(1) $(x, A) \in \widetilde{\mathcal{N}}$
(2) $x \in \mathrm{Lie}(A)$
(3) $\exp(x)$ preserves the flag F_A
(4) the vectorfield V_x (defined by x) on the flag manifold $\mathcal{F}$ vanishes on F_A
(5) $xA^j \subset A^{j-1}$ for $1 \le j \le n$.

So the Springer fiber $\pi^{-1}(x)$ is the zero set of the vectorfield V_x; it is the set of all flags that are preserved by x and is often referred to as the *variety of fixed flags*. For the subregular nilpotent $x \in \mathfrak{g}$ the Springer fiber turns out to be a string of $n-1$ copies of $\mathbb{P}^1$, each joined to the next at a single point. [picture]

16.3. **Lemma.** *The mapping $\phi : \widetilde{\mathcal{N}} \to \mathcal{B}$ identifies $\widetilde{\mathcal{N}}$ with the cotangent bundle to the flag manifold.*

Proof. The tangent space at the identity to $\mathcal{F}$ is $T_I(G/B) = \mathfrak{g}/\mathfrak{b}$. So its dual space is

$$T_I^*(G/B) = \{\phi : \mathfrak{g} \to \mathbb{C}|\ \phi(\mathfrak{b}) = 0\}.$$

The canonical inner product $\langle , \rangle : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ given by $\langle x, y\rangle = \mathrm{Trace}(xy)$ is symmetric and nondegenerate. Using this to identify $\mathfrak{g}^*$ with $\mathfrak{g}$ gives

$$T_I^*(G/B) \cong \{x \in \mathfrak{g}|\ \langle x, \mathfrak{b}\rangle = 0\} = \mathfrak{n}$$

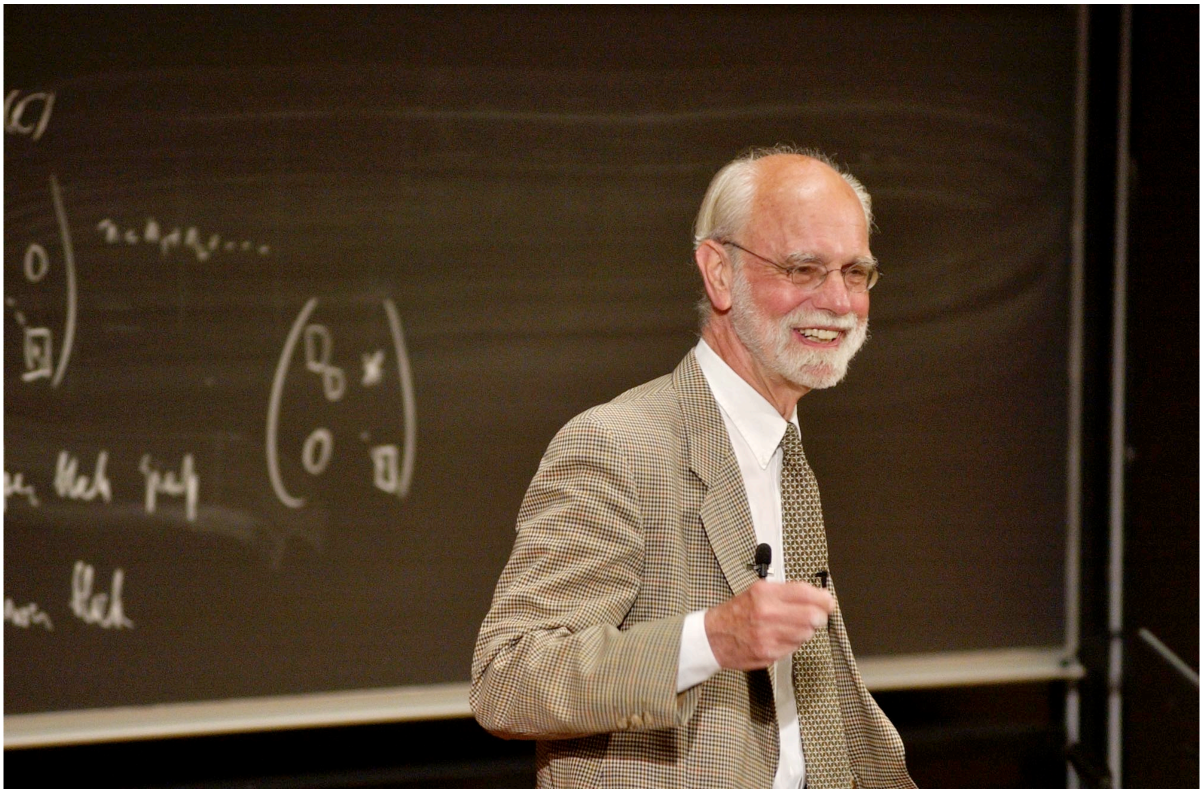
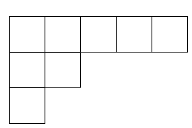


FIGURE 10. Tonny Springer

is the algebra of strictly upper triangular matrices, that is, the nilradical of $\mathfrak{b}$. So for each Borel subgroup $A \subset G$, the cotangent space $T^*_A(G/B) \cong \mathfrak{n}(A)$ is naturally isomorphic to the nilradical of Lie(A). But this is exactly the fiber, $\phi^{-1}(A)$. $\square$

16.4. The group G acts on everything in the diagram (16.2). It acts transitively on $\mathcal{B}$ and it acts with finitely many orbits on $\mathcal{N}$, each of which is a *nilpotent conjugacy class*. These form a Whitney stratification of $\mathcal{N}$ by complex algebraic strata. It follows from Jordan normal form that each nilpotent conjugacy class corresponds to a partition $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_r$ with $\sum \lambda_i = n$ or equivalently to a Young frame



Stratum closure relations correspond to refinement of partitions with the largest stratum corresponding to the case of a single Jordan block ($\lambda_1 = n$) and the smallest stratum corresponding to $0 \in \mathfrak{N}$, which is the partition $1 + 1 + 1 \cdots + 1 = n$.

16.5. The *Grothendieck simultaneous resolution* is the pair

$$\widetilde{\mathfrak{g}} = \{(x, A) \in \mathfrak{g} \times \mathcal{B} |\ x \in A\} \xrightarrow{\widehat{\pi}} \mathfrak{g}$$

Given $(x, A) \in \widetilde{\mathfrak{g}}$ choose $h \in G$ which conjugates A into the standard Borel subgroup B. It conjugates x into an element $x' \in \mathrm{Lie}(B)$ and the diagonal entries $\alpha = \alpha(x) \in \mathfrak{t}$ are well defined where $\mathfrak{t}$ is the set of diagonal matrices with trace $= 0$. On the other hand, the characteristic polynomial $ch(x)$ of x is determined by the diagonal matrix α but is independent of the order of the entries. The set of possible characteristic polynomials forms a vector space, with coordinates given by the coefficients of the characteristic polynomial, which provides an example of a remarkable theorem of Chevalley (cf. `http://www.math.ias.edu/~goresky/Borel.html`) that the quotient $\mathfrak{t}/W$ is again an affine space. The nilpotent elements $\mathcal{N}$ in $\mathfrak{g}$ map to zero in $\mathfrak{t}/W$. In summary we have a diagram

$$\begin{tikzcd}
\widetilde{\mathcal{N}} \arrow[r, hook] \arrow[d, "\pi"'] & \widetilde{\mathfrak{g}} \arrow[r, "\alpha"] \arrow[d, "\widehat{\pi}"] & \mathfrak{t} \arrow[d, "p"] \\
\mathcal{N} \arrow[r, hook] & \mathfrak{g} \arrow[r, "ch"'] & \mathfrak{t}/W
\end{tikzcd}$$

16.6. **Theorem** (Grothendieck, Lusztig, Slowdowy)**.** *The map* $\widehat{\pi} : \widetilde{\mathfrak{g}} \to \mathfrak{g}$ *is small. The map* $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ *is semi-small. For each* $A \in \mathfrak{t}$ *the map* $\widehat{\pi} : \alpha^{-1}(A) \to ch^{-1}(p(A))$ *is a resolution of singularities.*

16.7. **Adjoint quotient.** There is another way to view the map ch. Each $x \in \mathfrak{g}$ has a unique Jordan decomposition $x = x_s + x_n$ into commuting semisimple and nilpotent elements. Then x_s is conjugate to an element of $\mathfrak{t}$, and the resulting element is well defined up to the action of W. The quotient $\mathfrak{t}/W$ turns out to be isomorphic to the geometric invariant theory quotient $\mathfrak{g}//G$ and the map $ch : \mathfrak{g} \to \mathfrak{t}/W$ is called the adjoint quotient map.

The vector space $\mathfrak{t}$ consists of diagonal matrices $A = \mathrm{diag}(a_1, \cdots, a_n)$ with trace zero. The *reflecting hyperplanes* are the subspaces $H_{ij} = \{A | a_i = a_j\}$ where two entries coincide and they are permuted by the action of W. Their image in $\mathfrak{t}/W$ is the discriminant variety $Disc$ consisting of all (characteristic) polynomials with multiple roots. The complement of the set $\cup_{i \neq j} H_{ij}$ is sometimes called the *configuration space of n ordered points in* $\mathbb{C}$; its fundamental group is the *colored braid group.* The complement of the discriminant variety in $\mathfrak{t}/W$ is the configuration space of n unordered points, and its fundamental group is the braid group.

Suppose $x \in \mathfrak{g}^{rs} \subset \mathfrak{g}$ is regular and semisimple, meaning that its eigenspaces $E_1, E_2, \cdots, E_n$ are distinct and form a basis of $\mathbb{C}^n$. Then the flag $E_1 \subset E_1 \oplus E_2 \subset E_1 \oplus E_2 \oplus E_3 \cdots$ is fixed by x and every fixed flag has this form, for some ordering of the eigenspaces. Therefore there are $n!$ fixed flags and the symmetric group permutes them according to the regular representation.

