A GEOMETRIC PROOF OF THE EXISTENCE OF WHITNEY STRATIFICATIONS

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1. Introduction

A stratification of a set, e.g. an analytic variety, is, roughly, a partition of it into manifolds so that these manifolds fit together “regularly”. Stratification theory was originated by Thom and Whitney for algebraic and analytic sets. It was one of the key ingredients in Mather’s proof of the topological stability theorem [Ma] (see [GM] and [PW] for the history and further applications of stratification theory).

In this paper, given a partition of a singular set (which we know always exists), we prove that there is a “regular” partition. Our proof is based on a remark that if there are two parts of the partition $V$ and $W$ of different dimension and $V \subset W$, then irregularity of the partition at a point $x$ in $V$ corresponds to the existence of nonunique limits of tangent planes $T_y W$ as $y$ approaches $x$.

Consider either the category of (semi)analytic (or (semi)algebraic) sets. Call a subset $V \subset \mathbb{R}^m$ (or $\mathbb{C}^m$) a semivariety if locally at each point $x \in \mathbb{R}^m$ (or $\mathbb{C}^m$) it is a finite union of subsets defined by equations and inequalities

$$ f_1 = \cdots = f_k = 0 \quad \left\{ \begin{array}{ll} g_1 \neq 0, \ldots, g_l \neq 0 & \text{(complex case),} \\ g_1 > 0, \ldots, g_l > 0 & \text{(real case),} \end{array} \right.$$  

where $f_i$’s and $g_j$’s are real (or complex) analytic (or algebraic) depending on the case under consideration.

In the real algebraic case semivarieties are usually called semialgebraic sets; in the complex algebraic case they are called constructible, and in either analytic case they are called semianalytic sets. Semivarieties are closed under Boolean operations.

**Definition 1.** (Whitney) Let $V_i, V_j$ be disjoint manifolds in $\mathbb{R}^m$ (or $\mathbb{C}^m$), dim $V_J >$ dim $V_I$, and let $x \in V_i \cap \overline{V_j}$. A triple $(V_j, V_i, x)$ is called a (resp. $b$)-regular if

A) when a sequence $\{y_n\} \subset V_j$ tends to $x$ and $T_{y_n} V_j$ tends in the Grassmanian bundle to a subspace $\tau_x$ of $\mathbb{R}^m$ (or $\mathbb{C}^m$), then $T_x V_i \subset \tau_x$;

B) when sequences $\{y_n\} \subset V_j$ and $\{x_n\} \subset V_i$ each tend to $x$, the unit vector $(x_n - y_n)/|x_n - y_n|$ tends to a vector $v$, and $T_{y_n} V_j$ tends to $\tau_x$, then $v \in \tau_{x}^R$.

$V_j$ is called a (resp. $b$)-regular over $V_i$ if each triple $(V_j, V_i, x)$ is a (resp. $b$)-regular.

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1This way of defining $b$-regularity is due to Mather [Ma]. Whitney’s definition [Wh] is equivalent to this one provided of $a$-regularity.
Definition 2. (Whitney) Let $V$ be a semivariety in $\mathbb{R}^m$ (or $\mathbb{C}^m$). A disjoint decomposition

$$V = \bigsqcup_{i \in I} V_i, \quad V_i \cup V_j = \emptyset \quad \text{for} \quad i \neq j$$

into smooth semivarieties $\{V_i\}_{i \in I}$, called strata, is called an a (resp. b)-regular stratification if

1. each point has a neighborhood intersecting only finitely many strata;
2. the frontier $\overline{V}_j \setminus V_j$ of each stratum $V_j$ is a union of other strata $\bigsqcup_{i \in J(j)} V_i$;
3. any triple $(V_j, V_i, x)$ such that $x \in V_i \subset \overline{V}_j$ is a (resp. b)-regular.

Theorem 1. [Wh, Fl, Lo] For any semivariety $V$ in $\mathbb{R}^m$ (or $\mathbb{C}^m$) there is an a (resp. b)-regular stratification.

The existence of stratifications in the complex analytic case was proved by Whitney [Wh]. Later Thom published a sketch of a proof [Th]. Then Lojasiewicz [Lo] extended these results to the semianalytic case. The most illuminating proof is due to Wall [Wa], where based on Milnor’s curve selection lemma [Mi] he simplifies the above proofs. Hironaka [Hi] gave an elegant proof using his resolution of singularities, but it requires background in algebraic geometry. We give a geometric proof based on Milnor’s curve selection lemma [Mi], [Wa], Rolle’s lemma, and a transversality theorem. The rest of the paper is devoted to this proof.

