# A GEOMETRIC PROOF OF THE EXISTENCE OF WHITNEY STRATIFICATIONS 

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## 1. Introduction

A stratification of a set, e.g. an analytic variety, is, roughly, a partition of it into manifolds so that these manifolds fit together "regularly". Stratification theory was originated by Thom and Whitney for algebraic and analytic sets. It was one of the key ingredients in Mather's proof of the topological stability theorem (Ma] (see GM] and PW for the history and further applications of stratification theory).

In this paper, given a partition of a singular set (which we know always exists), we prove that there is a "regular" partition. Our proof is based on a remark that if there are two parts of the partition $V$ and $W$ of different dimension and $V \subset \bar{W}$, then irregularity of the partition at a point $x$ in $V$ corresponds to the existence of nonunique limits of tangent planes $T_{y} W$ as $y$ approaches $x$.

Consider either the category of (semi)analytic (or (semi)algebraic) sets. Call a subset $V \subset \mathbb{R}^{m}\left(\right.$ or $\left.\mathbb{C}^{m}\right)$ a semivariety if locally at each point $x \in \mathbb{R}^{m}\left(\right.$ or $\left.\mathbb{C}^{m}\right)$ it is a finite union of subsets defined by equations and inequalities

$$
f_{1}=\cdots=f_{k}=0 \quad \begin{cases}g_{1} \neq 0, \ldots, g_{l} \neq 0 & \text { (complex case) },  \tag{1}\\ g_{1}>0, \ldots, g_{l}>0 & \text { (real case) }\end{cases}
$$

where $f_{i}$ 's and $g_{j}$ 's are real (or complex) analytic (or algebraic) depending on the case under consideration.

In the real algebraic case semivarieties are usually called semialgebraic sets; in the complex algebraic case they are called constructible, and in either analytic case they are called semianalytic sets. Semivarieties are closed under Boolean operations.
Definition 1. (Whitney) Let $V_{i}, V_{j}$ be disjoint manifolds in $\mathbb{R}^{m}\left(\right.$ or $\left.\mathbb{C}^{m}\right)$, $\operatorname{dim} V_{j}>\operatorname{dim} V_{i}$, and let $x \in V_{i} \cap \overline{V_{j}}$. A triple $\left(V_{j}, V_{i}, x\right)$ is called a(resp. b)- regular if
A) when a sequence $\left\{y_{n}\right\} \subset V_{j}$ tends to $x$ and $T_{y_{n}} V_{j}$ tends in the Grassmanian bundle to a subspace $\tau_{x}$ of $\mathbb{R}^{m}\left(\right.$ or $\left.\mathbb{C}^{m}\right)$, then $T_{x} V_{i} \subset \tau_{x}$;
B) when sequences $\left\{y_{n}\right\} \subset V_{j}$ and $\left\{x_{n}\right\} \subset V_{i}$ each tends to $x$, the unit vector $\left(x_{n}-\right.$ $\left.y_{n}\right) /\left|x_{n}-y_{n}\right|$ tends to a vector $v$, and $T_{y_{n}} V_{j}$ tends to $\tau_{x}$, then $v \in \tau_{x} \rrbracket$.
$V_{j}$ is called a resp. b)- regular over $V_{i}$ if each triple $\left(V_{j}, V_{i}, x\right)$ is a (resp. b)- regular.

[^0]Definition 2. (Whitney) Let $V$ be a semivariety in $\mathbb{R}^{m}$ (or $\mathbb{C}^{m}$ ). A disjoint decomposition

$$
\begin{equation*}
V=\bigsqcup_{i \in I} V_{i}, \quad V_{i} \cup V_{j}=\emptyset \quad \text { for } \quad i \neq j \tag{2}
\end{equation*}
$$

into smooth semivarieties $\left\{V_{i}\right\}_{i \in I}$, called strata, is called an a (resp. b)-regular stratification if

1. each point has a neighborhood intersecting only finitely many strata;
2. the frontier $\overline{V_{j}} \backslash V_{j}$ of each stratum $V_{j}$ is a union of other strata $\bigsqcup_{i \in J(i)} V_{i}$;
3. any triple $\left(V_{j}, V_{i}, x\right)$ such that $x \in V_{i} \subset \overline{V_{j}}$ is a(resp. b)-regular.

Theorem 1. Wh, Th, Ld For any semivariety $V$ in $\mathbb{R}^{m}$ (or $\mathbb{C}^{m}$ ) there is an a (resp. b)-regular stratification.