16.8. **Springer's representation.** Since $\widehat{\pi}$ is a small map, we have a canonical isomorphism

$$R\widehat{\pi}_*(\underline{\mathbb{Q}}_{\widetilde{\mathfrak{g}}}[n]) \cong \underline{IC}^\bullet(\mathfrak{g};\mathcal{L}),$$

the intersection cohomology sheaf on $\mathfrak{g}$, constructible with respect to a stratification of $\widehat{\pi}$, with coefficients in the local system $\mathcal{L}$ over the regular semisimple elements whose fiber at $x \in \mathfrak{g}^{rs}$ is the direct sum $\oplus_F \mathbb{Q}_F$ of copies of $\mathbb{Q}$, one for each fixed flag. The symmetric group W acts on $\mathcal{L}$ which induces an action on $\underline{IC}^\bullet(\mathcal{L})$ and therefore also on the stalk cohomology at each point $y \in \mathfrak{g}$, that is, on

$$\underline{H}^r(R\widehat{\pi}_*(\underline{\mathbb{Q}}_{\widetilde{\mathfrak{g}}})_y = H^r(\widehat{\pi}^{-1}(y)) \cong \underline{IH}^r(\mathfrak{g};\mathcal{L})_y.$$

For $y \in \mathcal{N}$ this action of W on $H^*(\pi^{-1}(y))$ turns out to coincide with Springer's representation. The decomposition theorem for the semismall map π provides an enormous amount of information about these representations.

16.9. **Decomposition theorem for semismall maps.** Recall that a proper algebraic morphism $f : X \to Y$ is semismall if it can be stratified so that

$$2\dim_{\mathbb{C}} f^{-1}(y) \le \mathrm{cod(S)}$$

for each stratum $S \subset Y$, where $y \in S$. This implies that $\dim(X) = \dim(Y)$ and, if X is nonsingular, that $Rf_*(\mathbb{Q}_X)$ is perverse on Y. A stratum $S \subset Y$ is said to be *relevant* if $2d = \mathrm{cod(S)}$ where $d = \dim f^{-1}(y)$. In this case, the top degree cohomology $H^{2d}(f^{-1}(y)$ forms a local system L_S on the stratum S. The following result is due to W. Borho and R. MacPherson.

16.10. **Proposition.** *Suppose $f : X \to Y$ is semismall and X is nonsingular of complex dimension d. Then the decomposition theorem has the following special form:*

$$Rf_*(\mathbb{Q}_x)[d] \cong \bigoplus_S \underline{IC}^\bullet_{\overline{S}}(L_S)$$

where the sum is over those strata S that are relevant (with no shifts, if we use Deligne's numbering). In particular, the endomorphism algebra of this sheaf $\mathrm{End}(Rf_*(\mathbb{Q}_X)[d]) \cong \bigoplus_S \mathrm{End}_S(L_S)$ *is isomorphic to the direct sum of the endomorphism algebras of the individual local systems L_S.*

Proof. The top stratum, Y^o is always relevant. If no other strata is relevant then the map is *small*. In Lecture [??] we described a "standard" technique of proof (let us call this "Proof no. 1") which implies that $Rf_*(\mathbb{Q}_X[d]) \cong \underline{IC}^\bullet_Y(L_{Y^o})$. So there is only one term in the decompsition theorem. Now suppose the next relevant stratum has codimension c so that the fiber over points in this stratum has (complex) dimension $d = c/2$ (and in particular, the complex codimension c is even). One term in the decomposition theorem is $\underline{IC}^\bullet_Y(L_{Y^o})$. Consider the support diagram (e.g. if $c = 4$):

From this diagram we can see that a new summand must be added to the decomposition, and it is the local system $H^4(f^{-1}(y)) = H^{2d}(f^{-1}(y) = L_S$ arising from the top cohomology of the

i	j	$\mathrm{cod}_{\mathbb{C}}0$	$\mathrm{cod}_{\mathbb{C}}1$	$\mathrm{cod}_{\mathbb{C}}2$	$\mathrm{cod}_{\mathbb{C}}3$	$\mathrm{cod}_{\mathbb{C}}4$	$H^*(f^{-1}(y))$
8	4	c	c	c	c	c	
7	3			c	c	c	
6	2				c	c	
5	1					c	
4	0					0	$H^4(f^{-1}(y))$
3	-1					x	$H^3(f^{-1}(y))$
2	-2				x	x	$H^2(f^{-1}(y))$
1	-3			x	x	x	$H^1(f^{-1}(y))$
0	-4	x	x	x	x	x	$H^0(f^{-1}(y))$
		$\underline{IC}^{\bullet}$ support					

fiber, that is, from the irreducible components of the fiber. This local system on S gives rise to the summand $\underline{IC}^{\bullet}_{\bar{S}}(L_S)$ over the whole of the closure $\bar{S} \subset Y$. Let $U_S = \cup_{T \geq S}$ be the open set consisting of S together with the strata larger than S. We have constructed an isomorphism of $Rf_*(\mathbb{Q}_X)$ with

$$Rf_*(\mathbb{Q}_X[d]) \cong \underline{IC}^{\bullet}_Y(L_{Y^o}) \oplus \underline{IC}^{\bullet}_{\bar{S}}(L_S)$$

over the open set U_S. The exact same method as in Proof No.1 shows that this isomorphism extends uniquely to an isomorphism over the larger open set that contains additional (smaller) strata until we come to the next relevant stratum. Continuing in this way by induction gives the desired decomposition.

Finally, if L_R, L_S are local systems on distinct strata R, S of Y then

$$\mathrm{Hom}_{D^b_c(Y)}(\underline{IC}^{\bullet}_{\bar{R}}(L_R), \underline{IC}^{\bullet}_{\bar{S}}(L_S)) = 0$$

which implies that the endomorphism algebra of this direct sum decomposes into a direct sum of endomorphism algebras. □

16.11. **Some conclusions.** Let $d = \dim(G/B) = n(n-1)/2$. For the SL_n adjoint quotient $R\pi_*(\mathbb{Q}_{\widetilde{N}}[d]) \cong \oplus_S \underline{IC}^{\bullet}(\overline{S}; L_S)$

(1) Every stratum S is relevant: for $x \in S$, $2\dim(\mathcal{B}_x) = \mathrm{cod(S)}$.
(2) The odd cohomology of each Springer fiber $H^{2r+1}(\mathcal{B}_x) = 0$ vanishes.
(3) Let S be a stratum corresponding to some partition (or Young frame) $\lambda = \lambda(S)$. Then the Springer action on the top cohomology $H^{\mathrm{cod(S)}}(\mathcal{B}_x)$ is the irreducible representation ρ_λ corresponding to λ (modulo possible notational normalization involving transpose of the partition and tensoring with the sign representation). [Note: this representation is not (cannot) be realized via permutations of the components of $\mathcal{B}_x$, but the components give, nevertheless, a basis for the representation.]
(4) Every irreducible representation of W occurs in this decomposition, and it occurs with multiplicity one.

(5) The local systems occurring in the decomposition theorem (SL_n case only!) are all trivial.
(6) Putting these facts together, let S_λ denote the stratum in $\mathcal{N}$ corresponding to the partition λ. Then, using "classical" degree indices, the decomposition theorem in this case becomes:

$$R\pi_*(\mathbb{Q}_{\widetilde{d}}) \cong \oplus_\lambda \underline{IC}^\bullet(\overline{S}_\lambda)[-2d_\lambda] \otimes V_\lambda$$

where V_λ is the (space of the) irreducible representation ρ_λ of W and d_λ is the complex dimension of the stratum S_λ.
(7) Applying stalk cohomology at a point $x \in S_\mu$ to this formula gives:

$$H^i(\mathcal{B}_x) \cong \oplus_{\lambda \geq \mu} \underline{IH}_y^{i-2d_\lambda}(\overline{S}_\lambda) \otimes V_\lambda.$$

Consequently, if an irreducible representation ρ_λ of W occurs in $H^*(\mathcal{B}_x)$ then the stratum S_μ containing x is in the closure of the stratum S_λ corresponding to ρ.
(8) More generally, suppose $R < S$ are strata corresponding to partitions μ, λ respectively and let $x \in R$. Then $\dim \underline{IH}^{i-2d_\lambda}(\overline{S})_x$ is the multiplicity of the representation ρ_λ in the cohomology $H^i(\mathcal{B}_x)$ of the Springer fiber $\mathcal{B}_x$. In fact the Poincaré polynomial of these multiplicities

$$P_{\lambda,\mu}(t) = \sum_i \mathrm{mult}(\rho_\lambda, H^{2i}(\mathcal{B}_x))t^i = \sum_i \mathrm{rank}(\underline{IH}^{2i}(\overline{S})_x)t^i$$

turns out to be the Kostka-Foulkes polynomial.
(9) For $x = 0 \in \mathcal{N}$ the Springer fiber $\mathcal{B}_x = G/B$ is the full flag variety and the representation of W is the regular representation. Moreover, the full endomorphism algebra

$$\mathrm{End}(R\pi_*(\mathbb{Q}_{\widetilde{\mathcal{N}}}) \cong \mathbb{C}[W]$$

is isomorphic to the full group-algebra of the Weyl group, with its regular representation.

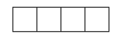
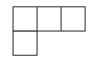
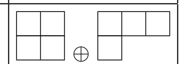
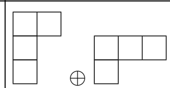
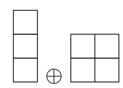
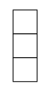
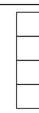
degee	rank	Young
$H^{12}(G/B)$	1	
$H^{10}(G/B)$	3	
$H^{8}(G/B)$	5	
$H^{6}(G/B)$	6	
$H^{4}(G/B)$	5	
$H^{2}(G/B)$	3	
$H^{0}(G/B)$	1	

FIGURE 11. Springer representations for SL_4. Each rep occurs as often as its dimension

Even for the W action on the full flag manifold $\mathcal{B} \cong G/B$ these results are startling. In this case it had been shown by Borel and Leray that the action of W was the regular representation, but which irreducible factors appeared in which degrees of cohomology had appeared to be a total mystery. Many computations were done by hand and the result appeared to be random. The above conclusions explain that the multiplicity of each representation ρ_λ in $H^i(G/B)$ is given by the rank of the local intersection cohomology, in degree i, at the origin $o \in \mathcal{N}$ of the stratum S that corresponds to the partition λ.

