Final Exam Answers

1 (a) This is the number of weak compositions of 10 into 3 parts (same as # of solutions to $x_1 + x_2 + x_3 = 10$, $x_i \geq 0$)

Formula: $\binom{10+3-1}{3-1} = \boxed{66}$

(b) Each x_i must be at least 2 so we only need to put $10 - 3 \cdot 2 = 4$ people into the 3 positions. This is weak compositions of 4 into 3 parts:

Formula: $\binom{4+3-1}{3-1} = \boxed{15}$

(c) We can pick one of 11 people to play goalkeeper. There are 10 remaining and each can get to go into one of 3 positions. There are 3^{10} ways to do this.

$\Rightarrow$ ans $= 11 \cdot 3^{10} = \boxed{649539}$

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2 (a) 10 digits, 6 slots $\Rightarrow$ $\boxed{10^6}$

(b) 10 choices of repeated digit.

$\times$ $\binom{6}{3}$ places to put it

$\times$ $9 \cdot 8 \cdot 7$ choices for remaining

3 slots

$= \boxed{100800}$

3) (a) Multinomial thm $\Rightarrow$

$$(x_1 + \cdots + x_m)^n = \sum \binom{n}{k_1 \ldots k_m} x_1^{k_1} \cdots x_m^{k_m}$$

set $x_i = 1$, $i = 1, \ldots, m$

and LHS $= m^n$, RHS $= \sum$ of multinom. coeffs

(b) consider

$$\left(\sum x_i\right)^n = \left(\sum x_i\right)\left(\sum x_i\right)^{n-1}$$

The coeff of $\prod x_i^{k_i}$ on LHS is $\binom{n}{k_1 \cdots k_m}$ and coeff on RHS is $\sum_{i=1}^{m} \binom{n-1}{k_1 \cdots k_i - 1 \cdots k_m}$

(c) $$\sum_j \frac{n!}{j!\,k!\,(n-j-k)!}$$

$$= \frac{n \cdot (n-1) \cdots (n-k+1)}{k!} \sum_j \frac{(n-k)!}{j!\,(n-k-j)!}$$

$$= \binom{n}{k} \cdot 2^{n-k}$$

[Other proofs of these are possible]

(d) $\frac{1}{1-A} = 1 + A + A^2 + A^3 + \cdots$ (4)
and we only need up to A^3
to get all contributions to x^3.
We get 108.

(4) (a) one 1×1 with 1 tree, 4 colors
$\Longrightarrow a_1 = \boxed{4}$

One 2×2 with 1 tree 4 colors
or two 1×1, each with 1 tree, each
with 4 possible colors.
$\Longrightarrow a_2 = 4 + 4\cdot 4 = \boxed{20}$

(b) a_3: 3 1×1s, each with 1 tree,
each in 4 diff colors.
$\Longrightarrow 4\cdot4\cdot4 = 64$
1 1×1, 1 2×2 each with
1 tree, 4 different colors,
2 possible orderings.
$\Longrightarrow 2\cdot4\cdot4 = 32$
1 3×3, 4 colors, 3
different trees
$\Longrightarrow 3\cdot 4 = 12$
total: $12 + 32 + 64 = \boxed{108}$

(c) let $A(x) = \sum_{n\geq1} a_n x^n$ where
a_n = # of ways to put 1
$n\times n$ flag on an n ft
pole
$= n^{n-2}\cdot 4$

So $A(x) = \sum_{n\geq1} 4\cdot n^{n-2} x^n$
Then by the composition formula
for OGFs, the series
$1/1-A$ is the OGF we want.
(d) also

5 (a) $a_0 = 1$

$a_1 = 4^1 + 1 \cdot 3 = 7$

$a_2 = 4^2 + 2 \cdot 4^1 \cdot 3 + 1 \cdot 3 = 43$

$a_3 = 4^3 + 3 \cdot 4^2 \cdot 1 \cdot 3$
$+ 3 \cdot 4^1 \cdot 1 \cdot 3$
$+ 4^0 \cdot 2 \cdot 3 = 250$

