MATH 513 EXAM I

This exam is worth 100 points, with each problem worth 20 points. There are problems on *both sides* of the page. Please complete Problems 1 and 2 and then *any three* of the remaining problems. Unless indicated, you must justify your answer to receive credit for a solution; correct answers alone are not necessarily sufficient for credit.

When submitting your exam, please indicate which problems (including Problems 1 and 2) you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly five problems; any unselected problems will not be graded, and if you select more than five only the first five (in numerical order) will be graded.

You may use a calculator to assist with arithmetic. You can only use the basic functions ($+$, $-$, $\times$, $/$, $=$) of the calculator; no programming is allowed. No phones, smartwatches, or other devices with connectivity can be used during the exam.

Let me know if you find any mistakes in the answers.

(1) (20 pts) **True/False.** Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem.
   - (a) (4 pts) Let X and Y be finite sets. A function $f\colon X \to Y$ is called *onto* or *surjective* if for every $x \in X$, there is a $y \in Y$ such that $f(x) = y$. **Answer:** False. Surjective means for every $y \in Y$ there is an $x \in X$ such that $f(x) = y$.
   - (b) (4 pts) The binomial coefficients satisfy the identity $\sum_{k=0}^{n} \binom{n}{k} = 2^n$ for all integral $n \geq 0$. **Answer:** True.
   - (c) (4 pts) The coefficient of $x^2y^2z^2w^4$ in $(x+y+z+w)^{10}$ is 210. **Answer:** False. The actual coefficient is 18900.
   - (d) (4 pts) The Stirling numbers of the second kind $S(n,k)$ satisfy the recursion $S(n,k) = kS(n-1,k-1) + nS(n-1,k)$. **Answer:** False. This is a corruption of the true recursion.
   - (e) (4 pts) For any two finite sets A, B, we have $|A||B| = |A \times B|$. **Answer:** True.

(2) (20 pts) **Short Answer.** Please give the answer to these short computations. There is no partial credit for this problem, and it is not necessary to show your work for this problem.
   - (a) (4 pts) Compute $(10)_4$. **Answer:** 5040
   - (b) (4 pts) Compute $\binom{10}{4}$. **Answer:** 210
   - (c) (4 pts) Compute $S(10,4)$. **Answer:** 34105. One way is to use the recursion formula. It takes some computation but is doable.
   - (d) (4 pts) Compute the number of *weak compositions* of 10 into 4 parts. **Answer:** $\binom{13}{3} = 286$

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(e) (4 pts) Compute the number of *compositions* of 10 into 4 parts. **Answer:** This is the same as the number of weak compositions of $10 - 4 = 6$ into 4 parts, which is $\binom{9}{3} = 84$.

(3) (20 pts) Consider the 13-letter word *SLEEPLESSNESS*.

(a) (8 pts) Count all possible ways to order the letters in this word. (Same letters are to be considered indistinguishable.) **Answer:** There are 5 Ss, 4 Es, 2 Ls, 1 P, and 1 N. The answer is the multinomial coefficient $\binom{13}{5\ 4\ 2\ 1\ 1} = 1081080$.

(b) (6 pts) Count all possible words of length 6 that can be made from this word. **Answer:** This is a challenging computation, because of all the different possibilities for distinguishable vs. indistinguishable letters. One way to organize it is first fix the number of Ss. There can be $s = 0, \ldots, 5$. Then for each one we can place the Ss, which gives a binomial coefficient $\binom{6}{s}$. This has to be multiplied by the remaining choices. For example for $s = 5$ we have one more letter left to pick so we get $\binom{6}{5} \cdot 4 = 24$ for 5 Ss. It is quite painful to go through all the possibilities. The final answer is 4934.

(c) (6 pts) Count all possible words of length 6 that can be made, if we require that all four Es should be adjacent. (This means that if we have any Es, we must have four and they must be adjacent.) **Answer:** We treat the sequence EEEE as a single character, and divide the cases into no Es and all four Es. The first computation is like what was already done, but not as painful. The second one is much easier. The total is $543 + 42 = 585$.

(4) (20 pts) A small-business owner wants to reward some of his five employees with extra days off. He wants to give away a total of 10 paid holidays to his five workers. No worker is to receive more than five of these holidays. How many choices does the owner have? **Answer:** Let $w(n,k) = \binom{n+k-1}{k-1}$ be the number of weak compositions of n into k parts. If there were no restrictions on the total number of hours a worker could receive, the answer would be $w(10,5) = 1001$. We start with this and take away the number of assignments with a maximum part $m = 10, 9, \ldots, 6$. If a weak composition has m for a maximum part, it must be in a unique position (why?), and there are 5 choices for it. What will be left over is a weak composition of $10 - m$ into 4 parts. So we must subtract $5(w(0,4) + w(1,4) + \cdots + w(4,4))$. The final answer is 651.

(5) (20 pts) Let S be the set of positive integers whose base 10 expansion only has 0 and 1 as digits. Prove that for any $n > 0$, there is an element of S that is divisible by n. (Hint: consider the sequence 1, 11, 111, 1111, ...) **Answer:** Use the pigeonhole principle. Let the boxes be the remainders upon division by n. Then if a_k is the number with k ones, eventually we must have two numbers in the sequence a_k, a_m in the same box with $k > m$. The number $a_k - a_m$ has the desired form and is divisible by n.

(6) (20 pts) Let $X = [\![2n]\!] = \{1, 2, \ldots, 2n\}$. A *pairing* on X is a set partition of X into blocks of order 2.

(a) (6 pts) How many pairings does X have for $n = 1, 2, 3$? **Answer:** 1, 3, 15. For instance for $n = 2$ they are $\{12, 34\}$, $\{13, 24\}$, $\{14, 23\}$.

(b) (7 pts) How many pairings does X for any n? **Answer:** $(2n-1)\cdot(2n-3)\cdots 5\cdot 3\cdot 1$. Let $P(n)$ be the number of pairings on $[\![2n]\!]$. Then $P(n) = (2n-1)\cdot P(n-1)$, since there after one chooses a partner for 1 there will be $2n-2$ things left over to pair. The result follows by induction.

(c) (7 pts) Let's say that a pairing is *good* if each block consists of an even and an odd number. How many good pairings does X have for any n? **Answer:** $n!$. Write the odd numbers in order: $1, 3, \ldots, 2n-1$. Then we have to choose a way to assign the even numbers to them. Any possible assignment gives a distinct good pairing, so there are $n!$ in total.

(7) Verify the following binomial coefficient identities. You may use any method of proof you like.

(a) (6 pts) $k\binom{n}{k} = n\binom{n-1}{k-1}$. **Answer:** Use the formula for the coefficients and simplify both sides.

(b) (7 pts) $\sum_{k=1}^{n} \frac{k}{n}\binom{n}{k} = 2^{n-1}$. **Answer:** Differentiate $(x+1)^n$ and its expansion using the binomial theorem and then substitute $x = 1$.

(c) (7 pts) $\sum_{j=0}^{k} \binom{n}{j}\binom{m}{k-j} = \binom{n+m}{k}$. **Answer:** Divide $[\![n+m]\!]$ into a block A of order n and a block B of order m. The LHS counts the number of ways to build a subset of $[\![n+m]\!]$ by choosing j from A, $k-j$ from B, where j runs over all possibilities. The RHS counts the subsets of $[\![n+m]\!]$ of order k.