MATH 411 FINAL EXAM

This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and then *any four* of the remaining problems. Unless indicated, you must show your work to receive credit for a solution. Make sure you answer every part of every problem you do.

Please note that there are problems on both sides of this page.

When submitting your exam, please indicate which problems you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly four problems; any unselected problems will not be graded, and if you select more than four only the first four (in numerical order) will be graded.

(1) Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem.
   (a) (4 pts) For each positive integer n, there is a cyclic group of order n.
   (b) (4 pts) The alternating group $A_n \subset S_n$ is the subgroup of all even permutations.
   (c) (4 pts) Let G be a group. For every divisor d of $|G|$, there exists a subgroup $H \subset G$ of order d.
   (d) (4 pts) A subset H of a group G is a subgroup if for all $h_1, h_2 \in H$, we have $h_1 h_2 \in H$.
   (e) (4 pts) The first isomorphism theorem states that if $\varphi\colon G \to H$ is an onto homomorphism, then H is isomorphic to G/K, where K is the kernel of φ.

(2) (20 pts) Let G be the set of ordered pairs $\{(x, y) \mid x, y \in \mathbb{R}\}$. Define a binary operation $*$ on G by $(x, y) * (x', y') = (xx', xy' + y)$. Show that $(G, *)$ is a group. (You do not need to verify that $*$ is actually a binary operation.)

(3) Let G be the cyclic group $\mathbb{Z}/30\mathbb{Z} = \{0, 1, \ldots, 29\}$.
   (a) (6 pts) Give a complete list of all the generators of G.
   (b) (7 pts) Give a complete list of all the subgroups of G.
   (c) (7 pts) For each subgroup H of G you gave, give a generator of H.

(4) Let σ be permutation given by

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 6 & 4 & 7 & 9 & 5 & 1 & 10 & 3 & 2 & 8 \end{pmatrix}$$

   (a) (5 pts) Compute σ^2.
   (b) (5 pts) Write σ as a product of disjoint cycles.

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(c) (5 pts) Compute the order of σ.
(d) (5 pts) Compute $|\langle\sigma^2\rangle|$ and $|\langle\sigma^4\rangle|$.

(5) Let G be the group $GL_2(\mathbb{R})$ and let H be the group $\mathbb{R}_{>0} = \{x \in \mathbb{R} \mid x > 0\}$. The binary operation on H is usual multiplication. Define a function $\varphi\colon G \to \mathbb{R}_{>0}$ by $\varphi(g) = (\det g)^2$.
(a) (6 pts) Show that φ is an onto homomorphism.
(b) (7 pts) Identify the kernel K of φ.
(c) (7 pts) For each right coset in G/K, find a representative in G.

(6) (20 pts) Give an example of each of the groups G_i, or explain why no such group exists.
(a) (4 pts) An *abelian* group G_1 of order 12.
(b) (4 pts) A *nonabelian* group G_2 of order 12.
(c) (4 pts) A group G_3 with no subgroups other than itself G_3 and the identity $\{e_{G_3}\}$.
(d) (4 pts) A group G_4 such that every subgroup is normal.
(e) (4 pts) A *nonabelian* group G_5 with an *abelian* normal subgroup G_6 such that G_5/G_6 is *abelian.*

(7) Let $n \geq 4$ and let S_n be the symmetric group. Let $\tau = (x_1, x_2)$ be a 2-cycle in S_n.
(a) (10 pts) Show that there is an element $\sigma \in S_n$ other than the identity satisfying $\sigma\tau = \tau\sigma$.
(b) (10 pts) Show that there is an element $\rho \in S_n$ satsifying $\rho\tau \neq \tau\rho$.

(8) Let G be a group and let $Z \subset G$ be the subset

$$Z = \{z \in g \mid zg = gz \text{ for all } g \in G\}.$$

(a) (10 pts) Show that Z is a subgroup of G.
(b) (10 pts) Show that Z is a normal subgroup G.