The existence of stratifications in the complex analytic case was proved by Whitney [Wh]. Later Thom published a sketch of a proof (Th]. Then Lojasiewicz [Ld] extended these results to the semianalytic case. The most illuminating proof is due to Wall [Wa], where based on Milnor's curve selection lemma Mi] he simplifies the above proofs. Hironaka [Hi] gave an elegant proof using his resolution of singularities, but it requires background in algebraic geometry. We give a geometric proof based on Milnor's curve selection lemma Mi], Wa, Rolle's lemma, and a transversality theorem. The rest of the paper is devoted to this proof.

Proof of theorem 17: A semivariety $V$ has well-defined dimension, say $d \leq m$. Denote by $V_{\text {reg }}$ the set of points, where $V$ is locally a real (or complex) analytic submanifold of $\mathbb{R}^{m}$ (or $\mathbb{C}^{m}$ ) of dimension $d . V_{\text {reg }}$ is a semivariety, moreover, $V_{\text {sing }}=V \backslash V_{\text {reg }}$ is a semivariety of positive codimension in $V$, i.e. $\operatorname{dim} V_{\text {sing }}<\operatorname{dim} V$. In the analytic case all these results may be found in Lojasiewicz [D]; in the algebraic case they are not difficult (see e.g. Mi]).

Step 1. There is a filtration of $V$ by semivarieties

$$
\begin{equation*}
V^{0} \subset V^{1} \subset \cdots \subset V^{d}=V \tag{3}
\end{equation*}
$$

where for each $k=1, \ldots, d$ the set $V^{k} \backslash V^{k-1}$ is a manifold of dimension $k$. This follows from the Lojasiewicz result. Indeed, consider $V_{\text {sing }} \subset V$, then $V \backslash V_{\text {sing }}$ is a manifold of dimension $d$ and $\operatorname{dim} V_{\text {sing }}<d$. Inductive application of these arguments completes the proof.

A refinement of a decomposition $V=\bigsqcup_{i \in I} V_{i}$ is a decomposition $V=\bigsqcup_{i^{\prime} \in I^{\prime}} V_{i^{\prime}}$ such that any stratum $V_{j}$ of the first decomposition is a union of some strata of the second one, i.e. there is a set $I^{\prime}(j) \subset I^{\prime}$ such that $V_{j}=\bigsqcup_{i^{\prime} \in I^{\prime}(j)} V_{i^{\prime}}$.

Step 2. Let $V \subset \mathbb{R}^{m}$ (or $\mathbb{C}^{m}$ ) be a manifold and $W \subset V$ be a semivariety. Denote by $\operatorname{Int}_{V}(W)$ the set of interior points of $W$ in $V$ w.r.t. the induced from $\mathbb{R}^{m}$ (resp. $\mathbb{C}^{m}$ ) topology. Let $V_{i}$ and $V_{j}$ be a pair of distinct strata. For each point $x \in V_{i} \cap \overline{V_{j}}$ denote by $V_{j}^{\text {con,x }}$ a local connected component of $V_{j}$ at $x$, i.e. a connected component of intersection of $V_{j}$ with a ball centered at $x$ and call it essential if the closure of $V_{j}^{\text {con, } x}$ has $x$ is in the interior, $x \in \operatorname{Int}_{V_{i}}\left(V_{i} \cap \overline{V_{j}^{\text {con,x}}}\right)$. Denote by $V_{j}^{\text {ess,x }}$ the union of all local essential components of $V_{j}$. Lojasiewicz Lo showed that $V_{j}$ has only a finitely many local connected components.

Theorem 2. For any two disjoint strata $V_{j}$ and $V_{i}$ the set of points
$\operatorname{Sing}_{a(\text { resp.b) }}\left(V_{j}, V_{i}\right)=\left\{x \in V_{i} \cap \overline{V_{j}}:\left(V_{j}^{\text {ess }, x}, V_{i}, x\right)\right.$ is not $a($ resp. $b)$ - regular $\}$,
is a semivariety in $V_{i}$ and $\operatorname{dim} \operatorname{Sing}_{a(\operatorname{resp} . b)}\left(V_{j}, V_{i}\right)<\operatorname{dim} V_{i}$.
Let us show that this theorem is sufficient to prove Theorem 1. Consider a decomposition $V=\bigsqcup_{i \in I} V_{i}$ and split the strata into two groups: the first group consists of strata of dimension at least $k$ and the second group is of the rest. Suppose that each stratum from the first group is $a$ (resp. b)-regular over each stratum from the second group. Then by definition of $a$ (resp. b)-regularity any refinement of a stratum from the second group preserves this $a$ (resp. b)-regularity.