Proof of theorem 1: A semivariety $V$ has well-defined dimension, say $d \leq m$. Denote by $V_{\text{reg}}$ the set of points, where $V$ is locally a real (or complex) analytic submanifold of $\mathbb{R}^m$ (or $\mathbb{C}^m$) of dimension $d$. $V_{\text{reg}}$ is a semivariety, moreover, $V_{\text{sing}} = V \setminus V_{\text{reg}}$ is a semivariety of positive codimension in $V$, i.e. $\dim V_{\text{sing}} < \dim V$. In the analytic case all these results may be found in Lojasiewicz [Lo]; in the algebraic case they are not difficult (see e.g. [Mi]).

Step 1. There is a filtration of $V$ by semivarieties

$$V^0 \subset V^1 \subset \cdots \subset V^d = V,$$

where for each $k = 1, \ldots, d$ the set $V^k \setminus V^{k-1}$ is a manifold of dimension $k$. This follows from the Lojasiewicz result. Indeed, consider $V_{\text{sing}} \subset V$, then $V \setminus V_{\text{sing}}$ is a manifold of dimension $d$ and $\dim V_{\text{sing}} < d$. Inductive application of these arguments completes the proof.

A refinement of a decomposition $V = \bigsqcup_{i \in I} V_i$ is a decomposition $V = \bigsqcup_{i' \in I'} V_{i'}$ such that any stratum $V_j$ of the first decomposition is a union of some strata of the second one, i.e. there is a set $I'(j) \subset I'$ such that $V_j = \bigsqcup_{i' \in I'(j)} V_{i'}$.

Step 2. Let $V \subset \mathbb{R}^m$ (or $\mathbb{C}^m$) be a manifold and $W \subset V$ be a semivariety. Denote by $\text{Int}_V(W)$ the set of interior points of $W$ in $V$ w.r.t. the induced from $\mathbb{R}^m$ (resp. $\mathbb{C}^m$) topology. Let $V_i$ and $V_j$ be a pair of distinct strata. For each point $x \in V_i \cap V_j$ denote by $V_{j, x}^{\text{con}}$ a local connected component of $V_j$ at $x$, i.e. a connected component of intersection of $V_j$ with a ball centered at $x$ and call it essential if the closure of $V_{j, x}^{\text{con}}$ has $x$ in the interior, $x \in \text{Int}_{V_i}(V_i \cap V_{j, x}^{\text{con}})$. Denote by $V_{j, x}^{\text{ess}}$ the union of all local essential components of $V_j$. Lojasiewicz [Lo] showed that $V_j$ has only a finitely many local connected components.
Theorem 2. For any two disjoint strata $V_j$ and $V_i$ the set of points 

$$\text{Sing}_{a(\text{resp. } b)}(V_j, V_i) = \{x \in V_i \cap \overline{V_j} : (V_j^{\text{ess}}, x) \text{ is not } a(\text{resp. } b) \text{- regular}\},$$

is a semivariety in $V_i$ and $\dim \text{Sing}_{a(\text{resp. } b)}(V_j, V_i) < \dim V_i$.

Let us show that this theorem is sufficient to prove Theorem 1. Consider a decomposition $V = \bigsqcup_{i \in I} V_i$ and split the strata into two groups: the first group consists of strata of dimension at least $k$ and the second group is of the rest. Suppose that each stratum from the first group is $a(\text{resp. } b)$-regular over each stratum from the second group. Then by definition of $a(\text{resp. } b)$-regularity any refinement of a stratum from the second group preserves this $a(\text{resp. } b)$-regularity.

Now apply this refinement inductively. Consider strata in $V^d \setminus V^{d-1}$ of dimension $d$. Using Theorem 2 and the result of Lojasiewicz [Lo] that a frontier of a semivariety has $d$-dimensional strata, we define a $d$-dimensional semivariety to the same problem for $(d - 1)$-dimensional semivarieties. Induction on dimension completes the proof of Theorem 1.

