PROBLEM SET 5
COMBINATORICS

MY SOLUTIONS

Problem 3.10.8:

Solve the recurrence relation $a_0 = 1$ and $a_n = \sum_{i=0}^{n-1}(i+1)a_i$ for $n \geq 1$

Solution. Note that

(i) $a_1 = 1$
(ii) For $n \geq 2$,

$$a_n = \sum_{i=0}^{n-1}(i+1)a_i = \sum_{i=0}^{n-2}(i+1)a_i + a_{n-1} = (n+1)a_{n-1}$$

Let

$$A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$$

and

$$a(x) = \sum_{n\geq 1} a_{n-1} \frac{x^n}{n!}$$

i.e. $a'(x) = A(x)$.
Using the our observations we can write:

$$\begin{aligned} A(x) &= 1 + x + \sum_{n=2}^{\infty} a_n \frac{x^n}{n!} \\ &= 1 + x + \sum_{n=2}^{\infty} (n+1)a_{n-1} \frac{x^n}{n!} \\ &= 1 + x + x\sum_{n=1} \frac{x^{n-1}}{(n-1)!} + \sum_{n=2} a_{n-1}\frac{x^n}{n!} \\ &= 1 + x + x(A(x) - 1) + (a(x) - x) \end{aligned}$$

After simplifying we obtain the ODE

$$a'(x) = xa'(x) - x + 1 + a(x)$$

which has solution

$$a(x) = \frac{x}{1-x} - \frac{1}{2}\frac{x^2}{1-x} + \frac{c}{1-x}$$

Using the fact that $a(0) = 0$ we conclude that $c = 0$. Thus,

$$a(x) = \frac{x}{1-x} - \frac{1}{2}\frac{x^2}{1-x}$$

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and after simplifying

$$A(x) = \frac{1}{2} + \frac{1}{2}\frac{1}{(1-x)^2} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(n+1)!}{2}\frac{x^n}{n!}$$

Thus, for $n \geq 1$

$$\boxed{a_n = \frac{(n+1)!}{2}}$$

Problem 3.10.9:

Let $f(n)$ be the number of ways to pay n cents using pennies, nickels, dimes, and quarters. Find the ordinary generating functions of the numbers $f(n)$.

$$\begin{aligned} F(x) = \sum_{n=0}^{\infty} f(n)x^n &= \underbrace{\left(\sum_{n=0}^{\infty} x^n\right)}_{pennies}\underbrace{\left(\sum_{n=0}^{\infty} x^{5n}\right)}_{nickles}\underbrace{\left(\sum_{n=0}^{\infty} x^{10n}\right)}_{dimes}\underbrace{\left(\sum_{n=0}^{\infty} x^{25n}\right)}_{quarters} \\ &= \underbrace{\left(\frac{1}{1-x}\right)}_{pennies}\underbrace{\left(\frac{1}{1-x^5}\right)}_{nickels}\underbrace{\left(\frac{1}{1-x^{10}}\right)}_{dimes}\underbrace{\left(\frac{1}{1-x^{25}}\right)}_{quarters} \end{aligned}$$

Problem 3.10.12:

Find the ordinary generating function for the number of partitions of n in which no part occurs more than 3 times.

Solution.

$$G(x) = \prod_{n=1}^{\infty}(1 + x + x^{2n} + x^{3n}) = \prod_{n=1}^{\infty} \frac{1-x^{4n}}{1-x^n}$$

Problem 3.10.17:

Deduce a formula for the number of compositions of n using Theorem 3.25.

Solution. We apply Theorem 3.25. In the situation of this problem using the some notation as in the book we have that:

$$A(x) = (x + x^2 + x^3 + \dots) = \sum_{x=1}^{\infty} x^n = \frac{x}{1-x}$$

Applying Theorem 3.25 the generating functions of the number of compositions of n is

$$B(x) = \frac{1}{1-A(x)} = \frac{1}{1-\frac{x}{1-x}} = \frac{1-x}{1-2x} = 1 + \frac{x}{1-2x} = 1 + \sum_{n=1}^{\infty} 2^{n-1}x^n$$

From where we conclude the number of compositions of n is 2^{n-1}

Problem 3.10.18:

> The semester of a college consists of n days. In how many ways can we separate the semester into sessions if each session has to consist of at least five days?

Solution. If there are k sessions the generating function of the number of ways to separate a semester into sessions when each session has at least 5 days is:

$$A(x)^k = \left(\frac{x^5}{1-x}\right)^k$$

Thus, the generating function for the number of separate a semester of n days into session if each session has to consist of at least 5 days is:

$$B(X) = \sum_{n=0}^{\infty}\left(\frac{x^5}{1-x}\right)^k = \frac{1}{1-A(x)} = \frac{1}{1-(x+x^5)}$$

> **Problem 3.10.23:**
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> Assume that in example 3.37 each table can order red wine, white wine, both, or neither. How many possibilities are there now for the seating arrangements and wine orders?

Solution. Same notation and explanations as in the book. We have

$$A(x) = \log\left(\frac{1}{1-x}\right)$$

$$B(x) = \sum_{k\geq 0} 4^k \frac{x^k}{k!} = e^{4k}$$

So our desired generating function is

$$\begin{aligned} H(x) &= \frac{1}{(1-x)^4} \\ &= \frac{1}{6}\left(\frac{1}{1-x}\right)''' \\ &= \frac{1}{6}\sum_{n=3}^{\infty} n(n-1)(n-2)x^{n-3} \\ &= \frac{1}{6}\sum_{n=0}^{\infty}(n+3)(n+2)(n+1)x^n \end{aligned}$$

Thus, the number of possibilities is

$$\frac{1}{6}(n+3)(n+2)(n+1)n! = \frac{(n+3)!}{6}$$

> **Problem 3.10.33:**
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> Let $h(n)$ be the number of ways to tile a $2\times n$ rectangle using 1×2 and 2×1 rectangles. Find the ordinary generating function of the numbers $h(n)$.

Solution. Let's start by making some observations:

(1) If $n = 0$, the statment is funny, but for the sake of the problem we will say that we have 1 possibility.
(2) If $n = 1$, we clearly have only one possibility.
(3) If $n = 2$, we have two possibilities, we can either put two vertical rectangles together or 2 horizontal recntangles together.
(4) Now, the a key of this problem is the following observation. If you put a horizontal tile, then a horizontal tile must be under it otherwise there is no chance that you can tile the entire rectangle.
(5) So, now consider the top left of the rectangle. There are tow possibilities for a tiling. Either this is covered by a vertical tile or a horizontal tile. In the first case we are left with a $2 \times (n-1)$ board to cover and in the second case by the previous observation there must be a horizontal rectangle under the one covering the top left corner and hence we are left with a $2 \times (n-2)$ rectangle to cover.

This shows that

$$h(n) = h(n-1) + h(n-2)$$

for $n \geq 2$ and $h(0) = h(1) = 1$. Thus, $h(n)$ is just the nth Fibonacci number. Thus (cf. Example 3.27),

$$H(x) = \sum_{n\geq 0} h(n)x^n = \frac{1}{1-x-x^2}$$

8, 9, 12, 17, 18, 23, 33.