17. Lecture 17: the Iwahori Hecke Algebra

Let G be a finite group. The convolution of two functions $f, f' : G \to \mathbb{C}$ is the function

$$(f * f')(x) = \frac{1}{|G|}\sum_{h \in G} f(xh^{-1})f'(h) = \frac{1}{|G|}\sum_{a \in G} f(a)f'(a^{-1}x).$$

this product is associative. If $H \subset G$ is a subgroup the Hecke algebra is

$$\mathcal{H}(G, H) = \{\phi : G \to \mathbb{C} |\ \phi(kgk') = \phi(g) \text{ for all } k, k' \in H\}$$

with algebra structure given by convolution. It is the convolution algebra of functions on the double coset space $H\backslash G/H$. If $\rho : H \to \mathrm{GL}(V)$ is a representation, the induced representation is

$$\mathrm{Ind}_H^G(\rho) = \{\phi : G \to V |\ \phi(hx) = \rho(h)\phi(x) \ \text{ for all } h \in H, x \in G\}$$

with action $(g.\phi)(x) = \phi(xg^{-1})$. Then

$$\mathrm{Hom}_G(\mathrm{Ind}_H^G V, W) \cong \mathrm{Hom}_H(V, \mathrm{Res}_H^G(W)) \ \text{ and } \mathcal{H}(G, H) \cong \mathrm{Hom}_G(\mathrm{Ind}_H^G(\mathbb{1}), \mathrm{Ind}_H^G(\mathbb{1})).$$

Now let $G = \mathrm{SL}_n(\mathbb{F}_q)$ and $H = B$ the collection of upper triangular matrices (or determinant one). Let $W = S_n$ be the symmetric group which may be thought of as acting on the standard basis vectors $\{e_1, \cdots, e_n\}$. It is generated by the "simple reflections" $S = \{s_1, \cdots, s_{n-1}\}$ where s_i exchanges e_i and e_{i+1}. The *length* $\ell(w)$ of an element $w \in W$ is the minimum number of elements required to express w as a product of simple reflections, and it is well defined. The Bruhat decomposition says that $G = \coprod_{w \in W} BwB$. Each B orbit $BwB/B \subset G/B$ is isomorphic to an affine space of dimension $\ell(w)$.

17.1. **Definition.** The Hecke algebra $\mathcal{H}$ is the algebra of B-bi-invariant functions on G. It has a basis consisting of functions

$$\phi_w = \mathbb{1}_{BwB}.$$

The unit element in $\mathcal{H}$ is the function $\phi_1 = \mathbb{1}_B$. In this algebra we will use the following normalization for convolution of bi-invariant functions $f, f' : G \to \mathbb{C}$,

$$(f * f')(x) = \frac{1}{|B|}\sum_{h \in G} f(xh^{-1})f'(h).$$

17.2. **Lemma.** *If $s \in S$ is a simple reflection and if $w \in W$ then the following holds:*

$$\begin{cases} \phi_w * \phi_{w'} = \phi_{ww'} & \text{if } \ell(w) + \ell(w') = \ell(ww') \\ \phi_s * \phi_s = (q-1)\phi_s + q\phi_1 & \\ \phi_s * \phi_w = (q-1)\phi_w + q\phi_{sw} & \text{if } \ell(sw) < \ell(w) \end{cases}$$

Proof. (The third equation follows from the second by induction.) The key nontrivial point (see for example, Bump's notes on Hecke algebras) is that

$$\begin{aligned} \ell(w) + \ell(w') = \ell(ww') &\implies (BwB)(Bw'B) = Bww'B \\ \ell(ws) = \ell(w) - 1 &\implies (BwB)(BsB) \subset (BwB) \cup (BwsB). \end{aligned}$$

Following Bump's notes, for $f \in \mathcal{H}$ let $\epsilon(f) = \frac{1}{|B|}\sum_{g\in G} f(g)$ so that $\epsilon(f * f') = \epsilon(f)\epsilon(f')$ and $\epsilon(\phi_w) = q^{\ell(w)}$. If $\ell(w)+\ell(w') = \ell(ww')$ then $\phi_w * \phi_{w'}$ is supported on $Bww'B$ and is B bi-invariant. Apply epsilon to conclude that $\phi_w * \phi_{w'} = \phi_{ww'}$. Similarly, $\phi_s * \phi_s$ is supported on $(BsB) \cup B$ so it equals $\alpha\phi_s + \beta\phi_1$ for some $\alpha, \beta \in \mathbb{C}$. Apply ϵ to conclude that $q^2 = \alpha q + \beta$. Evaluate at $x = I \in G$ to get $\phi_s * \phi_s(I) = |BsB|/|B| = q = \alpha.0 + \beta.1$ So $\alpha = q - 1$. □

17.3. **Remark.** The same holds for any semisimple algebraic group G defined over $\mathbb{F}_q$, where B is a Borel subgroup and W is the Weyl group. More generally, if W is any Coxeter group the Hecke algebra of W is defined to be the $\mathbb{Z}[q^{1/2}, q^{-1/2}]$ algebra generated by symbols ϕ_w and satisfying the relations in the box.

17.4. Kazhdan and Lusztig discovered a mysterious new basis for the Hecke algebra that appeared to be closely related to infinite dimensional representations of the Lie algebra $\mathfrak{g}$ of SL_n. Each element $\phi_w \in \mathcal{H}$ is invertible and the algebra $\mathcal{H}$ admits an involution defined by

$$\iota(q^{1/2}) = q^{-1/2} \text{ and } \iota(\phi_w) = (\phi_{w^{-1}})^{-1}.$$

17.5. **Theorem** (Kazhdan, Lusztig)**.** *For each $w \in W$ there is a unique element $c_w \in \mathcal{H}$ such that $\iota(c_w) = c_w$ and*

$$c_w = q^{-\ell(w)/2} \sum_{y \le w} P_{yw}(q)\phi_y$$

where $P_{ww} = 1$ and P_{yw} (for $y < w$) is a polynomial of degree $\le \frac{1}{2}(\ell(w) - \ell(y) - 1)$.

Existence and uniqueness of c_w is easily proven by induction. Kazhdan and Lusztig conjectured that the coefficients of P_{yw} were nonnegative integers. They further conjectured that, in the Grothendieck group of Verma modules,

$$[L_w] = \sum_{y \le w} (-1)^{\ell(w)-\ell(y)} P_{yw}(1)[M_y]$$

where M_w is the Verma module corresponding to highest weight $-\rho - w(\rho)$ and L_w is its unique irreducible quotient. This second conjecture became known as the *Kazhdan-Lusztig conjectures*; they were eventually proven by J. L. Brylinski and M. Kashiwara and independently by A. Beilinson and D. Bernstein. This circle of ideas became extremely influential in representation theory. But what exactly is the meaning of c_w and P_{yw}?

Algebra of correspondences

17.6. Let us return to the complex picture with $G = \mathrm{SL}_n(\mathbb{C})$, B the Borel subgroup of upper triangular matrices and W the symmetric group. let $X = G/B$ be the flag manifold, or equivalently, the variety of Borel subgroups of G. The group G decomposes as a disjoint union $G = \coprod_{w]\ inW} BwB$. It follows that the group B acts on X with finitely many orbits, and these are indexed by the elements of W. For each $w \in W$ the *Schubert cell* or *Bruhat cell* $X_w = BwB/B \subset G/B$ indexed by w contains X_y in its closure iff $y < w$ in the Bruhat order. Similarly the group G acts on $X \times X$ with finitely many orbits, each of which contains a unique point (B, wB) (thinking of the standard Borel subgroup B as being the basepoint in the flag manifold) for some $w \in W$. It consists of pairs of flags $(F_1 \subset F_2 \subset \cdots F_n = \mathbb{C}^n)$ and $(F'_1 \subset F'_2 \subset \cdots \subset F'_n = \mathbb{C}^n)$ that are in relative position $w \in W$, meaning that there exists an ordered basis $(e_1, e_2, \cdots, e_n)$ of $\mathbb{C}^n$ so that, for each $1 \le i \le n$

$$\langle e_1, e_2, \cdots, e_i \rangle = F_i \quad \text{and} \quad \langle e_{w(1)}, e_{w(2)}, \cdots, e_{w(i)} \rangle = F'_i.$$

If $q_2 : X \times X \to X$ denotes projection to the second factor then this orbit, let us denote it by $\mathcal{O}_w$ fibers over X with fiber equal to X_w; in particular it is simply connected. In other words, each G orbit on $X \times X$ intersects the fiber X in a single B orbit.

17.7. **Algebra of correspondences.** The following construction was discovered independently by R. MacPherson, G. Lusztig, Brylinski and Kashiwara, Beilinson and Bernsstein and is described in Springer's article [Springer]. It is convenient here to change notation in the Hecke algebra, setting

$$t = \sqrt{q}.$$

Consider the derived category $D^b_{c,even}(X \times X)$ of sheaves $A^\bullet$ on $X \times X$, cohomologically constructible with respect to this stratification such that $\underline{H}^i(A^\bullet) = 0$ for i odd. For for such a sheaf $A^\bullet$ let $\underline{H}^{2i}(A^\bullet)_w$ denote the stalk cohomology of $A^\bullet$ at a point in $\mathcal{O}_w$ and define

$$\begin{gathered} h : D^b_{c,even} \to \mathcal{H} \\ h(A) = \sum_{w \in W} \sum_{i \ge 0} \dim(\underline{H}^i(A^\bullet)_w) t^i \phi_w \end{gathered}$$

Note that all elements of $\mathcal{H}$ obtained in this way have nonnegative coefficients so the image of h ends up in a sort of "positive cone" in the Hecke algebra. Consider the following diagram of correspondences.

$$\begin{tikzcd} & X \times X \times X \arrow[dl, "q_{12}"'] \arrow[d, "q_{13}"] \arrow[dr, "q_{23}"] & \\ X \times X & X \times X & X \times X \end{tikzcd}$$

If $A^\bullet, B^\bullet \in D^b_c(X \times X)$ define their convolution product

$$A^\bullet \circ B^\bullet = Rq_{13*}\left(q_{12}^*(A^\bullet) \otimes q_{23}^*(B^\bullet)\right).$$

17.8. **Theorem.** *If $A^\bullet, B^\bullet \in D^b_{c,even}(X)$ then so is $A^\bullet \circ B^\bullet$. The mapping h satisfies*

$$h(A^\bullet + B^\bullet) = h(A^\bullet) + h(B^\bullet) \quad \text{and} \quad h(A^\bullet \circ B^\bullet) = h(A^\bullet)h(B^\bullet) \in \mathcal{H}.$$

Moreover $h(\mathbf{D}(A^\bullet)) = \iota(h(A^\bullet)$ (so ι corresponds to Verdier duality). Let $j_w : \mathcal{O}_w \to X$ denote the inclusion. Then

$$h\left(j_{w!}(\mathbb{C}_{\mathcal{O}_w})[\ell(w)]\right) = \phi_w \quad \text{and} \quad h\left(j_{w*}(\underline{IC}^\bullet_{\overline{\mathcal{O}}_w}(\mathbb{C}))\right) = c_w.$$

$$t^{\ell(w)}h(\underline{IC}^\bullet_{\overline{\mathcal{O}}_w}) = h(\underline{IC}^\bullet_{\overline{\mathcal{O}}_w}[-\ell(w)])$$

In other words, $P_{y,w}$ is the local intersection cohomology Poincaré polynomial of X_w at a point in X_y. It vanishes in odd degrees and its coefficients are non-negative. Using "classical" indexing for sheaf cohomology,

$$P_{yw}(t^2) = \sum_{i \geq 0} \dim \underline{IH}^{2i}_y(\overline{X}_w)t^{2i}.$$

This stalk cohomology vanishes in odd degrees and the highest power of t that can occur here is $t^{\ell(w)-\ell(y)-1}$. Besides making the essential connection with geometry this result is a "categorification" of the Hecke algebra: it replaces numbers and coefficients with (cohomology) groups. It implies that the coefficients of P_{yw} are non-negative integers.