Now apply this refinement inductively. Consider strata in $V^{d} \backslash V^{d-1}$ of dimension $d$. Using Theorem 2 and the result of Lojasiewicz [D] that a frontier of a semivariety has dimension less than a semivariety itself, refine $V^{d-1}$ so that each $d$-dimensional stratum is $a$ (resp. $b$ )-regular over each stratum in $V^{d-1}$. The above remark shows that any further refinement of the strata in $V^{d-1}$ preserves the $a\left(\right.$ resp. $b$ )-regularity of strata from $V^{d} \backslash V^{d-1}$ over it. This reduces the problem of the existence of stratification for $d$-dimensional semivarieties to the same problem for $(d-1)$-dimensional semivarieties. Induction on dimension completes the proof of Theorem 1 .

Our proof is based on the observation that if $V_{i} \subset \overline{V_{j}}$ are a pair of strata $a($ resp. b)regularity of $V_{j}$ over $V_{i}$ at $x$ in $V_{i}$ is closely related to whether the limit of tangent planes $T_{y} V_{j}$ is unique or not as $y$ from $V_{j}$ tends to $x$. The rest of the paper is devoted to the proof of Theorem 2 which consists of two steps. In section 1.1 we relate $a$ (resp. b)-regularity with (non) uniqueness of limits of tangent planes $T_{y} V_{j}$, then based on it and Rolle's lemma in section 1.3 we prove Theorem 2 .
1.1. The key definitions. Let $V_{i}$ and $V_{j}$ be a pair of distinct strata. Define

$$
\begin{equation*}
U n_{a}\left(V_{j}, V_{i}\right)=\left\{x \in V_{i} \cap \overline{V_{j}}: \text { for any } V_{j}^{\text {con,x }}, \text { there exists } \tau_{x} \subset T_{x} \mathbb{R}^{m}\right. \tag{4}
\end{equation*}
$$

(resp. $T_{x} \mathbb{C}^{m}$ ) such that for any $\left\{y_{n}\right\} \subset V_{j}^{\text {con,x }}$ tending to $\left.x, \quad T_{y_{n}} V_{j} \rightarrow \tau_{x}\right\}$,
Since $a$ (resp. b)-regularity is a local property, w.l.o.g. we can assume that locally $V_{i}$ is an $s$-plane with a basis of unit vectors $e_{1}, \ldots, e_{s}$. Using an idea of Kuo Ku (see also Wa ) we define $a$ Kuo map $\mathcal{P}^{a(\text { resp. b) }}: V_{j} \rightarrow \mathbb{R}$ which measures non $a($ resp. b)-regularity in terms of an angle between a vector or a plane and the tangent plane to $V_{j}$. Denote by $\pi_{i}: \mathbb{R}^{m} \rightarrow V_{i}^{\perp}$ (resp. $\pi_{i}^{\perp}: \mathbb{R}^{m} \rightarrow V_{i}$ ) the orthogonal projection along $V_{i}$ (resp. $V_{i}^{\perp}$ ) onto the complement $V_{i}^{\perp}$ (resp. $V_{i}$ ) with $x$ being the origin of $\mathbb{R}^{m}$ and by

$$
\begin{array}{r}
\pi_{j}: V_{j} \times \mathbb{R}^{m} \rightarrow \mathbb{R}^{m}, \pi_{j, t}: V_{j} \rightarrow \mathbb{R}^{m} \text { for } t=1, \ldots, s+1 \text { defined by } \\
\pi_{j}(y, v)=\pi_{T_{y} V_{j}}(v), \pi_{j, t}(y)=\pi_{j}\left(y, e_{t}\right), \pi_{j, s+1}(y)=\pi_{j}\left(y, \pi_{i}(y) /\left|\pi_{i}(y)\right|\right) \tag{5}
\end{array}
$$

where $\pi_{j}(y, v)$ is the orthogonal projection of $v$ along the tangent plane $T_{y} V_{j}$ to $V_{j}$ at $y$ naturally embedded into $\mathbb{R}^{m}$. Define analytic functions $\mathcal{P}^{a(r e s p . b)}: V_{j} \rightarrow \mathbb{R}$ by $\mathcal{P}^{a}(y)=$ $\sum_{t=1}^{s}\left|\pi_{j, t}(y)\right|^{2}\left(\right.$ resp. $\left.\mathcal{P}^{b}(y)=\sum_{t=1}^{s+1}\left|\pi_{j, t}(y)\right|^{2}\right)$. By the definition the level sets of $\mathcal{P}^{a(r e s p . b)}$ are semivarieties.

