Algebra M411 Exam II answers

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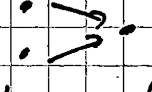
1.
   a) False. 
   b) True $|S_n| = n!$
   c) True
   d) False, it's 6
   e) True, they lie in the coset Hg.

2.
$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 4 & 10 & 5 & 8 & 3 & 9 & 2 & 1 & 6 \end{pmatrix}$$

   a) σ^2 : $1 \to 7 \to 9$
   $2 \to 4 \to 5$
   ... etc.
   $10 \to 6 \to 3$
$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 5 & 6 & 8 & 2 & 10 & 1 & 4 & 7 & 3 \end{pmatrix}$$

   b) σ^{-1}: read it from bottom to top. $7 \mapsto 1$
   $2 \to 4$
   ...
   $6 \to 10$
$$\sigma^{-1} = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 9 & 8 & 6 & 2 & 4 & 10 & 1 & 5 & 7 & 3 \end{pmatrix}$$

   c) use algorithm from class.
$$\sigma = (1,7,9)(2,4,5,8)(3,10,6)$$

   d) Recall: $\left\{\begin{matrix}\text{odd}\\ \text{even}\end{matrix}\right\}$ length cycles are $\left\{\begin{matrix}\text{even}\\ \text{odd}\end{matrix}\right\}$

$$\Longrightarrow \left.\begin{matrix}(1,7,9) \text{ even} \\ (2,4,5,8) \text{ odd} \\ (3,10,6) \text{ even}\end{matrix}\right\} \quad \text{even} \times \text{odd} \times \text{even} = \text{odd}$$

$$\Longrightarrow \sigma \notin A_{10}.$$

(2)

(3) (a) $H = \{00, 11, 02, 13\}$

(here ab means $(a, b) \in \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$)

cosets: H itself $\quad \{00, 11, \cancel{03}\ 02, 13\}$

$10 + H \quad \{10, 01, 12, 03\}$

That's all of them since $|\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}| = 8$

(b) $H = \{00, 11, 02, 13, 04, \ldots, 14\}$

= all of G. So there is just one coset, namely H itself.

(4) $\tau = (x_1, \ldots, x_r) \stackrel{?}{=} (x_1, x_2)(x_2, x_3) \cdots (x_{r-1}, x_r)$

Read RHS from R to left.

$x_1 \mapsto x_2$

$x_2 \mapsto x_3$

$\vdots$

$x_{r-1} \mapsto x_r$

$x_r \mapsto x_{r-1} \mapsto x_{r-2} \mapsto \cdots x_2 \mapsto x_1$

$\Longrightarrow$ RHS $= \tau$.

(1) (a) H is the subset of one-to-one onto functions $f: \{1, \ldots, 5\} \to \{1, \ldots, 5\}$ satisfying $f(5) = 5$. We must check:

enough for a subgp of a finite gp. $\Rightarrow$ subgp.

(a) contains identity? yes, the identity function takes $5 \mapsto 5$.

(b) closed under group operation? yes, if $f(5) = 5$ and $g(5) = 5$, then $(f \circ g)(5) = 5$ ✓

③

(b) $|H| = |S_4| = 4!$

Reason: any function that shuffles around $\{1,\dots,4\}$ in any way is in H. (The only requirement is that $5 \mapsto 5$).

So effectively we can perform any permutation in S_4 $\Rightarrow |H| = 4! = 24$.

(c) (a) ① Reflexive: $x \sim x$? yes, take $g = e$. Then $x = e x e^{-1}$.

② Symmetric: $x \sim y \Rightarrow y \sim x$?

If $x = gyg^{-1}$, then $y = g^{-1}xg$.

So apply the definition of $\sim$ with $g^{-1} \in G$ to see $y \sim x$.

③ Transitive? $x \sim y$ & $y \sim z \Rightarrow x \sim z$?

take g, g' s.t.

$$x = gyg^{-1}, \quad y = g'z(g')^{-1}.$$

Then

$$x = g(g'z(g')^{-1})g^{-1}$$

$$= (gg')z(gg')^{-1}$$

$$\Longrightarrow x \sim z.$$

(b) if G is abelian, then

$$x = gyg^{-1}$$

$$= gg^{-1}y$$

$$= y.$$

Thus $x \sim y \Leftrightarrow x = y$ and the equivalence classes are just the elements of G.

(7) (a) Write $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$ (4)

Then A, B in same right coset means $\exists \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} \in H$ s.t.

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

or $$\begin{pmatrix} a+mc & b+md \\ c & d \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}$$

$\Rightarrow c = z,\ d = w$, or same bottom row.

(b) Take $A = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

Same bottom row, but the matrices

$$\begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} B = \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

have entries that are always integers. $\Rightarrow$ A can't show up in this way and must lie in a different right coset.