17.9. The proof is tedious but does not require sophisticated methods; it is completely worked out in some great online notes by Konstanze Rietsch (following an outline of T. A. Springer [Bourbaki], as communicated to him by R. MacPherson). [**work out a few simple examples**] For brevity let $\underline{IC}^\bullet_e = \underline{IC}^\bullet_{\overline{\mathcal{O}}_w}$. The support condition for intersection cohomology implies that $h(\underline{IC}_w)$ is a linear combination of $h(j_{w!}\underline{\mathbb{C}}_w)$ with coefficients that are polynomials which satisfy the degree restriction. The proof that $h(\underline{IC}_w)$ is fixed under the involution ι takes some work.

First it is shown, if s is a simple reflection with corresponding orbit closure $\overline{\mathcal{O}}_s \subset X \times X$ that

$$h(\mathbb{C}_{\overline{\mathcal{O}}_s}) \circ A^\bullet = (\phi_s + 1).h(A^\bullet) \in \mathcal{H}.$$

The orbit closure $\overline{\mathcal{O}}_D$ fibers over X with fiber equal to $\mathbb{P}^1$. So this equation is believable but it takes a bit of work, considering the cases $sw > w$ and $sw < w$ separately.

One way to prove this is to consider the Bott-Samelson resolution of the Schubert variety X_w. It is obtained as a sequence of blowups by simple reflections. One checks at each stage of the induction that the result is preserved by ι. This proves that $h(\underline{IC}_w) = c_w \in \mathcal{H}$.

Finally, again using the Bott Samelsom resolution and induction, one proves that $h(\underline{IC}_w \circ \underline{IC}_{w'}) = c_w c_{w'}$. This proves that h is multiplicative.

Digression: Hecke algebra and modular forms

Let $G = \mathrm{SL}_n(\mathbb{R})$, let $K = O(n)$ and let $\Gamma_0 = \mathrm{SL}_n(\mathbb{Z})$. Then $D = G/K$ be the (contractible) symmetric space of positive definite symmetric matrices of determinant one. Let $X = \Gamma_0 \backslash D$. This is the moduli space of Riemannian tori. (For each lattice $L \subset \mathbb{R}^n$ of determinant one we get a torus $\mathbb{R}^n/L$ and an invariant Riemannian metric on it.) Each $g \in G_{\mathbb{Q}} = \mathrm{SL}_n(\mathbb{Q})$ gives a correspondence on this space as follows. Let $\Gamma' = \Gamma_0 \cap (g^{-1}\Gamma_0 g)$ and let $X' = \Gamma' \backslash D$. Then the correspondence $X' \to X \times X$ is given by $\Gamma' x \mapsto (\Gamma_0 x, \Gamma_0 g x)$. It is well defined and each of the projections $X' \to X$ is a finite covering. Moreover, the isomorphism class of this correspondence depends only on the double coset $\Gamma_0 g \Gamma_0$. (Replacing g by γg where $\gamma \in \Gamma_0$ does not change the correspondence. Replacing g by $g\gamma$ changes the correspondence but it gives an isomorphic correspondence.) Therefore points in the double coset space

$$\Gamma_0 \backslash \mathrm{SL}_n(\mathbb{Q}) / \Gamma_0$$

may be interpetred as defining correspondences on X, which therefore acts on the homology, cohomology, functions etc. of X. So the same is true of linear combinations of points. The Hecke algebra of compactly supported functions (meaning, functions with finite support) on $\Gamma_0 \backslash \mathrm{SL}_n(\mathbb{Q})/\Gamma_0$ also acts on $H^*(X)$. Such functions are called Hecke operators.

This construction makes more sense in the adèlic setting where natural Haar measures can be used in order to define the algebra structure and the action without resorting to correspondences. In this setting there is an equality

$$\mathrm{SL}_n(\mathbb{Z}) \backslash \mathrm{SL}_n(\mathbb{R})/K \cong \mathrm{SL}_n(\mathbb{Q}) \backslash \mathrm{SL}_m(\mathbb{A}_{\mathbb{Q}})/K.\mathrm{SL}_n(\widehat{\mathbb{Z}})$$

and the Hecke algebra is the convolution algebra of locally constant functions with compact support

$$f \in C_c^{\infty}(\mathrm{SL}_n(\mathbb{Q}) \backslash \mathrm{SL}_n(\mathbb{A}_f)/\mathrm{SL}_n(\mathbb{Q}))$$

where $\mathbb{A}_{\mathbb{Q}}$ denotes the adèles of $\mathbb{Q}$ and $\mathbb{A}_f$ the finite adèles.

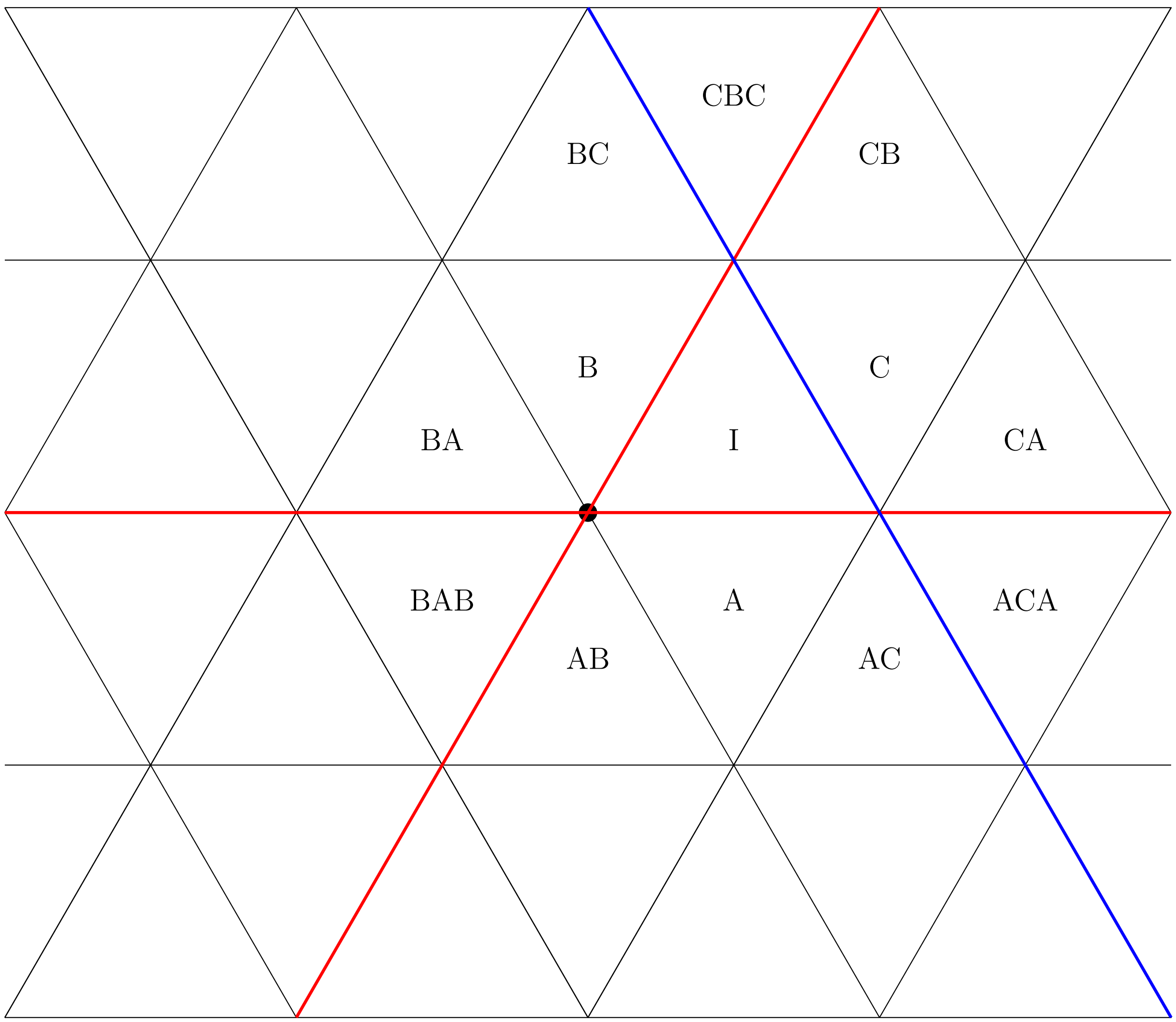
18. Lecture 19: The affine theory

18.1. **Affine Weyl group.** The symmetric group S_n has Dynkin diagram : [diagram] It is generated by simple reflections $s_1, \cdots, s_{n-1}$ with the relations

(1) $s_i^2 = 1$
(2) $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$ for $(1 \leq i \leq n-2)$
(3) $s_i s_j = s_j s_i$ if $|i - j| > 1$.

It can be interpreted as acting on $\mathbb{R}^{n-1} = \{\sum x_i = 0\} \subset \mathbb{R}^n$ with s_i acting as reflection across the hyperplane $x_i = x_{i+1}$. This decomposes $\mathbb{R}^{n-1}$ into Weyl chambers, one for each element of S_n.