Notice that the first $s$ terms of the function $\mathcal{P}^{a}(y)$ measure the angle between $T_{x} V_{i}=V_{i}$ and $T_{y} V_{j}$ and the last term measures the angle between the $V_{i}^{\perp}$ - component of $(y-x) /|y-x|$
and $T_{y} V_{j}$. Since any vector can be decomposed into $V_{i}$ and $V_{i}^{\perp}$ components, this proves the following

Fact 1. For any pair distinct strata $V_{j}$ and $V_{i}$ existence of a sequence $\left\{y_{n}\right\} \subset V_{j}$ tending to $x$ with a nonzero limit of $\mathcal{P}^{a(r e s p . ~ b) ~}\left(y_{n}\right)$ is equivalent to a(resp. b)-irregularity of $V_{j}$ over $V_{i}$ at $x$.

$$
\begin{array}{r}
U n_{b}\left(V_{j}, V_{i}\right)=\left\{x \in U n_{a}\left(V_{j}, V_{i}\right): \text { for any } V_{j}^{\text {con }, x} \text {, there exists } \epsilon \in \mathbb{R}\right. \\
\text { such that for any } \left.\left\{y_{n}\right\} \subset V_{j}^{\text {con }, x} \text { tending to } x, \mathcal{P}^{b}\left(y_{n}\right) \rightarrow \epsilon\right\}, \tag{6}
\end{array}
$$

Lemma 1. Let $V_{i}$ and $V_{j}$ be a pair of disjoint strata in $\mathbb{R}^{m}$ (or $\mathbb{C}^{m}$ ) with $V_{i} \cap \overline{V_{j}} \neq \emptyset$. Then $\operatorname{Sing}_{a(\mathrm{resp} . b)}\left(V_{j}, V_{i}\right)$ and $U n_{a(\mathrm{resp} . b)}\left(V_{j}, V_{i}\right)$ are semivarieties and

$$
\operatorname{Sing}_{a}\left(V_{j}, V_{i}\right) \subset \operatorname{Sing}_{b}\left(V_{j}, V_{i}\right), \quad \operatorname{Sing}_{a(\text { resp. } b)}\left(V_{j}, V_{i}\right) \subset V_{i} \backslash U n_{a(\text { resp.b) }}\left(V_{j}, V_{i}\right)
$$

Remark 1. The new result here is that $\operatorname{Sing}_{a(\mathrm{resp.b)}}\left(V_{j}, V_{i}\right) \subset V_{i} \backslash U n_{a(\text { resp.b) }}\left(V_{j}, V_{i}\right)$. The other inclusion may be found in Wh , Ma , [L0].

Proof: Let's first prove that $\operatorname{Sing}_{a}\left(V_{j}, V_{i}\right)$ is a semivariety. Consider $V_{i} \times T V_{j}=$ $\left\{\left(x, y, T_{y} V_{j}\right): x \in V_{i}, y \in V_{j}\right\}$. It is a semivariety in an appropriate Grassmanian bundle over $\mathbb{R}^{m} \times \mathbb{R}^{m}$ (resp. $\mathbb{C}^{m} \times \mathbb{C}^{m}$ ) and so is its closure. The condition $T_{x} V_{i} \not \subset \tau_{x}$ is semialgebraic and a projection of a semivariety is a semivariety. In the real (resp. complex) algebraic case it is called the Tarski-Seidenberg Principle (Ja) (resp. elimination theory Mul). In the real analytic case it depends on a generalization due to Lojasiewicz (L0) to varieties analytic in some variables and algebraic in others. In the complex analytic case, a proof may be found in Wh. Similar arguments show $\operatorname{Sing}_{b}\left(V_{j}, V_{i}\right)$ and $U n_{a(\text { resp.b) }}\left(V_{j}, V_{i}\right)$ are semivarieties.

Now let's see that $\operatorname{Sing}_{a}\left(V_{j}, V_{i}\right) \subset \operatorname{Sing}_{b}\left(V_{j}, V_{i}\right)$. For any sequence $\left\{y_{n}\right\} \subset V_{j}$ such that $T_{y_{n}} V_{j}$ has a limit $\tau_{x}$ as $y_{n}$ tends to $x$ and any $v \in T_{x} V_{i}$ there is a sequence $\left\{x_{n}\right\} \subset V_{i}$ such that $x_{n}$ tends to $x$ slower than the sequence $\left\{y_{n}\right\}$, i.e. $\left|y_{n}-x\right| /\left|x_{n}-x\right| \rightarrow 0$ and the unit vectors $\left(x_{n}-y_{n}\right) /\left|x_{n}-y_{n}\right|$ tends to $v$ as $\left.n \rightarrow \infty\right|^{2}$. If $x \notin \operatorname{Sing} g_{b}\left(V_{j}, V_{i}\right)$, then $v$ belongs to $\tau_{x}$. Since any $v \in T_{x} V_{i}$ belongs to $\tau_{x}, T_{x} V_{i}$ also belongs to $\tau_{x}$.