This data is arranged as (i) the power of 4 is showing how many practice test choices (ii) the coeff before that is the number of ways to split the students. (iii) the coeff before the 3 is the # of circular arrangements

(b) We can compute an EGF for what we want as $A \cdot B$, where A is EGF for the practice test group and B is the EGF for the discussion gp. So

$A = \sum_{n \geq 0} \frac{a_n}{n!} x^n$ a_n = # of practice test choice if n students take tests $= 4^n$

⑥

$$B = \sum_{n\geq 0} \frac{b_n}{n!} x^n \qquad b_n = 3 \cdot \#\text{ of circular arrangements}$$

$$3(n-1)! = \text{if } n > 0$$

$$b_0 = 1$$

(Note: $b_0 = 1$, not 3. We won't get const term of $A \cdot B$ to be 1 unless $b_0 = 1$, and that's part of the problem. Without $b_0 = 1$ we'll get other coeffs wrong too).

so $$A = \sum_{n\geq 0} \frac{4^n x^n}{n!} = e^{4x}$$

$$B = 1 + 3 \sum_{n\geq 1} x^n/n$$

$$= 1 - 3\log(1-x)$$

so the desired EGF is

$$e^{4x}(1 - 3\log(1-x))$$

$$= 1 + 7x + \frac{43}{2}x^2 + \frac{125}{3}x^3 + \cdots$$

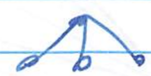
ⓒ coeff of x^3 is $\frac{125}{3}$, so we want $6! \frac{125}{3} = 250$ ✓

6 (a) Here are the trees and |auts|

$n=2$  2

$n=3$  2

$n=4$  2

 6

$n=5$  2

 2

 24

(b) The number of labellings of a graph G of order n is

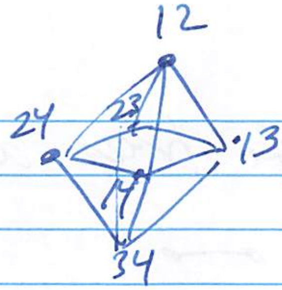
$$l(G) = n!/|\mathrm{aut}\,G|$$

so

$$n^{n-2} = \sum_{T\in\mathcal{T}_n}' l(T) = \sum_{T\in\mathcal{T}_n}' n!/|\mathrm{aut}\,T|$$

$$\Rightarrow n^{n-2}/n! = \sum_{T\in\mathcal{T}_n}' 1/|\mathrm{aut}\,T|$$

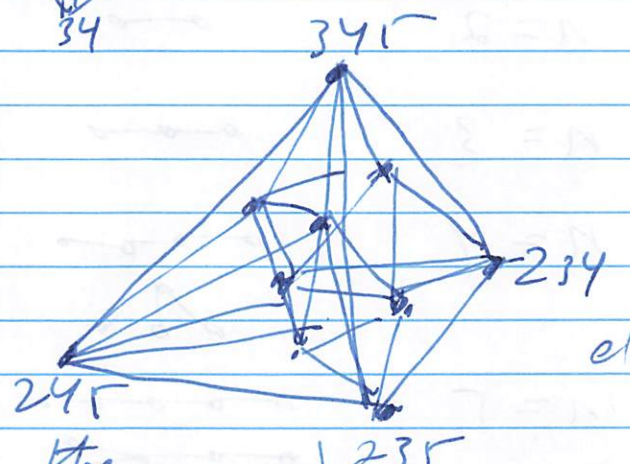
7 (a) $G(2,4)$



(octahedron)

$G(3,5)$

(octahedron inside a tetrahedron) (join opposite Δs on octahedron to the corners of the tetrahedron)



etc.