The affine symmetric group $\widetilde{S}_n$ has Dynkin diagram: It is generated by simple reflections $s_0, s_1, \cdots, s_n$ with the same relations as S_n and the additional relations (corresponding to edges $s_0 s_1$ and $s_n s_0$). It can be interpreted as acting on $\mathbb{R}^{n-1}$ by adding a reflecting hyperplane to the previous picture. Then it acts simply transitively on the alcoves. If we take the fundamental alcove as the basepoint (identity), then every alcove becomes labeled by a unique element of $\widetilde{S}_n$.



This diagram ilustrates the affine Weyl group for SL_3. The red lines are the reflecting hyperplanes of the finite Weyl group (generated by A and B). The blue line is the affine reflection, C.

18.2. The affine symmetric group can also be described as $S_n \ltimes A$ where A is the root lattice of translations,

$$\left\{(a_1, \cdots, a_n) \in \mathbb{Z}^n \middle| \sum a_i = 0\right\}$$

on which S_n acts by permutations. The group A may also be interpretred as the cocharacter group of the maximal torus $\mathcal{T}$ consisting of diagonal matrices of determinant one. The affine Weyl group can be described as

$$\{\omega : \mathbb{Z} \to \mathbb{Z} | \ \omega(i+n) = \omega(i) + n\}.$$

In this realization each element is determined by its value on $\{1, \cdots, n\}$ and so it may be written as $[\omega(1), \cdots, \omega(n)]$. Then elements in the lattice of translations are the elements $[a_1, \cdots, a_n]$ with $\sum a_i = 0$ and they act by addition, that is, $\omega(i) = a_i + i$ for $1 \le i \le n$. Then every element $\omega \in \widetilde{S}_n$ can be expressed as a permutation followed by a translation.

18.3. **Affine Hecke algebra.** Let $G_{\mathbb{Q}_p} = \mathrm{SL}_n(\mathbb{Q}_p)$, as a locally compact topological group and set $K = G(\mathbb{Z}_p) = \mathrm{SL}_m(\mathbb{Z}_p)$. Let $B_{\mathbb{F}_p}$ be the Borel subgroup of $\mathrm{SL}_n(\mathbb{Z}/p\mathbb{Z})$. The Iwahori subgroup I_p is the preimage $\pi^{-1}(B_p)$ under the $\pmod p$ mapping $\phi : K \to SL_n(\mathbb{Z}/p\mathbb{Z})$, that is, it consists of $n \times n$ matrices with entries in $\mathbb{Z}_p$, whose diagonal entries are invertible in $\mathbb{Z}_p$, and whose lower diagonal entries are multiples of p. It is compact and open in $G_{\mathbb{Q}_p}$.

The Iwahori Hecke algebra is the convolution algebra of locally constant complex valued functions

$$f \in C_c^\infty(I_p \backslash G_{\mathbb{Q}_p} / I_p)$$

with compact support on $G_{\mathbb{Q}_p}$ that are bi-invariant under I_p.

Haar measure μ on $G_{\mathbb{Q}_p}$ is normalized so that $\mu(I_p) = 1$.

The Bruhat decomposition in this case reads $G_{\mathbb{Q}_p} = \coprod_{w \in W_a} I_p w I_p$ where W_a is the affine Weyl group. Then $\mathcal{H}_I$ is generated by characteristic functions ϕ_w for $w \in W_a$ and with the same relations as before: $\phi_s^2 = (q-1)\phi_s + q\phi_1$. The Kazhdan Lusztig canonical basis c_w is defined exactly as before. The Kazhdan Lusztig theorem works in this context as well and it gives a basis of $\mathcal{H}_I$ consisting of elements c_w for $w \in W_a$ the affine Weyl group.

The field $\mathbb{Q}_p$ is analogous to the field $\mathbb{F}_q((T))$ of formal Laurent series (meaning formal power series with finitely many negative powers of T and coefficients in $\mathbb{F}_q$). The ring $\mathbb{Z}_p$ corresponds to $\mathbb{F}_q[[T]]$ (the ring of formal power series). Reduction modulo T gives a homomorphism $\phi : \mathbb{F}_q[[T]] \to \mathbb{F}_q$ and the Iwahori subgroup $I_{p((T))} = \phi^{-1}(B_q)$ is defined similarly. Then an Iwahori Hecke algebra over $\mathbb{F}_q$ is defined to be the convolution algebra of locally constant complex valued functions

$$f \in C_c^\infty(I_{p[[T]]} \backslash G_{\mathbb{F}_q((T))} / I_{p[[T]]})$$

with compact support that are bi-invariant under the Iwahori subgroup.

All of this has a complex analog following the same procedure as in the finite case. Instead of the flag manifold over $\mathbb{C}$ one uses the "affine flag manifold" $\mathrm{SL}_n(\mathbb{C}((T)))/I$ where $\mathbb{C}((T))$ is the field of formal Laurent series (that is, power series with finitely many negative powers of T) and where $I = \phi^{-1}(B)$ is the Iwahori subgroup defined by $\pmod T$ reduction,

$$\phi : \mathrm{SL}_n(\mathbb{C}[[T]]) \to \mathrm{SL}_n(\mathbb{C}).$$

The quotient $\mathrm{SL}_n\,((T))/I$ is infinite dimensional but it is an increasing limit of finite dimensional complex algebraic varieties, and each I orbit of an element $w \in W_a$ is a (generalized) "Schubert cell" or Bruhat cell, of dimension $\ell(w)$. The Kazhdan Lusztig polynomials P_{yw} have non-negative coefficients and they may be interpreted as the local intersection cohomology Poincaré polynomials of one Schubert cell at a point in another Schubert cell.

However, the sheaf-convolution construction does not work in this setting because the orbits of I on $X \times X$ (where X denotes the affine flag manifold) have infinite dimension and infinite codimension. Instead, another approach is needed, which will be described later in the case of the affine Grassmannian.

18.4. **Overview.** There are strong analogies between these constructions over different fields. The following chart gives some idea of the parallels between the different cases.

Field	$\mathbb{C}$	$\mathbb{F}_q$	$\mathbb{Q}_p$	$\mathbb{C}((T))$	$\mathbb{F}_q((T))$
Group	$SL_n(\mathbb{C})$	$\mathrm{SL}_n(\mathbb{F}_q)$	$\mathrm{SL}_n(\mathbb{Q}_p)$	$\mathrm{SL}_n(\mathbb{C}((T)))$	$\mathrm{SL}_n(\mathbb{F}_q((T)))$
symbol	$G_{\mathbb{C}}$	G_q	$G_{\mathbb{Q}_p}$	$G_{((T))}$	$G_{q((T))}$
Borel/Iwahori	$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}$	$\begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}$	$\phi^{-1}(B)$	$\phi^{-1}(B)$	$\phi^{-1}(B_q)$
symbol	$B_{\mathbb{C}}$	$B_{\mathbb{F}_q}$	I_p	$I_{[[T]]}$	$I_{q[[T]]}$
Weyl group	S_n	S_n	W_a	W_a	W_a
Bruhat decomp	$\coprod_{w\in W} BwB$	$\coprod_{w\in W} BwB$	$\coprod_{w\in W_a} IwI$	$\coprod_{w\in W_a} IwI$	$\coprod_{w\in W_a} IwI$
Flag manifold	$G_{\mathbb{C}}/B_{\mathbb{C}}$	$G_{\mathbb{F}_q}/B_{\mathbb{F}_q}$	$G_{\mathbb{Q}_p}/I_p$	$G_{((T))}/I_{((T))}$	$G_{q((T))}/I_{q[[T]]}$
Parabolic/Parahoric	$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}$	$\begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}$	$\mathrm{SL}_n(\mathbb{Z}_p)$	$\mathrm{SL}_n(\mathbb{C}[[T]])$	$\mathrm{SL}_n(\mathbb{F}_q[[T]])$
symbol	$P_{\mathbb{C}}$	$P_{\mathbb{F}_q}$	K_p	$G_{[[T]]}$	$G_{q[[T]]}$
Bruhat decomp			$\coprod_{a\in A} K_p a K_p$	$\coprod_{a\in A} G_{[[T]]} a G_{[[T]]}$	$\coprod_{a\in A} G_{q[[T]]} a G_{q[[T]]}$
Grassmannian	$G_{\mathbb{C}}/P_{\mathbb{C}}$	$G_{\mathbb{F}_q}/P_{\mathbb{F}_q}$	$G_{\mathbb{Q}_p}/K_p$	$G_{((T))}/G_{[[T]]}$	$G_{q((T))}/G_{q[[T]]}$

18.5. **Definition.** The affine Grassmannian is the quotient

$$X = \mathrm{SL}_n(\mathbb{C}((T)))/\,\mathrm{SL}_n(\mathbb{C}[[T]]).$$

If we think of $\mathbb{C}((T))^n = \cup_{N=0}^{\infty} t^{-N}\mathbb{C}[[T]]$ then a *lattice* in $\mathbb{C}((T))^n$ is a $\mathbb{C}[[T]]$ submodule $M \subset \mathbb{C}((T))^n$ (meaning that it is preserved under multiplication by T) such that

$$T^{-N}\mathbb{C}[[T]]^n \supset M \supset T^N\mathbb{C}[[T]]^n$$

for sufficiently large N, and which satisfies the determinant one conditon, $\wedge^n M = \mathbb{C}[[T]]$. The affine Grassmannian is the set of all such lattices. In fact the group $\mathrm{SL}_n(\mathbb{C}((T)))$ acts transitively on the set of such lattices and the stabilizer of the standard lattice $\mathbb{C}[[T]]^n$ is the parahoric subgroup $\mathrm{SL}_n(\mathbb{C}[[T]])$.

$$\begin{pmatrix} T^{-1} & & & \\ 0 & T^2 & & \\ 0 & 0 & T^{-3} & \\ 0 & 0 & 0 & T^2 \end{pmatrix} K = \begin{array}{|c|c|c|c|c|}\hline T^{-4} & & & & \\ \hline T^{-3} & & & \bullet & \\ \hline T^{-2} & & & \bullet & \\ \hline T^{-1} & \bullet & & \bullet & \\ \hline T^{0} & \bullet & & \bullet & \\ \hline T^{1} & \bullet & & \bullet & \\ \hline T^{2} & \bullet & \bullet & \bullet & \bullet \\ \hline T^{3} & \bullet & \bullet & \bullet & \bullet \\ \hline \end{array}$$

18.6. **Its stratification.** The affine Grassmannian X is an infinite increasing union of projective varieties. It has two interesting stratifications. The first, is by orbits of the Iwahori subgroup $I_{[[T]]}$. These orbits are indexed by the group of translations A in affine Weyl group and the orbit $X_a = I_{[[T]]}aK$ is a Schubert cell: it is an affine space of dimension $\ell(a)$ with $X_y \subset \overline{X}_w$ iff $y < w$ in the Bruhat order on W_a. In other words (setting $I = I_{[[T]]}$ for brevity)

$$X = \coprod_{a\in A} IaK/K \text{ because } \mathrm{SL}_n(\mathbb{C}((T))) = IAK$$

which is the analog of the Iwasawa decomposition of G. The lattice of translations A may be identified with the group of all diagonal matrices $\mathrm{diag}(T^{a_1}, T^{a_2}, \cdots T^{a_n})$ such that $\sum_i a_i = 0$. Such an element may be interpreted as a cocharacter of the maximal torus $\mathfrak{T}$ of diagonal matrices, that is, $A \cong \chi_*(\mathfrak{T})$.