To see that $\operatorname{Sing}_{a}\left(V_{j}, V_{i}\right) \subset V_{i} \backslash U n_{a}\left(V_{j}, V_{i}\right)$, suppose $x \in \operatorname{Sing}_{a}\left(V_{j}, V_{i}\right) \cap U n_{a}\left(V_{j}, V_{i}\right)$. Fix an $a$-irregular essential local connected component $V_{j}^{\text {con,x }}$ of $V_{j}$ at $x$. There is a $\operatorname{dim} V_{j^{-}}$ plane $\tau_{x}$ such that for any sequence $\left\{y_{n}\right\} \subset V_{j}^{\text {con, } x}$ tending to $x$ we have $T_{y_{n}} V_{j} \rightarrow \tau_{x}$. Since $x \in \operatorname{Sing}_{a}\left(V_{j}, V_{i}\right)$, we have $T_{x} V_{i} \not \subset \tau_{x}$, i.e. there is a unit vector $v \in T_{x} V_{i}$ which has a positive angle with $\tau_{x}$, i.e. $<\left(v, \tau_{x}\right)=2 \delta>0$. Denote by $C_{\delta, v}(x)=\left\{y \in \mathbb{R}^{m}:\left(\frac{y-x}{|y-x|}, v\right)>\right.$ $1-\delta\}$ the $\delta$-cone around $v$ centered at $x$ and by $l_{v}(x)$ the ray starting at $x$ in the direction of $v$. The intersection $V_{j}^{\text {con, } x} \cap C_{\delta, v}(x)=V_{j, \delta, v}^{c o n, x}$ is a semivariety and $l_{v}(x)$ is in its closure. By the Lojasiewicz result $V_{j, \delta, v}^{\text {con,x }}$ consists of a finite number of connected components. So one can choose a connected component $W_{j, \delta, v}^{c o n, x} \subset V_{j, \delta, v}^{c o n, x}$ which contains $l_{v}(x)$ in the closure. By Milnor's curve selection lemma Mi], Wa] there is an analytic curve $\gamma$ which belongs to $W_{j, \delta, v}^{c o n, x} \cup\{x\}$. Since $\gamma$ is analytic, it has a limiting tangent vector $w$ at $x$

[^1]which is by our construction should belong to $\tau_{x}$ and $C_{\delta, v}(x)$. This is a contradiction with $<\left(v, \tau_{x}\right)=2 \delta$.

To see that $\operatorname{Sing}_{b}\left(V_{j}, V_{i}\right) \subset V_{i} \backslash U n_{b}\left(V_{j}, V_{i}\right)$ it is sufficient to prove that $\operatorname{Sing}_{b}\left(V_{j}, V_{i}\right)$ $\cap U n_{a}\left(V_{j}, V_{i}\right) \subset V_{i} \backslash U n_{b}\left(V_{j}, V_{i}\right)$. Let $x \in \operatorname{Sing}_{b}\left(V_{j}, V_{i}\right) \cap U n_{a}\left(V_{j}, V_{i}\right)$ and $V_{j}^{\text {con,x }}$ be a $b$-irregular essential local connected component at $x$. Since $x \in U n_{a}\left(V_{j}, V_{i}\right)$, there is a unique limiting tangent plane $\tau_{x}=\lim T_{y_{n}} V_{j}$ independent of $\left\{y_{n}\right\} \subset V_{j}^{\text {con }, x}$ tending to $x$ and by the previous passage $x$ is $a$-regular, i.e. $V_{i} \subset \tau_{x}$. By Fact 1 and $b$-irregularity of $x$ there is a sequence $\left\{y_{n}\right\} \in V_{j}^{\text {con,x }}$ such that $\left|\mathcal{P}^{b}\left(y_{n}\right)\right| \rightarrow 2 \delta \neq 0$. Let's prove existence of a sequence $\left\{y_{n}^{\prime}\right\} \in V_{j}^{c o n, x}$ such that $\left|\mathcal{P}^{b}\left(y_{n}\right)\right| \rightarrow \epsilon<\delta$ which shows that $x \notin U n_{b}\left(V_{j}, V_{i}\right)$.

