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INTRODUCTION TO THE PAPERS OF R. THOM AND J. MATHER

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Immediately following the commentary below, the previously published article by R. Thom is printed in its entirety: R. Thom, *Ensembles et morphismes stratifiés*, Bulletin of the American Mathematical Society **75** (1969), no. 2, 240–284 (French). This is followed by the first publication of the 1970 lecture notes of J. Mather, *Notes on topological stability*.

For any closed set $E \subset \mathbb{R}^n$ there exists a C^∞ function $f : \mathbb{R}^n \to \mathbb{R}$ such that $E = f^{-1}(0)$. This includes the Cantor set, the Sierpiński sponge, the Snowflake, and other sets of fractional Hausdorff dimension. How does one prove that this sort of behavior cannot happen when f is an analytic function or an algebraic function? These questions were approached about 80 years ago when it was shown ([38, 15, 13, 16]) that algebraic sets could be triangulated.[1] For many years the 1932 paper [13] was cited as the only known proof that complex algebraic sets were locally contractible. But these papers are difficult to follow, and for decades the triangulability of algebraic and analytic sets was treated with some suspicion. Later articles, such as [30, 3, 18] and especially [10, 11, 8], finally put these questions to rest. However, the interesting structure of the singularities of an algebraic set is not easily2 described using simplices, and people began to look for a more intelligent way to decompose an algebraic set into (fewer, larger) pieces, starting with the *nonsingular* part.

Hassler Whitney struggled with these questions for decades. In 1946 he wrote *Complexes of manifolds* [40], in which he considered spaces that were glued together out of smooth manifolds much in the way that a cell complex is glued together from cells. In 1957 Whitney showed in [41] that it is possible, in any algebraic set, to choose an open dense nonsingular part such that its complement is an *algebraic* set of *smaller dimension*. Therefore this procedure can be repeated so as to give a finite filtration by closed subsets $X_0 \subset X_1 \subset \cdots \subset X_n$ such that X_{r+1} is obtained from X_r by attaching a (possibly empty) smooth manifold $S_{r+1} := X_{r+1} - X_r$ of dimension $r+1$.

One might hope that the resulting decomposition into pieces $X = \coprod_r S_r$ is *locally trivial*—that any two sufficiently nearby points $x, y \in S_r$ should have neighborhoods that are *isomorphic* (in a sense to be made precise below) by an isomorphism (a homeomorphism, or perhaps, a diffeomorphism) that preserves the induced filtrations. In 1962 René Thom made a first attempt in [34] to make precise such a notion of a locally trivial stratification. He proposed that each stratum S should have a

[1] In other words, every compact algebraic set is homeomorphic to a finite simplicial complex.

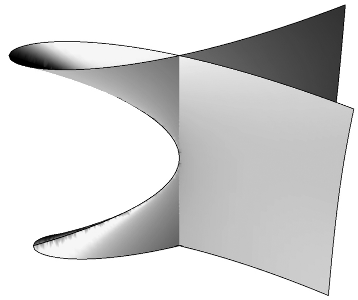

FIGURE 1

"good" tubular neighborhood T_S, together with a projection function $T_S \to S$ and a "carpeting" function[2] $S \to \mathbb{R}_{>0}$, both of which should have maximal rank when restricted to any larger stratum.

It turns out that Whitney's procedure as described above is not enough to give such a locally trivial *stratification*. In a remarkable 1965 paper [42] Whitney described several important examples that illustrate the problems. The first example (see Figure 1) is a two-dimensional algebraic set with an obvious nonsingular part that appears to consist of several sheets of paper. If we throw away this two-dimensional part, then what remains is a smooth one-dimensional manifold. But clearly there is one point on this *singular stratum* that is special. The problem is that we "threw away" the nonsingular part too soon, because the way that it twists around the singular stratum changes at the special point. It was necessary to find a way to identify that point as special, and Whitney [42, Sect. 8, p. 228] proposed his Conditions A and B as possible candidates.[3] He also outlined a proof (with details in [43]) that complex analytic sets can be *Whitney stratified* (that is, stratified so as to satisfy Whitney's conditions A and B) with complex analytic strata. If we stratify the preceding example with two strata, then condition B fails at the origin, forcing us to (correctly) consider the origin as a third stratum.

Large classes of naturally occurring sets have since been shown to admit Whitney stratifications; see (for example) [19, 6, 5, 7, 9, 12, 14, 1, 31, 32]. However, conditions A and B still allow for some dangerous pathologies. Thom points out in [35] that the spiral $r = \exp(-\theta^2)$ satisfies the Whitney conditions, but it turns infinitely many times as it approaches the origin.

An even more worrisome example is described in Whitney's article [42]. Typically, a singular space X might arise as a subset of Euclidean space, or as a subset of a smooth manifold M. A reasonable goal is to ask for a stratification of X so that if $x, y \in X$ lie in a single (connected) stratum, then there should exist open neighborhoods $U_x, U_y \subset M$ and a diffeomorphism $\phi : U_x \to U_y$ which takes $X \cap U_x$ to $X \cap U_y$ in a stratum-preserving way. It turns out that this is too much to hope for, even if X is a complex algebraic set. Whitney's example is shown in Figure 2: any smooth isotopy that flows along the x-axis will have the property that, when restricted to a normal slice through that stratum, the derivative will map four lines

[2]In [26], Mather replaced this with the more intuitive and useful notion of a function that measures the "distance" from the stratum S.

[3]In [26], Mather showed that condition B implies condition A.

