M513 Exam II Solutions

① Put A_i, $i=1,\dots,4$ to be the sets of students enrolled in the 4 different subjects. We want $|A_1 \cap \dots \cap A_4| =: x$
I/E tells us $|\bigcup_i A_i| = \sum_i |A_i|$

$$- \sum_{i<j} |A_i \cap A_j|$$
$$+ \sum_{i<j<k} |A_i \cap A_j \cap A_k|$$
$$- x$$

We also know $x + 400 - |\bigcup A_i| = 100$
Plugging in all the numbers give $x = 20$.
Note there is a bug in the problem, which was taken from a textbook:
The number of students in one of the triple intersections is ~~bigger~~ less than 20.

(2) Let $A_p = |\{1 \le x \le 10000 \mid p \text{ divides } x\}|$

We want $\left| [\![10000]\!] \setminus \bigcup_{p=2,7,11} A_p \right|$

If $f(n) := \lfloor 10000/n \rfloor$, then $|A_p| = f(p)$.

So we do i.e. and get

$$10000 - (f(2) + f(7) + f(11)) + (f(2\cdot 7) + f(2\cdot 11) + f(7\cdot 11)) - f(2\cdot 7\cdot 11) = 3896.$$

③ ⓐ $t(0)=1$. Clearly $t(n)=0$ unless $3|n$ because each part is a multiple of 3. Thus we only need to compute $t(n)$ for $n=3,6,9$.

$n=3$: 3 $\Rightarrow t(3)=1$

$n=6$: 6, 33 $\Rightarrow t(6)=2$

$n=9$: 9, 63, 333 $\Rightarrow t(9)=3$.

ⓑ Let D be a Ferrers diagram of a partition contributing to $t(n)$, and let D' be its conjugate. Each row of D is divisible by 3, so each column of D' is divisible by 3. This is exactly the condition that each part appears a multiple of 3 times.

ⓒ We have $t(n)=0$ unless $3|n$, and in that case $t(3n)=p(n)$. We simply multiply each part of any partition contributing to $p(n)$ by 3 to produce one contributing to $t(3n)$. This is a bijection.

④ $a_n = n a_{n-1} + n!$, $a_0 = 1$

the first few a_n's suggest

$a_n = (n+1)!$. Put $A(x) = \sum_{n \geq 0} \frac{a_n x^n}{n!}$.

Then

$$A = 1 + \sum_{n \geq 1} a_n x^n / n!$$

$$= 1 + x \sum_{n \geq 1} \frac{n a_{n-1}}{n!} x^n + \sum_{n \geq 1} x^n$$

$$= 1 + x \cdot A + x/(1-x)$$

$$\Rightarrow A = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + \cdots$$

$$= \sum_{n \geq 0} (n+1) x^n$$

so $a_n / n! = n+1 \Rightarrow a_n = (n+1)!$

(5) $a_n = (n+1)a_{n-1} + 2^{n+1}$, $a_0 = 2$.
We compute

n	0	1	2	3
a_n	2	8	32	144

} Incidentally, you should always do this to check your answer.

Let $A = \sum a_n x^n / n!$, then

$$A = 2 + \underbrace{\sum_{n\geq 1} \frac{n a_{n-1}}{n!} x^n}_{\mathrm{I}} + \underbrace{\sum_{n\geq 1} \frac{a_{n-1}}{n!} x^n}_{\mathrm{II}} + \underbrace{2\sum_{n\geq 1} \frac{2^n x^n}{n!}}_{\mathrm{III}}$$

I : xA
II : $\int A\,dx =: a$
III : $2(e^{2x}-1) = 2e^{2x}-2$.

$\Rightarrow$ $A = xA + a + 2e^{2x}$
or $(1-x)A - a = 2e^{2x}$ Let $y = a$, $\frac{dy}{dx} = A$

$\Rightarrow$ ODE $(1-x)\,dy/dx - y = 2e^{2x}$
or $\frac{d}{dx}\big((1-x)y\big) = 2e^{2x}$

so $y = a = \frac{e^{2x}}{1-x} + \frac{C}{1-x}$.

We need to compute $A = a'$ and then determine C. There are 2 ways to go:

(1) we can diff. these functions then expand into a power series to get a formula for a_n

(2) we can convert these functions to power series and then differentiate.

The second is actually <u>easier</u>. Either way we'll get the same answer.

$$a = e^{2x} \cdot \frac{1}{1-x} + \frac{C}{1-x}$$

$$= \sum_{n\geq 0} \left(\sum_{i=0}^{n} 2^i/i! \right) x^n + \sum_{n\geq 0} C x^n$$

$$\Rightarrow A = a' = \sum_{n\geq 0} n \left(\sum_{i=0}^{n} \frac{2^i}{i!} \right) x^{n-1} + \sum_{n\geq 0} n \cdot C \cdot x^{n-1}$$

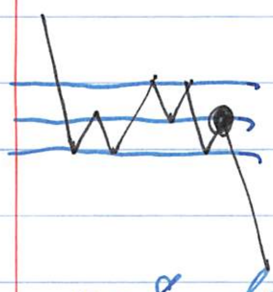
comparing with $A = \sum_{n\geq 0} a_n x^n/n!$ we get

$$a_n = (n+1)! \left(\sum_{i=0}^{n+1} 2^i/i! \right) + (n+1) C$$

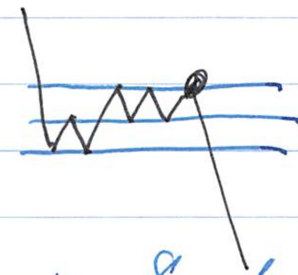
plug in $n=0$ & $a_0 = 2$ to get $C = -1$.
That gives our formula.