For each $\tilde{x} \in V_{i}$ close to $x$ consider the "level" set $V_{j}^{\text {con,x }}(\tilde{x})=V_{j}^{\text {con,x }} \cap\left(V_{i}^{\perp}+\{\tilde{x}\}\right)$ over $\tilde{x}$. Transversality of $\tau_{x}$ with $V_{i}^{\perp}$ and uniqueness of $\lim T_{y_{n}} V_{j}^{\text {con }, x}$ imply that $V_{j}^{\text {con,x }}(\tilde{x})$ is a manifold and $\tau_{j}(y)=T_{y} V_{j} \cap V_{i}^{\perp}$ depends continuously on $y$ in $V_{j}^{\text {con,x }}$. Consider the set of $\tilde{x} \in V_{i}$ for which have the corresponding "level" set $V_{j}^{\text {con,x }}(\tilde{x})$ has $\tilde{x}$ in the closure, i.e. $\tilde{x} \in \overline{V_{j}^{\text {con,x }}(\tilde{x})}$. Since $V_{j}^{\text {con }, x}$ is essential, the set of such $\tilde{x}$ 's is everywhere dense in a neighborhood of $x$ in $V_{i}$. Moreover, the "angle" function $\mathcal{P}^{b}$ is bounded in absolute value by $\delta$ on each local connected "level" component of $V_{j}^{\text {con, } x}(\tilde{x})$ having $\tilde{x}$ in its closure. Thus, one can find a sequence of points $\left\{y_{n}\right\} \subset V_{j}^{\text {con, } x}$ tending to $x$ each point $y_{n}$ of which belongs to a "level" connected component of $V_{j}^{\text {con, }}\left(\pi_{i}^{\perp}\left(y_{n}\right)\right)$, having $\pi_{i}^{\perp}\left(y_{n}\right) \in V_{i}$ in the closure. By construction $\left|\mathcal{P}^{b}\left(y_{n}\right)\right|<\delta$ for all $n$. Q.E.D.
1.2. Separation of Planes. Consider the real case. The complex case can be done in a similar way. Let $\tau_{0}$ and $\tau_{1}$ be two distinct orientable $k$-dimensional planes in $\mathbb{R}^{m}$. An orientable ( $m-k$ )-dimensional plane $l$ in $\mathbb{R}^{m}$ separates $\tau_{0}$ and $\tau_{1}$ if $l$ is transversal to $\tau_{0}$ and $\tau_{1}$ and the orientations induced by $\tau_{0}+l$ and $\tau_{1}+l$ in $\mathbb{R}^{m}$ are different. Notice that there always exists an open set of orientable $(m-k)$-planes separating any two distinct orientable $k$-plane.

Rolle's Lemma. If a continuous family of orientable $k$-planes $\left\{\tau_{t}\right\}_{t \in[0,1]}$ connects $\tau_{0}$ and $\tau_{1}$ and an orientable $(m-k)$-plane $l$ separates $\tau_{0}$ and $\tau_{1}$. Then for some $t^{*} \in(0,1)$ transversality of $\tau_{t^{*}}$ and $l$ fails.

In what follows we use the transversality theorem [GM] which says : if $V \subset \mathbb{R}^{m}$ is a manifold, then almost every plane of dimension $k$ is transversal to $V$.

### 1.3. A reduction lemma.

Lemma 2. Let $V_{j}$ and $V_{i}$ be a distinct strata and $\operatorname{dim} V_{j}>\operatorname{dim} V_{i}$. Then there is a set of strata $\left\{V_{j}^{p}\right\}_{p \in \mathbb{Z}}$ (resp. $\left\{V_{i}^{p}\right\}_{p \in \mathbb{Z}}$ ) in $V_{j}$ (resp. in $V_{i}$ ) each of positive codimension in $V_{j}$ (resp. in $V_{i}$ ) such that

$$
\begin{equation*}
\operatorname{Sing}_{a(\text { resp. b) }}\left(V_{j}, V_{i}\right) \subset \bigcup_{p \in \mathbb{Z}} \operatorname{Sing}_{a(\text { resp. b) }}\left(V_{j}^{p}, V_{i}\right) \bigcup_{p \in \mathbb{Z}} V_{i}^{p} \text {. } \tag{7}
\end{equation*}
$$

Remarks. 1. Inductive application of this lemma to the right-hand side of (7) reduces dimensions of $V_{j}^{p}$ 's up to $\operatorname{dim} V_{i}$.
2. By the result of Lojasiewicz [D] dimension of the frontier of a semivariety $\left(\operatorname{Sing}_{a(\mathrm{resp} . b)}\left(V_{j}^{p}, V_{i}\right) \subset\right.$ $\left.V_{i} \cap \overline{V_{j}^{p}}\right)$ has dimension strictly smaller that a semivariety itself.
3. By lemma 0 the set $\operatorname{Sing}_{a(\text { resp. b) }}\left(V_{j}, V_{i}\right)$ is a semivariety. Since a countable union of semivarieties of positive codimension in $V_{i}$ contains $\operatorname{Sing}_{a(\text { resp. b) }}\left(V_{j}, V_{i}\right)$, $\operatorname{Sing}_{a(\text { resp. b) }}\left(V_{j}, V_{i}\right)$ has a positive codimension in $V_{i}$ which proves Theorem $\mathbb{Z}_{6}$.