Our proof is based on the observation that if $V_i \subset \overline{V_j}$ are a pair of strata $a(\text{resp. } b)$-regularity of $V_j$ over $V_i$ at $x$ in $V_i$ is closely related to whether the limit of tangent planes $T_y V_j$ is unique or not as $y$ from $V_j$ tends to $x$. The rest of the paper is devoted to the proof of Theorem 2 which consists of two steps. In section 1.1 we relate $a(\text{resp. } b)$-regularity with (non)uniqueness of limits of tangent planes $T_y V_j$, then based on it and Rolle’s lemma in section 1.2 we prove Theorem 2.

1.1. The key definitions. Let $V_i$ and $V_j$ be a pair of distinct strata. Define

$$Un_a(V_j, V_i) = \{x \in V_i \cap \overline{V_j} : \text{for any } V_j^{\text{con}}, \text{ there exists } \tau_x \subset T_x \mathbb{R}^m \text{ (resp. } T_x \mathbb{C}^m) \text{ such that for any } \{y_n\} \subset V_j^{\text{con}} \text{ tending to } x, \text{ } T_{y_n} V_j \to \tau_x\},$$

(4)

Since $a(\text{resp. } b)$-regularity is a local property, w.l.o.g. we can assume that locally $V_i$ is an $s$-plane with a basis of unit vectors $e_1, \ldots, e_s$. Using an idea of Kuo [Ku] (see also [Wa]) we define a Kuo map $P^{a(\text{resp. } b)} : V_j \to \mathbb{R}$ which measures non $a(\text{resp. } b)$-regularity in terms of an angle between a vector or a plane and the tangent plane to $V_j$. Denote by $\pi_i : \mathbb{R}^m \to V_i^\perp$ (resp. $\pi_{i^\perp} : \mathbb{R}^m \to V_i$) the orthogonal projection along $V_i$ (resp. $V_i^\perp$) onto the complement $V_i^\perp$ (resp. $V_i$) with $x$ being the origin of $\mathbb{R}^m$ and by

$$\pi_j : V_j \times \mathbb{R}^m \to \mathbb{R}^m, \text{ } \pi_{j,t} : V_j \to \mathbb{R}^m \text{ for } t = 1, \ldots, s + 1 \text{ defined by}$$

$$\pi_j(y, v) = \pi_{T_y V_j}(v), \text{ } \pi_{j,t}(y) = \pi_j(y, e_t), \text{ } \pi_{j,s+1}(y) = \pi_j(y, \pi_i(y)/|\pi_i(y)|),$$

(5)

where $\pi_j(y, v)$ is the orthogonal projection of $v$ along the tangent plane $T_y V_j$ to $V_j$ at $y$ naturally embedded into $\mathbb{R}^m$. Define analytic functions $P^{a(\text{resp. } b)} : V_j \to \mathbb{R}$ by $P^a(y) = \sum_{t=1}^s |\pi_{j,t}(y)|^2$ (resp. $P^b(y) = \sum_{t=1}^{s+1} |\pi_{j,t}(y)|^2$). By the definition the level sets of $P^{a(\text{resp. } b)}$ are semivarieties.

Notice that the first $s$ terms of the function $P^a(y)$ measure the angle between $T_y V_i = V_i$ and $T_y V_j$ and the last term measures the angle between the $V_i^\perp$-component of $(y-x)/|y-x|$...
and $T_y V_j$. Since any vector can be decomposed into $V_i$ and $V_i^⊥$ components, this proves the following

**Fact 1.** For any pair distinct strata $V_j$ and $V_i$ existence of a sequence $\{y_n\} \subset V_j$ tending to $x$ with a nonzero limit of $\mathcal{P}^{a(\text{resp. } b)}(y_n)$ is equivalent to $a(\text{resp. } b)$-irregularity of $V_j$ over $V_i$ at $x$.

$$Un_b(V_j, V_i) = \{x \in Un_a(V_j, V_i) : \text{for any } V_j^{\text{con}, x}, \text{ there exists } \epsilon \in \mathbb{R} \text{ such that for any } \{y_n\} \subset V_j^{\text{con}, x} \text{ tending to } x, \mathcal{P}^b(y_n) \rightarrow \epsilon\},$$

(6)

**Lemma 1.** Let $V_i$ and $V_j$ be a pair of disjoint strata in $\mathbb{R}^m$ (or $\mathbb{C}^m$) with $V_i \cap \overline{V_j} \neq \emptyset$. Then $Sing_a(\text{resp. } b)(V_j, V_i)$ and $Un_a(\text{resp. } b)(V_j, V_i)$ are semivarieties and

$$Sing_a(V_j, V_i) \subset Sing_b(V_j, V_i), \quad Sing_a(\text{resp. } b)(V_j, V_i) \subset V_i \setminus Un_a(\text{resp. } b)(V_j, V_i).$$

**Remark 1.** The new result here is that $Sing_a(\text{resp. } b)(V_j, V_i) \subset V_i \setminus Un_a(\text{resp. } b)(V_j, V_i)$. The other inclusion may be found in [WL], [Ma], [Lo].