(6) (a) $a_3 = 5$, $a_4 = 8$. This suggests we have a shifted Fibonacci sequence: $1, 2, 3, 5, 8, \ldots$

(b) We claim $a_n = a_{n-1} + a_{n-2}$. Given any path, the last reflection is either on the middle line or the outer line (outer = top or bottom depending on the parity of n).



$n=8$, last is middle



$n = 8$, last is outer (= top)

If delete the last reflection when outer, we get an arbitrary element for a_{n-1}. If delete the last reflection when inner we get an arbitrary element for a_{n-2}

$$\Longrightarrow a_n = a_{n-1} + a_{n-2}, \quad n \geq 2.$$

(c) This is a shifted Fib. sequence and we know the G.F. for the usual Fibs.

$$F = \frac{1}{1-x-x^2} = 1 + x + 2x^2 + 3x^3 + 5x^4 + \dots$$

we want $\frac{1}{x}(F-1) = \boxed{\frac{x+1}{1-x-x^2}}$.

7 (a) $a_1 = 0$ (tiles too big)
$a_2 = 2$ (can only use one 1×2)
$a_3 = 4$ (can only use one 1×3)
$a_4 = 4$ (can use 2 1×2's)

(b) The last tile is either 1×2 or 1×3. Delete it to make an arb. tiling from a_{n-2} or a_{n-3}. Since have 2 1×2s and 4 1×3s, we get

(*) $$a_n = 2a_{n-2} + 4a_{n-3} \quad n \geq 3.$$

Note that (*) holds when $n = 2$ if we put $a_1 = 0$ (which it of course is).

We can follow our usual method or apply the theorem stated in class to get the OGF
$$A(x) = \frac{1}{1-2x^2-4x^3}.$$

(c) This is challenging. We know that we can make a P. Frac. expansion
$$\frac{1}{1-2x^2-4x^3} = \frac{P}{1-\alpha x} + \frac{Q}{1-\beta x} + \frac{R}{1-\gamma x} \qquad (**)$$

where P, Q, R are certain constants and α, β, γ are the <u>inverses</u> of the roots of the denom.

(**) $\Rightarrow a_n = P\cdot\alpha^n + Q\cdot\beta^n + R\cdot\gamma^n$

If the answer has the form
$a_n \approx \frac{2}{5} \cdot 2^n$ for n large, then

(i) one of α, β, γ must be 2, say $\alpha = 2$.

(ii) P_n must therefore be $2/5$, and

(iii) β^n, γ^n must be much smaller than 2^n in the limit.

(i) $\Rightarrow (1-2x) \,\big|\, (1-2x^2-4x^3)$ and indeed
$$(1-2x)(1+2x+2x^2) = 1-2x^2-4x^3$$
Inverse Roots of the quadratic poly are (using Q.F.) $\beta = -1+i$, $\gamma = -1-i$

Now $|\alpha| = 2$ and $|\beta|, |\gamma| = \sqrt{2}$, so when n is large the term $P\alpha^n$ dominates. (**) Finally we need P. We can multiply through by $1-2x^2-4x^3$ and plug in $x = 1/2$ to get

$$1 = P\overbrace{(1+2x+2x^2)}^{\neq 0 \,@\, x=\frac{1}{2}} + Q\overbrace{(\cancel{\cdots})}^{0} + R\overbrace{(\cancel{\cdots})}^{0 \,@\, x=\frac{1}{2}}$$

$\Rightarrow P = 2/5 \Rightarrow a_n \approx 2/5 \; 2^n$ when n is large.

(8) (a) $a_1 = 3$, $a_2 = 7$, $a_3 = 17$. We can put $a_0 = 1$ for convenience.

(b) Two ways to proceed:

① We are given that the recurrence has the form

$$a_n = Aa_{n-1} + Ba_{n-2}.$$

we can use data from (a) to solve for $(A,B) = (2,1)$.

② We can consider how words can end in two letters. There are 7 possibilities since $a_2 = 7$. We can group them like this:

_____ A | A
_____ B | A
_____ C | A
_____ A | C
_____ B | B
_____ C | C

any word of length $n-1$ ends in one of A, B, C. So we have 2 ways to extend an arb word of length $n-1$ according to these rules.

_____ A C $\Longleftarrow$ any word of length $n-2$ can have an A then a C appended

$$\Rightarrow \quad a_n = 2a_{n-1} + a_{n-2}.$$

we can solve the recursion using our standard techinque and get

$$A = \sum a_n x^n = {x+1}/{1-2x-x^2}.$$

ⓒ we do partial fractions.

$$A = \frac{P}{1-\alpha x} + \frac{Q}{1-\beta x}$$

where α, β are inverse roots of the denom of A. These are

$$\alpha = 1+\sqrt{2}, \ \beta = 1-\sqrt{2}.$$

so we have a formula of this form

(*) $$a_n = P \cdot (1+\sqrt{2})^n + Q(1-\sqrt{2})^n.$$

we can either find P, Q using standard techniques of Part. Fracs. or can use (*) together with some values of a_n to solve for P, Q. Either way we get

$$P = {\sqrt{2}+1}/{2}, \quad Q = {\sqrt{2}-1}/{2}$$

so $$a_n = \frac{\sqrt{2}+1}{2} \cdot (1+\sqrt{2})^n - \frac{\sqrt{2}-1}{2}(1-\sqrt{2})^n$$