Proof: If $x \in \operatorname{Sing}_{a(\text { resp. b) }}\left(V_{j}, V_{i}\right)$, then by the construction of $\mathcal{P}^{a(\text { resp. } b)}$, for some $\epsilon>0$ there is a sequence $\left\{y_{n}\right\} \subset V_{j}^{\text {con, } x}$ with $\mathcal{P}^{a(\text { resp.b) }}\left(y_{n}\right) \rightarrow \epsilon$. There are two cases:

1) there are different limits: $\mathcal{P}^{a(\text { resp.b) }}\left(y_{n}^{\prime}\right) \rightarrow \epsilon^{\prime}, \mathcal{P}^{a(\text { resp. } b)}\left(y_{n}\right) \rightarrow \epsilon^{\prime \prime}$, and $\epsilon^{\prime} \neq \epsilon^{\prime \prime}$;
2) the limit $\mathcal{P}^{a(\text { resp.b })}\left(y_{n}\right)$ is unique, positive, and independent of $\left\{y_{n}\right\}$.

Consider case 1). By Sard's lemma there is a regular value $\epsilon^{*} \in\left(\epsilon^{\prime}, \epsilon^{\prime \prime}\right)$ of $\mathcal{P}^{a(\text { resp.b) }}$. By the rank theorem $V_{j}^{\epsilon^{*}}=\left(\mathcal{P}^{a(\text { resp.b })}\right)^{-1}\left(\epsilon^{*}\right)$ is a smooth semivariety of codimension 1 in $V_{j}$. Let's show that $x \in \overline{V_{j}^{\epsilon^{*}}}$. Consider a local connected component $V_{j}^{\text {con,x }}$ and two sequences $\left\{y_{n}^{\prime}\right\}$ and $\left\{y_{n}^{\prime \prime}\right\}$ in $V_{j}^{\text {con, } x}$ converging to $x$ such that $\mathcal{P}^{a(\text { resp.b) }}\left(y_{n}^{\prime}\right) \rightarrow \epsilon^{\prime}$ and $\mathcal{P}^{a(\text { resp.b })}\left(y_{n}^{\prime \prime}\right) \rightarrow \epsilon^{\prime \prime}$ as $n \rightarrow \infty$. $\mathcal{P}^{a(\text { resp.b })}$ is continuous and $V_{j}^{\text {con,x }}$ is connected, thus we can connect each $y_{n}^{\prime}$ and $y_{n}^{\prime \prime}$ in $\underline{V}_{j}^{\text {con,x }}$ by a curve and find a sequence $\tilde{y}_{n} \rightarrow x$ for which $\mathcal{P}^{a(\text { resp. } b)}\left(\tilde{y}_{n}\right)=\epsilon^{*}$. Thus $x \in \overline{V_{j}^{\epsilon^{*}}}$. Consider a countable dense set $\left\{\epsilon_{p}\right\}_{p \in \mathbb{Z}_{+}}$in $[0, k+1]$ of regular values of $\mathcal{P}^{a(\text { resp. } b)}$ so that for any two $\epsilon^{\prime} \neq \epsilon^{\prime \prime}$, there is a separating $\epsilon_{p} \in\left(\epsilon^{\prime}, \epsilon^{\prime \prime}\right)$. Define $V_{j}^{p}=\left(\mathcal{P}^{a(\text { resp. } b)}\right)^{-1}\left(\epsilon_{p}\right)$. Thus any $b$-irregular point $x$ is in the closure of the union $\cup_{p \in \mathbb{Z}_{+}} V_{j}^{p}$. After consideration of case 2), we will prove that $V_{j}^{p}$ is $b$-irregular over $V_{i}$ at those $x$.

Consider case 2). By Lemma 1 in this case if $x \in \operatorname{Sing}_{a(\text { resp. b) }}\left(V_{j}, V_{i}\right)$, then $x$ belongs to $V_{i} \backslash U n_{a}\left(V_{j}, V_{i}\right)$. Therefore, there are two sequences $\left\{y_{n}^{0}\right\},\left\{y_{n}^{1}\right\}$ in a local connected component $V_{j}^{\text {con, } x}$ tending to $x$ such that $T_{y_{n}^{0}} V_{j} \rightarrow \tau_{0}, T_{y_{n}^{1}} V_{j} \rightarrow \tau_{1}$, and $\tau_{0} \neq \tau_{1}$. Choose an orientation of $T_{y_{0}^{0}} V_{j}$. By connecting $y_{0}^{0}$ locally with all other points $\left\{y_{n}^{s}\right\}$ one can induce an orientation on all other $T_{y_{n}^{s}} V_{j}$ so that the orientations of $\tau_{0}$ and $\tau_{1}$ coincide with the orientations of the limits.