**Proof:** Let’s first prove that $Sing_a(V_j, V_i)$ is a semivariety. Consider $V_i \times TV_j = \{(x, y, T_y V_j) : x \in V_i, y \in V_j\}$. It is a semivariety in an appropriate Grassmanian bundle over $\mathbb{R}^m \times \mathbb{R}^m$ (resp. $\mathbb{C}^m \times \mathbb{C}^m$) and so is its closure. The condition $T_x V_i \not\subset \tau_x$ is semialgebraic and a projection of a semivariety is a semivariety. In the real (resp. complex) algebraic case it is called the Tarski-Seidenberg Principle [La] (resp. elimination theory [Mu]). In the real analytic case it depends on a generalization due to Lojasiewicz [Lo] to varieties analytic in some variables and algebraic in others. In the complex analytic case, a proof may be found in [WL]. Similar arguments show $Sing_b(V_j, V_i)$ and $Un_a(\text{resp. } b)(V_j, V_i)$ are semivarieties.

Now let’s see that $Sing_a(V_j, V_i) \subset Sing_b(V_j, V_i)$. For any sequence $\{y_n\} \subset V_j$ such that $T_{y_n} V_j$ has a limit $\tau_x$ as $y_n$ tends to $x$ and any $v \in T_x V_i$ there is a sequence $\{x_n\} \subset V_i$ such that $x_n$ tends to $x$ slower than the sequence $\{y_n\}$, i.e. $|y_n - x|/|x_n - y_n| \rightarrow 0$ and the unit vectors $(x_n - y_n)/|x_n - y_n|$ tends to $v$ as $n \rightarrow \infty$ If $x \not\in Sing_b(V_j, V_i)$, then $v$ belongs to $\tau_x$. Since any $v \in T_x V_i$ belongs to $\tau_x$, $T_x V_i$ also belongs to $\tau_x$.

To see that $Sing_a(V_j, V_i) \subset V_i \setminus Un_a(V_j, V_i)$, suppose $x \in Sing_a(V_j, V_i) \cap Un_a(V_j, V_i)$. Fix an $a$-irregular essential local connected component $V_j^{\text{con}, x}$ of $V_j$ at $x$. There is a dim $V_j$-plane $\tau_x$ such that for any sequence $\{y_n\} \subset V_j^{\text{con}, x}$ tending to $x$ we have $T_{y_n} V_j \rightarrow \tau_x$. Since $x \in Sing_a(V_j, V_i)$, we have $T_x V_i \not\subset \tau_x$, i.e. there is a unit vector $v \in T_x V_i$ which has a positive angle with $\tau_x$, i.e. $<(v, \tau_x) = 2\delta > 0$. Denote by $C_{\delta, v}(x) = \{y \in \mathbb{R}^m : (\frac{y - x}{|y - x|}, v) > 1 - \delta\}$ the $\delta$-cone around $v$ centered at $x$ and by $l_v(x)$ the ray starting at $x$ in the direction of $v$. The intersection $V_j^{\text{con}, x} \cap C_{\delta, v}(x) = V_j^{\text{con}, x} \cap l_v(x)$ is a semivariety and $l_v(x)$ is in its closure. By the Lojasiewicz result $V_j^{\text{con}, x}$ consists of a finite number of connected components. So one can choose a connected component $W_{j, \delta, v}^{\text{con}, x} = V_j^{\text{con}, x} \cap l_v(x)$ which contains $l_v(x)$ in the closure. By Milnor’s curve selection lemma [Ma], [Wa] there is an analytic curve $\gamma$ which belongs to $W_{j, \delta, v}^{\text{con}, x} \cup \{x\}$. Since $\gamma$ is analytic, it has a limiting tangent vector $w$ at $x$.

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2 This was first noticed by J. Mather [Ma]
which is by our construction should belong to $\tau_x$ and $C_{\delta,v}(x)$. This is a contradiction with $<v,\tau_x> = 2\delta$.