The second stratification is by orbits of the subgroup $K = \mathrm{SL}_n(\mathbb{C}[[T]])$. The (finite dimensional) Bruhat decomposition $\mathrm{SL}_n(\mathbb{C}) = \coprod_{w\in W} BwB$ implies that $K = \coprod_{w\in W} IwI$. So each K orbit is a union of $n!$ Schubert cells. Let $\mathfrak{T}$ be the torus of diagonal matrices (of determinant one) in SL_n. Then W acts on $\mathfrak{T}$ and on its group of cocharacters $A = \chi_*(\mathfrak{T})$, which we have identified with the lattice of translations in the affine Weyl group W_a. A fundamental domain for this action is the *positive cone* $A_+ \subset \mathcal{T}_+$. So the Bruhat decomposition becomes, in this case:

$$\mathrm{SL}_n(\mathbb{C}((T))) = \coprod_{a \in A_+} KaK \quad \text{and } X = \coprod_{a \in A_+} KaK/K.$$

The strata are no longer cells, but each stratum has the structure of a vector bundle over a nonsingular projective algebraic variety, so it is simply connected.

18.7. **Two more views of the affine Grassmannian.** Let $\mathbb{C}[T]$ be the ring of polynomials and let $\mathbb{C}(T)$ be the field of rational functions $p(T)/q(T)$. There is a natural map

$$G(\mathbb{C}(T))/G(\mathbb{C}[T]) \to G(\mathbb{C}((T)))/G(\mathbb{C}[[T]]).$$

It turns out to be an isomorphism. We can also consider $G(\mathbb{C}(T))$ to be the *loop group*

$$LG = \left\{ f : S^1 \to G | f \in \mathbb{C}(T) \right\}$$

consisting of all mappings which are rational functions. (Similarly one could consider analytic, smooth, or continuous functions; the results are homotopy equivalent). If LG^+ denotes mappings that can be extended (holomorphically, or as a polynomial) over the origin in $\mathbb{C}$ then the quotient

$$LG/LG^+ \cong G(\mathbb{C}(T))/G(\mathbb{C}[T])$$

is sometimes referred to (by physicists) as the *fundamental homogeneous space*.

19. Perverse sheaves on the affine Grassmannian

19.1. **Spherical Hecke algebra.** The Hecke algebra

$$\mathcal{H}(G(\mathbb{Q}_p)//G(\mathbb{Z}_p)) \text{ resp. } \mathcal{H}(G(\mathbb{F}_q((T))//G(\mathbb{F}_q[[T]])) \text{ etc.}$$

of locally constant compactly supported bi-invariant functions is called the *spherical Hecke algebra*, that is,

$$\boxed{\mathcal{H}(G(\mathbb{Q}_p)//G(\mathbb{Z}_p)) = C_c^\infty(\mathrm{SL}_n(\mathbb{Z}_p)\backslash \mathrm{SL}_n(\mathbb{Q}_p)/\mathrm{SL}_n(\mathbb{Z}_p)).}$$

Recall that the representation ring $R(G)$ of a (complex) reductive group G is isomorphic isomorphic to the Weyl invariants

$$R(G) \cong \mathbb{Z}[\chi^*(\mathcal{T})]^W$$

in the group of characters of a maximal torus $\mathcal{T}$. In fact, a fundamental domain for the action of W on $\chi^*(\mathcal{T})$ is given by the positive Weyl chamber, $\chi^*(\mathcal{T})_+$. To such a character $\lambda \in \chi^*(\mathcal{T})_+$ one associates the irreducible representation V_λ with highest weight λ. Its trace is a character of $\mathcal{T}$.

As a consequence, there are many equivalent ways to view this Hecke algebra.

(1) By theorems of Satake and MacDonald, there is a natural isomorphism

$$\mathcal{H}(G(\mathbb{Q}_p)//G(\mathbb{Z}_p)) \cong \mathbb{C}[X_*(\mathcal{T})]^W$$

of the Hecke algebra with the Weyl invariants in the group algebra of the cocharacter group of the maximal torus.

(2) This in turn may be identified with the Weyl invariants $\mathbb{C}[X^*(\widehat{\mathcal{T}})]^W$ in the characters of the dual torus.

(3) Which, by the adjoint quotient map, is isomorphic to the group of conjugation-invariant polynomial functions on PGL_n. (Recall that we previously identified this as a polynomial algebra, given by the coefficients of the characteristic polynomial.)

(4) This may be identified with $\mathbb{C} \otimes K(\mathrm{Rep}_{\mathrm{PGL_n}})$ (that is, the Grothendieck group of the category of finite dimensional (rational) representations of PGL_n) by associating, to any representation ρ its character (or trace), which is a Weyl invariant polynomial function.

(5) In fact, these identifications can be made over the integers. See [Gross] who considers the Hecke algebra $\mathcal{H}_G$ of $\mathbb{Z}$-valued functions on this double coset and describes isomorphisms

$$\mathcal{H}_G \longrightarrow \Big(\mathcal{H}_{\mathcal{T}} \otimes \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]\Big)^W \longleftarrow \mathrm{Rep}(\widehat{G}) \otimes \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}].$$

Here $\widehat{G}$ is the *Langlands dual group* of the group G. (The dual of SL_n is PGL_n.)

Digression: Dual group

(from Wikipedia, who took it from [Springer])

A root datum consists of a quadruple $(X^*, \Phi, X_*, \Phi^\vee)$ where X^*, X_* are free abelian groups of finite rank together with a perfect pairing $\langle,\rangle : X^* \times X_* \to \mathbb{Z}$, where $\Phi \subset X^*$ and $\Phi^\vee \subset X_*$

are finite subsets, and where there is a bijection $\Phi \to \Phi^\vee$, denoted $\alpha \mapsto \alpha^\vee$, and satisfying the following conditions:

(1) $\langle \alpha, \alpha^\vee \rangle = 2$ for all $\alpha \in \Phi$
(2) The map $x \mapsto x - \langle x, \alpha^\vee \rangle \alpha$ takes Φ to Φ and
(3) the induced action on X_* takes $\Phi^\vee$ to $\Phi^\vee$.

If G is a reductive algebraic group over an algebraically closed field then it defines a root datum where X^* is the lattice of characters of a (split) maximal torus T, where X_* is the lattice of cocharacters of T, where Φ is the set of roots and $\Phi^\vee$ is the set of coroots. A connected reductive algebraic group over an algebraically closed field K is determined up to isomorphism by its root datum and every root datum corresponds to such a group. Let G be a connected reductive algebraic group with root datum $(X^*, \Phi, X_*, \Phi^\vee)$. Then the connected reductive algebraic group with root datum $(X_*, \Phi^\vee, X^*, \Phi)$ is called the Langlands dual group and it is denoted ${}^LG$.

Langlands duality switches adjoint groups with simply connected groups. Takes type A_n to A_n but it switches types $Sp(2n)$ with $SO(2n+1)$. It preserves the type $SO(2n)$.

19.2. **Lusztig's character formula.** As above let $G = SL_n(\mathbb{C}((T)))$, let $K = \mathrm{SL}_n(\mathbb{C}[[T]])$, let $I \subset K$ be the Iwahori subgroup. The affine flag manifold $Y = G/I$ fibers over the affine Grassmannian $X = G/K$ with fiber isomorphic to $K/I \cong \mathrm{SL}_n(\mathbb{C})/B(\mathbb{C})$ the finite dimensional flag manifold, which is smooth. Consequently the singularities of I-orbit closures in X are the same as the singularities of I orbit closures in Y. The K orbits on X are indexed by cocharacters in the positive cone. If $\lambda = \mathrm{diag}(a_1, a_2, \cdots, a_n) \in \mathbb{Z}^n$ is in the positive cone (and $\sum a_i = 0$) let $x_\lambda = \mathrm{diag}(T^{a_1}, T^{a_2}, \cdots, T^{a_n}) \in \mathrm{SL}_n(\mathbb{C}((T)))$. The K orbit X_λ corresponding to a is $X_\lambda = Kx_\lambda K/K \subset G/K$. If $\mu \le \lambda$ then the point x_μ lies in the closure of the stratum X_λ and the local intersection cohomology Poincaré polynomial

$$\sum_{i \ge 0} \dim(\underline{IH}^{2i}_{x_\mu}(\overline{X}_\lambda))t^i = P_{\mu,\lambda}(t)$$

is given by the Kazhdan Lusztig polynomial $P_{\mu,\lambda}$ for the affine Weyl group. Lusztig [Lu] proves

19.3. **Theorem.** *Let $\mu \le \lambda \in \chi_*(\mathcal{T})_+ \cong \chi^*(\mathcal{T}^*)_+$. Let V_λ be the representation (of ${}^LG(\mathbb{C})$) of highest weight λ. It decomposes into weight spaces $(V_\lambda)(\mu)$ under the action of the maximal torus. Then*