Denote $\operatorname{dim} V_{j}$ by $k$. There is an orientable $(m-k)$-plane $l_{j}$ separating $\tau_{0}$ and $\tau_{1}$ and transversal to $V_{j}$ (by the transversality theorem). Consider the orthogonal projection $\pi_{l_{j}}$ along $l_{j}$ onto its orthogonal complement $l_{j}^{\perp}$. Denote by $p_{l_{j}, j}$ its restriction to $V_{j}$, $p_{l_{j}, j}=\left.\pi_{l_{j}}\right|_{V_{j}}: V_{j} \rightarrow l_{j}^{\perp}$. Denote by $\operatorname{Crit}\left(l_{j}, V_{j}\right)$ the set of critical points of $p_{l_{j}, j}$ in $V_{j}$ where the rank of $p_{l_{j}, j}$ is not maximal. Then $\operatorname{Crit}\left(l_{j}, V_{j}\right)$ is a semivariety in $V_{j}$ and $\operatorname{dim} \operatorname{Crit}\left(l_{j}, V_{j}\right)<\operatorname{dim} V_{j}$. Connect two points $y_{n}^{0}$ and $y_{n}^{1}$ by a curve in $V_{j}$, then $T_{y_{n}^{0}} V_{j}$ deformates continuously to $T_{y_{n}^{0}} V_{j}$. Then by Rolle's Lemma there is a critical point of $p_{l_{j}, j}$ in $V_{j}^{\text {con }, x}$ arbitrarily close to $x$. Thus $x \in \overline{\operatorname{Crit}\left(l_{j}, V_{j}\right)}$.

By the transversality theorem there is a countable dense set of orientable $(m-k)$-planes $\left\{l_{j}^{r}\right\}_{r \in \mathbb{Z}_{+}}$transversal to $V_{j}$ and separating any two distinct orientable $k$-planes $\tau_{0}$ and $\tau_{1}$. Therefore, we have

$$
\begin{equation*}
V_{i} \backslash U n_{a}\left(V_{j}, V_{i}\right) \subset \bigcup_{r \in \mathbb{Z}_{+}}\left\{\overline{\operatorname{Crit}\left(l_{j}^{r}, V_{j}\right)} \backslash \operatorname{Crit}\left(l_{j}^{r}, V_{j}\right)\right\} . \tag{8}
\end{equation*}
$$

[^2]By lemma 11 we know that $V_{i} \backslash U n_{a}\left(V_{j}, V_{i}\right)$ is a semivariety. We know that $\operatorname{Crit}\left(l_{j}, V_{j}\right)$ $\subset V_{j}$ is a semivariety and $\operatorname{dim} \operatorname{Crit}\left(l_{j}, V_{j}\right)<\operatorname{dim} V_{j}$. Thus we can decompose it into strata $\operatorname{Crit}\left(l_{j}, V_{j}\right)=\bigsqcup_{p \in L_{j}} V_{j}^{p}$. Renumerate these $V_{j}^{p}$,s to have $\left\{V_{j}^{p}\right\}_{p \in \mathbb{Z}_{-}}$.

Consider strata $\left\{V_{j}^{p}\right\}_{p \in \mathbb{Z}} \subset V_{j}$ which we constructed in the cases 1 and 2. Then

$$
\begin{equation*}
\operatorname{Sing}_{a(\text { resp. b) }}\left(V_{j}, V_{i}\right) \subset \bigcup_{p \in \mathbb{Z}}\left\{\overline{V_{j}^{p}} \backslash V_{j}^{p}\right\} \tag{9}
\end{equation*}
$$

The definitions of $\mathcal{P}^{a(\text { resp.b) }}$ and $\pi_{j, s}$ explicitly imply that (7) is satisfied, because $\mathcal{P}^{a(\text { resp.b) }}\left(y_{n}\right)$ has a positive limit point for any $\left\{y_{n}\right\} \subset V_{j}^{p}$. If one projects along a smaller plane $\left(T_{y_{n}} V_{j}^{p} \subset T_{y_{n}} V_{j}\right)$, then the size of the projection is larger. Thus for the Kuo map $\mathcal{P}_{j^{p}, i}^{a(\text { resp.b) }}: V_{j}^{p} \rightarrow \mathbb{R}$, defined in (5), the sequence $\mathcal{P}_{j^{p}, i}^{a(\text { resp.b) }}\left(y_{n}\right)$ also has a positive limit point. Now to separate interior and boundary points of the closures $\overline{V_{j}^{p}}$ in $V_{i}$ define the set $V_{i}^{p}=\left(V_{i} \cap \overline{V_{j}^{p}}\right) \backslash I n t_{V_{i}}\left(V_{i} \cap \overline{V_{j}^{p}}\right)$. This completes the proof of the lemma and Theorem 2. Q.E.D.

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    ${ }^{1}$ This way of defining $b$-regularity is due to Mather Ma. Whitney's definition Wh is equivalent to this one provided of $a$-regularity

[^1]:    ${ }^{2}$ This was first noticed by J.Mather Ma

[^2]:    ${ }^{3}$ one can show that this case is impossible