To see that $\text{Sing}_b(V_j, V_i) \subset V_i \setminus U_{\delta b}(V_j, V_i)$ it is sufficient to prove that $\text{Sing}_b(V_j, V_i) \cap U_{\delta a}(V_j, V_i) \subset V_i \setminus U_{\delta b}(V_j, V_i)$. Let $x \in \text{Sing}_b(V_j, V_i) \cap U_{\delta a}(V_j, V_i)$ and $V_j^{\text{con},x}$ be a $b$-irregular essential local connected component at $x$. Since $x \in U_{\delta a}(V_j, V_i)$, there is a unique limiting tangent plane $\tau_x = \lim_{y \to x} T_{y*} V_j$ independent of $\{y_n\} \subset V_j^{\text{con},x}$ tending to $x$ and by the previous passage $x$ is $a$-regular, i.e. $V_i \subset \tau_x$. By Fact 1 and $b$-irregularity of $x$ there is a sequence $\{y_n\} \subset V_j^{\text{con},x}$ such that $|P^b(y_n)| \to 2\delta \neq 0$. Let’s prove existence of a sequence $\{y_n\} \subset V_j^{\text{con},x}$ such that $|P^b(y_n)| \to \epsilon < \delta$ which shows that $x \notin U_{\delta b}(V_j, V_i)$.

For each $\tilde{x} \in V_i$ close to $x$ consider the “level” set $V_j^{\text{con},x}(\tilde{x}) = V_j^{\text{con},x} \cap (V_i + \{\tilde{x}\})$ over $\tilde{x}$. Transversality of $\tau_x$ with $V_i^\perp$ and uniqueness of $\lim T_{y*} V_j^{\text{con},x}$ imply that $V_j^{\text{con},x}(\tilde{x})$ is a manifold and $\tau_j(y) = T_{y*} V_j \cap V_i^\perp$ depends continuously on $y$ in $V_j^{\text{con},x}$. Consider the set of $\tilde{x} \in V_i$ for which have the corresponding “level” set $V_j^{\text{con},x}(\tilde{x})$ has $\tilde{x}$ in the closure, i.e. $\tilde{x} \in V_j^{\text{con},x}(\tilde{x})$. Since $V_j^{\text{con},x}$ is essential, the set of such $\tilde{x}$’s is everywhere dense in a neighborhood of $x$ in $V_i$. Moreover, the “angle” function $P^b$ is bounded in absolute value by $\delta$ on each local connected “level” component of $V_j^{\text{con},x}(\tilde{x})$ having $\tilde{x}$ in its closure. Thus, one can find a sequence of points $\{y_n\} \subset V_j^{\text{con},x}$ tending to $x$ each point $y_n$ of which belongs to a “level” connected component of $V_j^{\text{con},x}(\pi^1_i(y_n))$, having $\pi^1_i(y_n) \in V_i$ in the closure. By construction $|P^b(y_n)| < \delta$ for all $n$. Q.E.D.

1.2. Separation of Planes. Consider the real case. The complex case can be done in a similar way. Let $\tau_0$ and $\tau_1$ be two distinct orientable $k$-dimensional planes in $\mathbb{R}^m$. An orientable $(m-k)$-dimensional plane $l$ in $\mathbb{R}^m$ separates $\tau_0$ and $\tau_1$ if $l$ is transversal to $\tau_0$ and $\tau_1$ and the orientations induced by $\tau_0 + l$ and $\tau_1 + l$ in $\mathbb{R}^m$ are different. Notice that there always exists an open set of orientable $(m-k)$-planes separating any two distinct orientable $k$-plane.

Rolle’s Lemma. If a continuous family of orientable $k$-planes $\{\tau_t\}_{t \in [0,1]}$ connects $\tau_0$ and $\tau_1$ and an orientable $(m-k)$-plane $l$ separates $\tau_0$ and $\tau_1$. Then for some $t^* \in (0,1)$ transversality of $\tau_{t^*}$ and $l$ fails.

In what follows we use the transversality theorem [GM] which says : if $V \subset \mathbb{R}^m$ is a manifold, then almost every plane of dimension $k$ is transversal to $V$.

1.3. A reduction lemma.

Lemma 2. Let $V_j$ and $V_i$ be a distinct strata and $\dim V_j > \dim V_i$. Then there is a set of strata $\{V_j^p\}_{p \in \mathbb{Z}}$ (resp. $\{V_i^p\}_{p \in \mathbb{Z}}$) in $V_j$ (resp. in $V_i$) each of positive codimension in $V_j$ (resp. in $V_i$) such that

$\text{Sing}_a(\text{resp. } b)(V_j, V_i) \subset \bigcup_{p \in \mathbb{Z}} \text{Sing}_a(\text{resp. } b)(V_j^p, V_i) \bigcup V_i^p$.