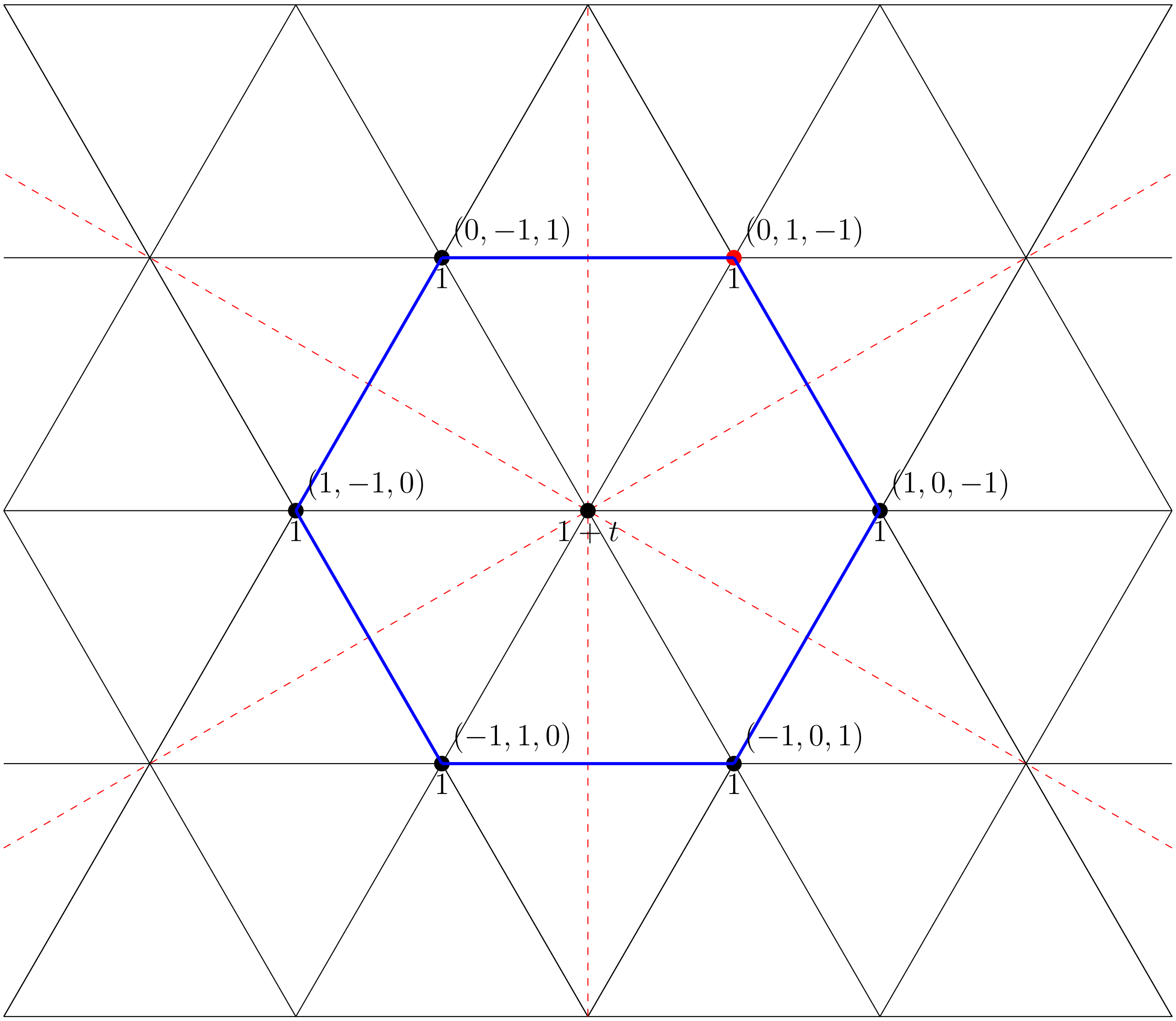
$$\dim(V_\lambda(\mu)) = P_{\mu\lambda}(1).$$

That is, the local intersection cohomology Euler characteristic of the affine Schubert varieties (and of the affine K orbits) equals the weight multiplicity in the irreducible representation. (If you wish to add up these polynomials in order to get the intersection cohomology of the whole orbit closure, then you must do so with a shift $t^{\ell(\lambda)-\ell(\mu)}$ corresponding to the codimension in X_λ of the I-orbit that contains the point x_μ.) Consequently Lusztig considers the full Kazhdan Lusztig polynomial $P_{\mu\lambda}(q)$ be a *q-analog of the weight multiplicity.* The individual coefficients were eventually shown (by R. Brylinski, Lusztig, others) to equal the multiplicity of the weight μ in a certain layer $V_\lambda^r/V_\lambda^{r-1}$ of the filtration of V_λ that is induced by the principal nilpotent element.

19.4. **Moment map.** The complex torus $\{(a_1, \cdots, a_n) \in \mathbb{C}^n | \ \prod_i a_i = 1\} \cong (\mathbb{C}^\times)^{n-1}$ acts on X with a the moment map (for the action of $(S^1)^{n-1}$) $\mu : X \to \mathfrak{a}^*$. Each fixed point $x_\lambda = (T^{\lambda_1}, \cdots, T^{\lambda_n})$ corresponds to a cocharacter $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{Z}^n$ (with $\sum_i \lambda_i = 0$). The torus action preserves both stratifications and the image of each stratum closure is a convex polyhedron. We can put all this information on the same diagram. Fix $\lambda \in \chi_*(\mathcal{T})_+$. Let V_λ be the irreducible representation of PGL_3 with highest weight λ. Let X_λ be the K-orbit of the point x_λ. Then $\mu(X_\lambda)$ is the convex polyhedron spanned by the W orbit of the point λ. The lattice points μ inside this polyhedron correspond to the weight spaces $V_\lambda(\mu)$. At each of these points the Kazhdan Lusztig polynomial $P_{\mu,\lambda}(t)$ gives the dimension of this weight space.

For SL_3 the moment map image gives the triangular lattice which can be interpreted as the weight lattice for PGL_3. The moment map image of the first stratum is a hexagon and in fact the first stratum has the structure of a vector bundle over the flag manifold $F_{1,2}(\mathbb{C}^3)$. The closure of this stratum consists of adding a single point, so it is the Thom space of this bundle. The local intersection cohomology at the vertex (which appears at the origin in the diagram) is the primitive cohomology of the flag manifold (that is, $1 + t$).

The following diagram represents the moment map image of the affine Grassmannian for SL_3; it also equals the weight diagram for $\mathrm{PGL}_3(\mathbb{C})$. The dotted red lines are reflecting hyperplanes for the Weyl group. The red dot is a highest weight λ for PGL_e; the other dots are their Weyl images. The blue hexagon is the outline of the moment map images; it is also the collection of weights in the irreducible representation of highest weight λ. The 1 and $1 + t$ beneath the dots are the Kazhdan Lusztig polynomials.



19.5. **Perverse sheaves on X.** Throughout this section, to simplify notation, let $G = \mathrm{SL}_n(\mathbb{C}((T)))$, $K = \mathrm{SL}_n(\mathbb{C}[[T]])$ and $X = G/K$. We would like to imitate the construction with the flag manifold, and create a convolution product for sheaves on $X \times X$ that are constructible with respect to the stratification by G orbits, that is, if $p_{ij} : X \times X \times X \to X \times X$ as before, set

$$A^\bullet \circ B^\bullet = Rp_{13*}(p_{12}^*(A^\bullet) \otimes p_{23}^*(B^\bullet)).$$

Unfortunately the orbits of G on $X \times X$ have infinite dimension and infinite codimension so this simply does not make sense. V. Ginzburg and (later) K. Vilonen and I. Mirkovič found a way around this problem.

Let $\mathcal{P}(X)$ denote the category of perverse sheaves, constructible with respect to the above orbit stratification of X. (Ginzburg shows this is equivalent to the category of K-equivariant perverse sheaves on X.) Since each stratum is simply connected the local systems associated to these sheaves are trivial. It turns out that the intersection cohomology sheaves live only in even degrees and this implies that every perverse sheaf is isomorphic to a direct sum of $\underline{IC}^\bullet$ sheaves of stratum closures.

Mirkovič and Vilonen define a tensor product structure on $\mathcal{P}(X)$ as follows. Consider the diagram

$$X \times X \xleftarrow{\ p\ } G \times X \xrightarrow{\ q\ } G \times_K X \xrightarrow{\ m\ } X$$

Here, $k.(g, x) = (gk, k^{-1}x)$ so that $G \times_K X$ is a bundle over X whose fibers are copies of X, and $m(g, x) = gx$. If $A^\bullet, B^\bullet \in P(X)$ it turns out that there exists $C^\bullet$ a perverse sheaf on $G \times_K X$, constructible with respect to the G orbits on this space, such that

$$q^*(C^\bullet) = p^*(\pi_1^*(A^\bullet) \otimes \pi_2^*(B^\bullet))$$

where $\pi_1, \pi_2 : X \times X \to X$ are the two projections. Then set $A^\bullet \circ B^\bullet = Rm_*(C^\bullet)$.

19.6. **Theorem.** *If $A^\bullet, B^\bullet \in \mathcal{P}(X)$ then so is $A^\bullet \circ B^\bullet$. The functor $h : A^\bullet \mapsto H^*(X; A^\bullet)$ is exact and it induces an equivalence of categories*

$$\mathcal{P}(X) \sim \mathrm{Rep}(^L G)$$

which takes $A^\bullet \circ B^\bullet$ to the tensor product $h(A^\bullet) \otimes h(B^\bullet)$ of the associated representations. If $\lambda \in \chi_*(\mathcal{T})_+$ then $h(\underline{IC}^\bullet(\overline{X}_\lambda)) = V_\lambda$ is the irreducible representation of highest weight λ.

19.7. Although it sounds intimidating, the convolution product of sheaves is exactly parallel to the previous case of the finite (dimensional) flag manifold. Consider the weight diagram for SL_3 and the moment map image of torus fixed points in the affine Grassmannian for PGL_3. The coordinate lattices are indicated on Figure 12, for example, the point $(1, 0, 0)$ corresponds to the lattice $sK = \begin{array}{|c|c|c|c|} \hline T^{-2} & & & \\ \hline T^{-1} & \bullet & & \\ \hline T^0 & \bullet & \bullet & \bullet \\ \hline T^1 & \bullet & \bullet & \bullet \\ \hline T^2 & \bullet & \bullet & \bullet \\ \hline \end{array}$ where s is a certain simple reflection in the affine Weyl group. So the orbit of $G(\mathbb{C}[[T]])$ in $X \times X$ corresponding to this element consists of the set of pairs of lattices $L_0 \xrightarrow{s} L$ that are in relative position s, that is,

$$\left\{(L_0, L_1) |\ L_0 \subset L_1 \subset T^{-1}L_0 \text{ and } \ \dim(L_1/L_0) = 1\right\}.$$

If we fix L_0 to be the standard lattice, then we see that the orbit $\mathcal{O}_s \subset X$ is isomorphic to $\mathbb{P}^1$ so its $\underline{IC}$ sheaf is the constant sheaf.

