PROBLEM SET 4
COMBINATORICS

MY SOLUTIONS

Problem 3.10.3:
Find an explicit formula for a_n if $a_0 = 3$ and $a_n = 7a_{n-1} + 2$ for $n \geq 1$.

Solution. Let

$$A(x) = \sum_{n\geq 0} a_n x^n$$

Using the definition of a_n we can rewrite

$$\begin{aligned} A(x) &= 3 + 7x \sum_{n\geq 1} 7a_{n-1}x^n + \sum_{n\geq 1} 2x^n \\ &= 3 + 7xA(x) + 2\frac{x}{1-x} \\ &= \frac{3}{1-7x} + 2\frac{x}{(1-x)(1-7x)} \\ &= \frac{3}{1-7x} + \frac{1}{3}\frac{1}{x-1} - \frac{1}{3}\frac{1}{7x-1} \\ &= 3\sum_{n\geq 0} 7^n x^n - \frac{1}{3}\sum_{n\geq 0} x^n + \frac{1}{3}\sum_{n\geq 0} 7^n x^n \\ &= \sum_{n\geq 0} (\frac{10}{3} \cdot 7^n - \frac{1}{3})x^n \end{aligned}$$

And therefore

$$\boxed{a_n = \frac{10}{3} \cdot 7^n - \frac{1}{3}}$$

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Problem 3.10.4:

Find an explicit formula for a_n if $a_0 = 1$, $a_1 = 2$ and $a_n = 2a_{n-1} - a_{n-2}$ for $n \geq 1$.

Solution. Let

$$A(x) = \sum_{n\geq 0} a_n x^n$$

Using the definition of a_n we can rewrite

$$\begin{aligned}
A(x) &= a_0 + a_1 x + 2\sum_{n\geq 2} a_{n-1}x^n - \sum_{n\geq 2} a_{n-2}x^n \\
&= a_0 + a_1 x + 2x(A(x) - a_0) - x^2 A(x) \\
&= 1 + 2x + 2x(A(x) - 1) - x^2 A(x) \\
&= \frac{1}{1 - 2x + x^2} \\
&= \frac{1}{(x-1)^2} \\
&= \sum_{n\geq 0} (n+1)x^n
\end{aligned}$$

And therefore

$$\boxed{a_n = n + 1}$$

Problem 3.10.6:

Find an explicit formula for a_n if $a_0 = 1$ and $a_n = (n+1)a_{n-1} + 3^n$ for $n \geq 1$.

Solution. Let

$$A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$$

and

$$a(x) = \sum_{n\geq 1} a_{n-1} \frac{x^n}{n!}$$

i.e. $a'(x) = A(x)$.
Using the definition of a_n we can rewrite

$$\begin{aligned}
A(x) = a'(x) &= \sum_{n\geq 0} a_n \frac{x^n}{n!} \\
&= 1 + \sum_{n\geq 1} n a_{n-1} \frac{x^n}{n!} + \sum_{n\geq 1} a_{n-1} \frac{x^n}{n!} + \sum_{n\geq 1} 3^n \frac{x^n}{n!} \\
&= 1 + \sum_{n\geq 1} a_{n-1} \frac{x^n}{(n-1)!} + \sum_{n\geq 1} a_{n-1} \frac{x^n}{n!} + \sum_{n\geq 1} \frac{(3x)^n}{n!} \\
&= 1 + xA(x) + a(x) + e^{3x} + 1 \\
&= xa'(x) + a(x) + e^{3x}
\end{aligned}$$

So we get the ODE

$$a'(x) = \frac{a(x) + e^{3x}}{1-x}$$

we find that

$$a(x) = \frac{1}{3}\frac{e^{3x}}{1-x} + \frac{C}{1-x}$$

where $C \in \mathbb{R}$. Thus, we have that

$$A(x) = \frac{1}{3}\frac{e^{3x}}{(1-x)^2} + \frac{e^{3x}}{1-x} + \frac{C}{(1-x)^2}$$