Remarks. 1. Inductive application of this lemma to the right-hand side of (3) reduces dimensions of $V_j^p$’s up to $\dim V_i$.

2. By the result of Lojasiewicz [L] dimension of the frontier of a semivariety $(\text{Sing}_a(\text{resp. } b)(V_j^p, V_i) \subset V_i \cap \overline{V_j^p})$ has dimension strictly smaller that a semivariety itself.
3. By lemma 2 the set \( \text{Sing}_a(\text{resp. } b)(V_j, V_i) \) is a semivariety. Since a countable union of semivarieties of positive codimension in \( V_i \) contains \( \text{Sing}_a(\text{resp. } b)(V_j, V_i) \), \( \text{Sing}_a(\text{resp. } b)(V_j, V_i) \) has a positive codimension in \( V_i \) which proves Theorem 2.

**Proof:** If \( x \in \text{Sing}_a(\text{resp. } b)(V_j, V_i) \), then by the construction of \( p^{\text{a}(\text{resp. } b)} \), for some \( \epsilon > 0 \) there is a sequence \( \{y_n\} \subset V_j^{\text{con}, x} \) with \( p^{\text{a}(\text{resp. } b)}(y_n) \to \epsilon \). There are two cases:

1) there are different limits: \( p^{\text{a}(\text{resp. } b)}(y_n') \to \epsilon' \), \( p^{\text{a}(\text{resp. } b)}(y_n) \to \epsilon'' \), and \( \epsilon' \neq \epsilon'' \);

2) the limit \( p^{\text{a}(\text{resp. } b)}(y_n) \) is unique, positive, and independent of \( \{y_n\} \).

Consider case 1). By Sard’s lemma there is a regular value \( \epsilon^* \in (\epsilon', \epsilon'') \) of \( p^{\text{a}(\text{resp. } b)} \). By the rank theorem \( V_j^{\epsilon^*} = (p^{\text{a}(\text{resp. } b)})^{-1}(\epsilon^*) \) is a smooth semivariety of codimension 1 in \( V_j \). Let’s show that \( x \in V_j^{\epsilon^*} \). Consider a local connected component \( V_j^{\text{con}, x} \) and two sequences \( \{y_n'\} \) and \( \{y_n''\} \) in \( V_j^{\text{con}, x} \) converging to \( x \) such that \( p^{\text{a}(\text{resp. } b)}(y_n') \to \epsilon' \) and \( p^{\text{a}(\text{resp. } b)}(y_n'') \to \epsilon'' \) as \( n \to \infty \). \( p^{\text{a}(\text{resp. } b)} \) is continuous and \( V_j^{\text{con}, x} \) is connected, thus we can connect each \( y_n' \) and \( y_n'' \) in \( V_j^{\text{con}, x} \) by a curve and find a sequence \( \tilde{y}_n \to x \) for which \( p^{\text{a}(\text{resp. } b)}(\tilde{y}_n) = \epsilon^* \). Thus \( x \in V_j^{\epsilon^*} \). Consider a countable dense set \( \{\epsilon_p\}_{p \in \mathbb{Z}^+} \) in \([0, k + 1]\) of regular values of \( p^{\text{a}(\text{resp. } b)} \) so that for any two \( \epsilon' \neq \epsilon'' \), there is a separating \( \epsilon_p \in (\epsilon', \epsilon'') \). Define \( V_j^p = (p^{\text{a}(\text{resp. } b)})^{-1}(\epsilon_p) \). Thus any \( b \)-irregular point \( x \) is in the closure of the union \( \cup_{p \in \mathbb{Z}^+} V_j^p \). After consideration of case 2), we will prove that \( V_j^p \) is \( b \)-irregular over \( V_i \) at those \( x \).