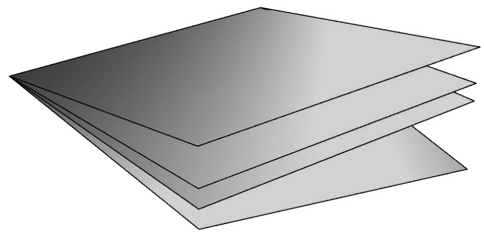

FIGURE 2

in $\mathbb{R}^2$ to four lines in $\mathbb{R}^2$, three of which do not move but the fourth does. However the derivative is determined by the fact that the first three lines do not move; therefore, it cannot move the fourth line either.

Thus, the best we can hope for is that a stratification should be *topologically* locally trivial: given points x, x' in a connected component of a single stratum, there should be a continuous map (which is smooth on each stratum, but not necessarily differentiable on the ambient manifold) which maps a neighborhood of x homeomorphically, in a stratum-preserving way, to a neighborhood of x'. But what hope is there to construct such a homeomorphism that is continuous, stratum preserving, smooth on each stratum, and yet is not differentiable, on an n-dimensional space that might be as pathological as Thom's spiral? Thom had the daring idea to address this problem by constructing a *discontinuous* vector field on the stratified space whose flow is continuous but not differentiable. The flow of such a *controlled vector field* is then the desired homeomorphism. In some sense, this is the key idea that separates a *reasonable* (i.e., a stratified) singular space from an *unreasonable* space (such as a Cantor set): the reasonable spaces have locally constant topological type.

Thom had in mind a particular application to the study of smooth mappings. Let M and N be smooth manifolds with M compact. A smooth mapping $f : M \to N$ is said to be *differentiably stable* if, for every sufficiently nearby mapping $g : M \to N$, there exist diffeomorphisms $\phi : M \to M$ and $\psi : N \to N$ that convert f into g; that is, $g = \psi^{-1} \circ f \circ \phi$. One might guess that such differentiably stable maps form an open and dense set in the space of all smooth mappings. For $\dim(N) = 1$ this is a basic result in Morse theory, and for $\dim(M) = \dim(N) = 2$ it follows from Whitney's 1955 article [39]. But in 1960, Thom gave a counterexample [17, §11] when $\dim(M) = \dim(N) = 16$.

The complete answer to this question was determined in a remarkable series of articles by John Mather [20, 21, 22, 23, 24, 25] in which he determined necessary and sufficient conditions, depending only on $\dim(M)$ and $\dim(N)$, for the differentiably stable mappings to form an open and dense set in the space of all smooth mappings. Those conditions are that the pair $(m = \dim(M), n = \dim(N))$ should satisfy any one of the following relations:

$$\begin{array}{rcrcl}
n-m & \geq & 4 & \text{and} & m < \frac{6}{7}n + \frac{8}{7}, \\
3 \geq n-m & \geq & 0 & \text{and} & m < \frac{6}{7}n + \frac{9}{7}, \\
n-m & = & -1 & \text{and} & n < 8, \\
n-m & = & -2 & \text{and} & n < 6, \\
n-m & = & -3 & \text{and} & n < 7.
\end{array}$$

It turns out that the difficulty is caused, in some sense, by the smoothness requirement on ϕ, ψ, which we now relax. Two smooth mappings $f, g : M \to N$ are said to be *topologically equivalent* if there exist *homeomorphisms* ϕ, ψ such that $g = \psi^{-1} \circ f \circ \phi$. A smooth mapping $f : M \to N$ is said to be *topologically stable* if every sufficiently nearby mapping g is topologically equivalent to f. Thom conjectured that for *any* two smooth manifolds M, N (with M compact), the set of topologically stable mappings forms an open and dense subset of $C^\infty(M, N)$.

In a series of papers that spanned a decade (including [34, 33, 35]), Thom proposed the whole apparatus of stratification theory and controlled vector fields in order to establish this result. His most detailed treatment of stratification theory appears in the paper, *Ensembles et morphismes stratifiés* [35], which is reprinted here. Unfortunately, Thom's treatment does not provide a complete proof of the density of topologically stable mappings.[4] Moreover, his proof that Whitney stratifications are topologically *locally trivial* is extremely difficult to understand and lacks essential details. Thom's outline is geometric and full of wonderful ideas, but it is written in a way that gives the reader very little guidance on how to start filling in the gaps.

In 1970 John Mather worked out a proof of Thom's conjecture (that the set of topologically stable mappings is open and dense in the space $C^\infty(M, N)$), and he began the task of writing a book that would explain his proof. In 1970 Mather gave a series of lectures on this material at Harvard, and his notes (which were to be part of the first chapter of the book) were written up [26]. These notes are precise, clear and very readable; they place the theory of stratifications on a firm foundation. They contain proofs of the *isotopy lemmas* and of the fact that every Whitney stratification is locally trivial. The arguments involve delicate existence and extension theorems for tubular neighborhoods, combined with double inductions among the strata.

Although the later chapters of Mather's book were never completed, the *Notes* circulated widely in mimeographed and photocopied form for many years. They became the standard source for the foundations of stratification theory, and they have often been summarized and paraphrased (for example, in [4, Chapter 2], [37, Chapter 1], [29, Chapter 3]). They are being formally published here for the first time (with minor corrections by John Mather). The rest of the project to prove that topologically stable mappings are dense was outlined in [27] with details in [28]; see also [4] and [2].

The two papers (re)printed here represent a turning point in the theory of stratifications. Their significance extends well beyond the study of the stability of smooth mappings. Indeed, stratification theory has become an indispensable tool in many branches of topology, geometry, algebra, combinatorics, and number theory, and the word *stratification* is now part of the standard vocabulary of modern mathematics. The subject itself continues to grow and evolve.

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[4]Thom acknowledges this in [36, p. 204].

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