ALGEBRAIC NUMBER THEORY PROBLEMS I

For problems marked with ($\star$), a computer will be helpful for some of the computations.

(1) (a) Prove that all quadratic extensions of $\mathbb{Q}$ have the form $\mathbb{Q}(\sqrt{m})$ for some $m \in \mathbb{Z}$.
    (b) Suppose m and n are squarefree and are not equal. Prove that $\mathbb{Q}(\sqrt{m}) \neq \mathbb{Q}(\sqrt{n})$.

(2) Prove or disprove: an algebraic number α is integral if and only if the norm and trace of α lie in $\mathbb{Z}$.

(3) For any N, let ζ_N be a primitive Nth root of unity. Then the cyclotomic field $\mathbb{Q}(\zeta_N)$ is an Galois extension of degree $\varphi(N)$, with cyclic Galois group. The ring of integers in $\mathbb{Q}(\zeta_N)$ is $\mathbb{Z}[\zeta_N]$
    (a) Find all proper subfields K of $\mathbb{Q}(\zeta_7)$ and their signatures, discriminants, and rings of integers $\mathcal{O}_K$.
    (b) The Kronecker–Weber theorem asserts that all abelian fields are contained in a cyclotomic field. Let θ be a root of $x^3 - 3x + 1$. The field $L = \mathbb{Q}(\theta)$ is Galois with cyclic Galois group. Can you find a cyclotomic field containing it? Compute the signature of $\mathcal{O}_L$ and its ring of integers $\mathcal{O}_L$.

(4) ($\star$) Let p and q be distinct primes, and let $L = \mathbb{Q}(\sqrt{p}, \sqrt{q})$. L is called a *biquadratic* field.
    (a) Compute the Galois group of L and classify the subgroups of L.
    (b) Formulate a conjecture about the structure of $\mathcal{O}_L$ and prove it. *Hints:* Work in the $\mathbb{Q}$-basis $1, \sqrt{p}, \sqrt{q}, \sqrt{pq}$. Use what you know about quadratic fields to construct some integral elements.

(5) Let $\beta_1, \ldots, \beta_n$ be the roots of a polynomial f of degree n. Prove that $f'(\beta_i) = \prod_{i \neq j}(\beta_i - \beta_j)$.

(6) (A long problem) Let m be a cubefree integer. Compute the ring of integers of $\mathbb{Q}(\sqrt[3]{m})$. *Hint:* This is Marcus, Ch.2, Ex.41. He breaks the problem down into many steps to lead you through it.

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