Consider case 2). By Lemma 3 in this case if \( x \in \text{Sing}_a(\text{resp. } b)(V_j, V_i) \), then \( x \) belongs to \( V_i \setminus U_{na}(V_j, V_i) \). Therefore, there are two sequences \( \{y_n^0\}, \{y_n^1\} \) in a local connected component \( V_j^{\text{con}, x} \) tending to \( x \) such that \( T_{y_0}V_j \to \tau_0 \), \( T_{y_1}V_j \to \tau_1 \), and \( \tau_0 \neq \tau_1 \). Choose an orientation of \( T_{y_0}V_j \). By connecting \( y_0^0 \) locally with all other points \( \{y_n^1\} \) one can induce an orientation on all other \( T_{y_n^1}V_j \) so that the orientations of \( \tau_0 \) and \( \tau_1 \) coincide with the orientations of the limits.

Denote \( \dim V_j \) by \( k \). There is an orientable \((m - k)\)-plane \( l_j \) separating \( \tau_0 \) and \( \tau_1 \) and transversal to \( V_j \) (by the transversality theorem). Consider the orthogonal projection \( \pi_{l_j} \) along \( l_j \) onto its orthogonal complement \( l_j^{\perp} \). Denote by \( p_{l_j} \) its restriction to \( V_j \), \( p_{l_j} | V_j : V_j \to l_j^{\perp} \). Denote by \( \text{Crit}(l_j, V_j) \) the set of critical points of \( p_{l_j} \) in \( V_j \) where the rank of \( p_{l_j} \) is not maximal. Then \( \text{Crit}(l_j, V_j) \) is a semivariety in \( V_j \) and \( \dim \text{Crit}(l_j, V_j) < \dim V_j \). Connect two points \( y_0^0 \) and \( y_1^0 \) by a curve in \( V_j \), then \( T_{y_n^0}V_j \) deforms continuously to \( T_{y_n^1}V_j \). Then by Rolle’s Lemma there is a critical point of \( p_{l_j} \) in \( V_j^{\text{con}, x} \) arbitrarily close to \( x \). Thus \( x \in \text{Crit}(l_j, V_j) \).

By the transversality theorem there is a countable dense set of orientable \((m - k)\)-planes \( \{l_j^r\}_{r \in \mathbb{Z}^+} \) transversal to \( V_j \) and separating any two distinct orientable \( k \)-planes \( \tau_0 \) and \( \tau_1 \). Therefore, we have

\[
(8) \quad V_i \setminus U_{na}(V_j, V_i) \subset \bigcup_{r \in \mathbb{Z}^+} \left( \text{Crit}(l_j^r, V_j) \setminus \text{Crit}(l_j^r, V_j) \right).
\]

\(^3\) One can show that this case is impossible.
By lemma 1 we know that $V_i \setminus \text{Un}(V_j, V_i)$ is a semivariety. We know that $\text{Crit}(l_j, V_j) \subset V_j$ is a semivariety and $\dim \text{Crit}(l_j, V_j) < \dim V_j$. Thus we can decompose it into strata $\text{Crit}(l_j, V_j) = \bigsqcup_{p \in L_j} V^p_j$. Renumerate these $V^p_j$'s to have $\{V^p_j\}_{p \in \mathbb{Z}}$.

Consider strata $\{V^p_j\}_{p \in \mathbb{Z}} \subset V_j$ which we constructed in the cases 1 and 2. Then

(9) \[
\text{Sing}_{a(\text{resp. b})}(V_j, V_i) \subset \bigcup_{p \in \mathbb{Z}} \left\{ V^p_j \setminus V^p_j \right\}.
\]

The definitions of $P_{\text{a(resp.b)}}$ and $\pi_{j,s}$ explicitly imply that (7) is satisfied, because $P_{\text{a(resp.b)}}(y_n)$ has a positive limit point for any $\{y_n\} \subset V^p_j$. If one projects along a smaller plane ($T_{y_n}V^p_j \subset T_{y_n}V_j$), then the size of the projection is larger. Thus for the Kuo map $P_{j^p,i}^{a(\text{resp.b})} : V^p_j \to \mathbb{R}$, defined in (5), the sequence $P_{j^p,i}^{a(\text{resp.b})}(y_n)$ also has a positive limit point. Now to separate interior and boundary points of the closures $\overline{V^p_j}$ in $V_i$ define the set $\overline{V^p_i} = (V_i \cap \overline{V^p_j}) \setminus \text{Int}_{V_i}(V_i \cap \overline{V^p_j})$. This completes the proof of the lemma and Theorem 2. Q.E.D.

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References


[Ja] Jakobson, N.; Basic Algebra, vol. 1, 1974;

[Ku] Kuo, T.-C.; The ratio test for analytic Whitney stratifications, Lecture Notes, No. 192, pp.141-149;


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