Note that, $A(0) = \frac{1}{3} + 1 + C = a_0 = 1$, and hence $C = -1/3$. Thus after playing a little bit with power series we find that:

$$\begin{aligned}
A(x) &= \sum_{n\geq 0} \left(\sum_{k=0}^{n} \frac{n!(n-k+1)}{k!} 3^{k-1} \right) \frac{x^n}{n!} + \sum_{n\geq 0} \left(\sum_{k=0}^{n} \frac{n!}{k!} 3^k \right) \frac{x^n}{n!} - \frac{1}{3} \sum_{n\geq 0} (n+1)x^n \\
&= \sum_{n\geq 0} \left(\sum_{k=0}^{n} \frac{n!(n-k+1)}{k!} 3^{k-1} \right) \frac{x^n}{n!} + \sum_{n\geq 0} \left(\sum_{k=0}^{n} \frac{n!}{k!} 3^k \right) \frac{x^n}{n!} - \frac{1}{3} \sum_{n\geq 0} (n+1)! \frac{x^n}{n!}
\end{aligned}$$

Thus,

$$\boxed{a_n = n! \sum_{k=0}^{n} \frac{(n-k+1)3^{k-1} + 3^k}{k!} - \frac{1}{3}(n+1)!}$$

or if you wrote a little different your power series you could also have obtained something like

$$\boxed{a_n = \frac{1}{3}\sum_{k=0}^{n}\binom{n}{k}3^{n-k}(k+1)! + n!\sum_{k=0}^{n}\frac{3^k}{k!} - \frac{1}{3}(n+1)!}$$

Problem 3.10.7:
Find an explicit formula for a_n if $a_0 = 1$, $a_1 = 2$ and
$$a_n = n(a_{n-1} + a_{n-2})$$
for $n \geq 1$.

Solution. Let
$$A(x) = \sum_{n\geq 0} a_n \frac{x^n}{n!}$$
and
$$a(x) = \sum_{n\geq 1} a_{n-1} \frac{x^n}{n!}$$
i.e. $a'(x) = A(x)$.
Using the definition of a_n we can rewrite

$$\begin{aligned} A(x) = a'(x) &= \sum_{n\geq 0} a_n \frac{x^n}{n!} \\ &= 1 + 2x + \sum_{n\geq 2} n(a_{n-1} + a_{n-2}) \frac{x^n}{n!} \\ &= 1 + 2x + x \sum_{n\geq 1} a_n \frac{x^n}{n!} + x \sum_{n\geq 1} a_{n-1} \frac{x^n}{n!} \\ &= 1 + 2x + x\,(A(x) - a_0) + xa(x) \\ &= 1 + 2x + xa'(x) - x + xa(x) \\ &= 1 + x + xa'(x) + xa(x) \end{aligned}$$

So we get the ODE
$$a'(x) = \frac{1 + x + xa(x)}{1 - x}$$
We find that
$$a(x) = \frac{x}{1-x} - C\frac{e^{-x}}{1-x}$$
where $C \in \mathbb{R}$. Note that, $A(0) = C = 0$. Thus, we have that
$$a(x) = \frac{x}{1-x}$$
and hence
$$A(x) = a'(x) = \frac{1}{(1-x)^2} = \sum_{n\geq 0}(n+1)x^n = \sum_{n\geq 0}(n+1)!\frac{x^n}{n!}$$
Thus,
$$\boxed{a_n = (n+1)!}$$

1. How to solve ODEs of the form of the previous Questions

This are just some quick notes assuming eveything works ”nice” on how to solve ODEs of the form of the previous two problems. Consider an ODE of the form:

$$y' + p(x)y = r(x)$$

Let $h(x) = \int p(x)dx$ and $F(x) = e^h$.

Note, that $F'(x) = e^{h(x)} \cdot h'(x) = e^{h(x)}p(x) = F(x)p(x)$. Now, if we multiply our ecuation by $F(x)$ we find that:

$$F(x)y'(x) + F(x)p(x)y(x) = F(x)r(x)$$

Replacing $F(x)$ and $F'(x)$ we find that

$$e^h y' = h'e^h y = e^h y' + (e^h)'y = (e^h y)' = re^h$$

Thus,

$$y = e^{-h}(\int e^h r dx + C)$$

We can use this to solve the ODEs in this problem set. For example in Q6 the ODE (using y instead of a) can be written as

$$a' - \frac{a}{1-x} = \frac{e^{3x}}{1-x}$$

So, in this case

$$\begin{aligned}
p(x) &= -\frac{1}{1-x} \\
r(x) &= \frac{e^{3x}}{1-x} \\
h(x) &= \int p(x)dx = \log(x-1) \\
y(x) &= e^{-h}\left(\int e^h r(x)dx + C\right. \\
&= \frac{1}{1-x}\left(\int e^{3x}dx + C\right) \\
&= \frac{1}{3}\frac{e^{3x}}{1-x} + \frac{C}{1-x}
\end{aligned}$$