(b) Given 2 subsets $\{a_1, \dots a_k\}$, $\{b_1, \dots b_k\}$ suppose they have a subset $K = \{c_1, \dots, c_\ell\}$ in common. (This could be empty; doesn't matter) Write them as follows

$a_1, \dots a_{k-\ell}, c_1, \dots, c_\ell, b_1, \dots, b_{k-\ell}$

Then move a "window" over one element at a time:

$\boxed{a_1 \cdots a_{k-\ell}\, c_1 \cdots c_\ell}\; b_1\, b_2 \cdots b_{k-\ell}$

$a_1\, \boxed{a_2 \cdots c_\ell\, b_1}\; b_2 \cdots b_{k-\ell}$

$\vdots$

$a_1 \cdots \boxed{c_1 \cdots c_\ell\, b_1 \cdots b_{k-\ell}}$

7 b $\boxed{\text{cont'd}}$ We get a sequence of verts joined by edges (the verts are labelled by the elements in the boxes).
$\Rightarrow$ $\exists$ a path from I to J.

c Since the graph is regular, we only have to compute the degree of one vertex. Let $I = \{a_1, \dots, a_k\}$. We can move to any of its neighbors by deleting one of $a_1, \dots, a_k$ and adjoining one of $[n] \setminus I$, which is of order $n-k$. ~~Thus~~ There are $\boxed{k(n-k)}$ neighbors and that's the degree.

d We know $\text{size}(G) = \frac{1}{2} \sum_v' \deg(v)$.
There are $\binom{n}{k}$ verts each of degree $k(n-k)$. Thus

$$\text{size } G(k,n) = \boxed{\frac{1}{2} k(n-k) \binom{n}{k}}$$

Remark: This is related to the "Johnson scheme". These graphs are also the edges of polyhedra called "hypersimplices"

8 (a) Unfortunately my graph was missing an edge from top to bottom so I didn't penalize people whose graphs for $n=4$ were missing similar edges.

(b) Let π be the partition of $[n]$ where every part has order 1. Then π is a refinement of any set partition of $[n]$, so there is an edge from every vertex to π. Therefore P_n is connected.

(c) Q_n, $n \geq 4$ is still connected. We no longer have the partition with one part, so each partition has at least 2 parts. Thus any such partition can be refined to a partition with one part of order 2 and the rest of order 1. So we have to show that any 2 such partitions π_1, π_2 can be connected by a path. To see it we form the partition π_3 that has one part the union of the 2 parts of order 2, with the rest order 1, then there is a path

$$\pi_2 \rightarrow \pi_3 \rightarrow \pi_1$$

and Q_n is connected.

9 (a) Each function f determines an ordered set partition of $[\![n]\!]$. The i^{th} part is the inverse image $f^{-1}(i)$. So we just have to count the ordered set partitions of $[\![3]\!]$. There are 5 set partitions

$\{1,2,3\}$, $\{\{12\},\{3\}\}$, $\{\{1\},\{2\},\{3\}\}$
$\{\{13\},\{2\}\}$
$\{\{23\},\{1\}\}$

ordering the parts, we get

$$1 + \begin{matrix} 2 \\ +2 \\ +2 \end{matrix} + 6$$

$$= \boxed{13}.$$

(b) By the composition Formula,

$$A(x) = B(C(x)),$$

where B, C are EGFs that count the following:

(i) $C(x) = \sum_{n \geq 1} \frac{c_n}{n!} x^n$ and c_n counts the set partitions with one block on $[\![n]\!]$.

$B(x) = \sum_{n \geq 0} \frac{b_n}{n!} x^n$ and b_n counts the # of orderings of n things.

We have $c_n = 1$, $b_n = n!$

so $C(x) = e^x - 1$

$$B(x) = 1 + x + x^2 + \cdots$$
$$= \frac{1}{1-x}$$

$$\Longrightarrow B(C(x)) = \frac{1}{1-(e^x-1)}$$

$$= \frac{1}{2-e^x}$$

$$= 1 + x + 3 \cdot \frac{x^2}{2!} + 13 \frac{x^3}{3!} + \cdots$$

so we recover $S_3 = 13$.