Let us consider the convolution product of this sheaf with itself. Thus the total space of the correspondence consists of triples of lattices

$$C = \left\{(L_0, L_1, L_2) |\ L_0 \xrightarrow{s} L_1 \xrightarrow{s} L_2\right\}$$

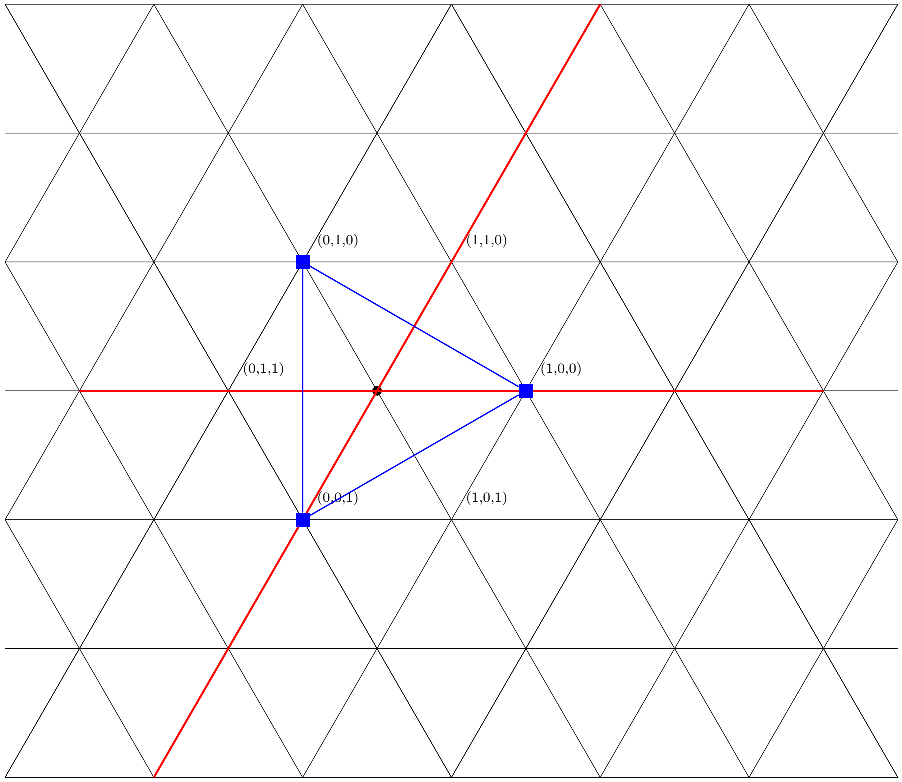


FIGURE 12. SL_3: Reflecting hyperplanes and moment map image of $\mathcal{O}_{(1,1,0)}$

in their appropriate relative positions. Apply π_{13} to obtain the correspondence

$$\pi_{23}(C) = \left\{(L_0, L_2) |\ L_0 \subset L_2 \subset T^{-2}L_0 \ \text{ and } \ \dim(L_2/L_0) = 2\right\}.$$

Again, taking L_0 to be the standard lattice, we need to understand the decomposition into orbits of $R\pi_{23*}(\underline{\mathbb{Q}}_C)$. There are two types of such lattices: the first type consists of those lattices in the orbit xK that was considered in §18.6, that is, lattices like this:

T^{-3}			
T^{-2}	$\bullet$		
T^{-1}	$\bullet$	$\star$	$\star$
T^{0}	$\bullet$	$\bullet$	$\bullet$
T^{1}	$\bullet$	$\bullet$	$\bullet$
T^{2}	$\bullet$	$\bullet$	$\bullet$

which project under the moment map to an image that contains the W-translates of $(2, 0, 0)$. The second type is lattices like this:

T^{-3}			
T^{-2}			
T^{-1}	$\bullet$	$\bullet$	
T^{0}	$\bullet$	$\bullet$	$\bullet$
T^{1}	$\bullet$	$\bullet$	$\bullet$
T^{2}	$\bullet$	$\bullet$	$\bullet$

which project under the moment map to an image containing $(1,1,0)$. That is,

$$m : C \to \mathcal{O}_{(2,0,0)} \cup \mathcal{O}_{(1,1,0)}.$$

The map m is guaranteed to be semi-small and $Rm_*(\underline{\mathbb{Q}}_C)$ breaks into a direct sum of copies of $\underline{IC}$ sheaves of these two strata the multiplicities equal to the number of components of the fiber. One checks that the multiplicity equals one.

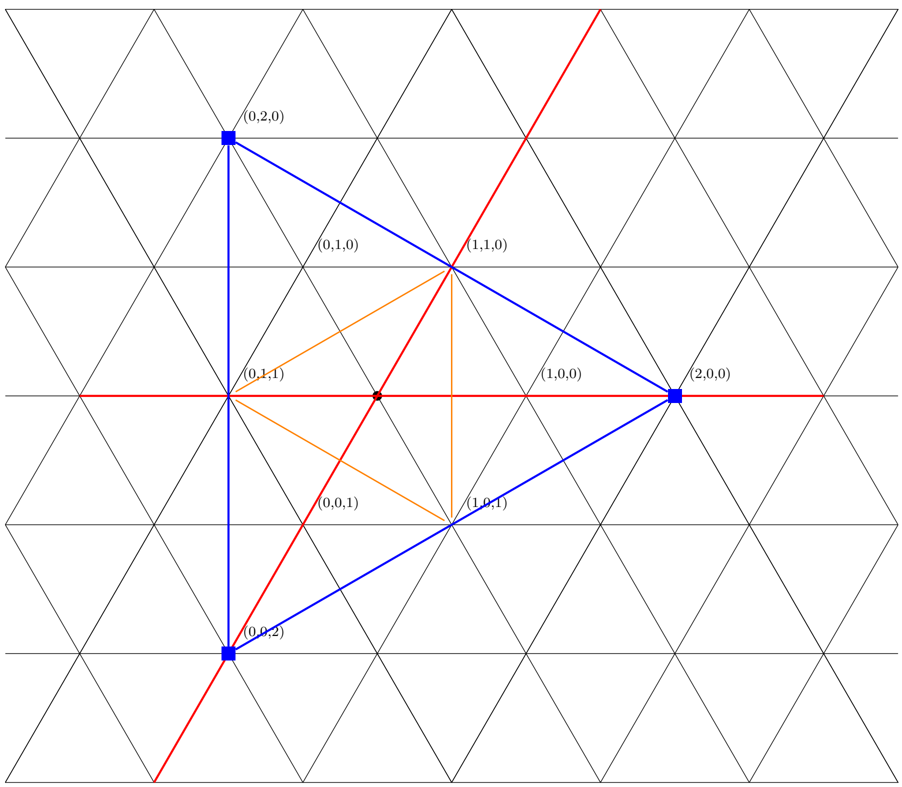


FIGURE 13. $\mathcal{O}_{(1,0,0)} \circ \mathcal{O}_{(1,0,0)} = \mathcal{O}_{(2,0,0)} + \mathcal{O}_{(1,1,0)}$

By taking the cohomology of these sheaes we obtain highest weight representations: $V_{(1,0,0)} = \mathrm{std}$, $V_{(1,1,0)} = \mathrm{std}^\vee$, and $V_{(2,0,0)} = \wedge^2(\mathrm{std})$ (where std is the standard representation) so that

$$\mathrm{std} \otimes \mathrm{std} \cong \mathrm{std}^\vee \oplus \wedge^2(\mathrm{std}).$$

19.8. **Remarks.** Theorem 19.6 should be regarded as a categorification of Satake's isomorphism. In fact, taking the K group of the Grothendieck group on both sides gives

$$K(\mathcal{P}(X)) \cong \mathcal{H}(G, K) \cong \chi_*(\mathcal{T})^W$$

which is the classical Satake isomorphism.

If we could duplicate the construction in the finite dimensional case we would consider G orbits on $X \times X$. Ginzburg, Mirkovič and Vilonen replace $X \times X$ with $G \times_K X$ which is a fiber bundle over X with fiber isomorphic to X. They replace the G orbits with the strata $S_{\lambda,\mu}$ which is a fiber bundle over X_μ with fiber isomorphic to X_λ.

It is totally nonobvious that the convolution of perverse sheaves is perverse. This depends on the fact that the mapping $m : G \times_K X \to X$ is semi-small in a very strong sense. For each $\lambda, \mu \in \chi_*(\mathcal{T})_+$ let $S_{\lambda,\mu} = p^{-1}(X_\lambda) \times_K X_\mu$. These form a stratification of $G \times_K X$. It turns out that the restriction $m : \overline{S}_{\lambda,\mu} \to X$ is semi-small (onto its image, which is a union of strata X_τ). This implies that $Rm_*(\mathcal{L}[d_{\lambda,\mu}])$ is perverse, for any locally constant sheaf $\mathcal{L}$ on $S_{\lambda,\mu}$ (where $d_{\lambda,\mu} = \dim(S_{\lambda,\mu})$.

19.9. **Tannakian category.** (see [Deligne & Milne]) There is a general theorem that a reductive algebraic group (say, over an algebraically closed field) can be recovered from its cartegory of representations. More generally if $\mathcal{C}$ is a tensor category, that is, an abelian category together with a "tensor" structure ($A, B \in \mathcal{C} \implies A \circ B \in \mathcal{C}$) that is commutative and associative in a functorial way (satisfies the em associativity constraint and the *commutativity constraint*, and if $h : \mathcal{C} \to \{VS\}$ is a rigid *fiber functor* (that is, an exact functor to vector spaces such that $h(A \circ B) = h(A) \otimes h(B)$) then the group of automorphisms of h (that is, the group of natural transformations $h \to h$) is an algebraic group G whose category of representations is equivalent to the original category $\mathcal{C}$.

It turns out that $\mathcal{P}(X)$ is such a Tannakian category and that h is a rigid fiber functor. Therefore the group of automorphisms of h is isomorphic to the Langlands dual group ${}^LG$.