MATH 411 EXAM II

This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and then *any four* of the remaining problems. Unless indicated, you must show your work to receive credit for a solution. Make sure you answer every part of every problem you do.

Please note that there are problems on both sides of this page.

When submitting your exam, please indicate which problems you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly four problems; any unselected problems will not be graded, and if you select more than four only the first four (in numerical order) will be graded.

(1) Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem.
   (a) (4 pts) If $f\colon X \to Y$ is an onto function, then an inverse function $f^{-1}\colon Y \to X$ exists.
   (b) (4 pts) $|S_4| = 24$.
   (c) (4 pts) The function $f\colon \mathbb{Z} \to \mathbb{Z}$ defined by $f(x) = x^3$ is one to one.
   (d) (4 pts) The order of $(1,2)(3,4,5)$ in S_5 is 5.
   (e) (4 pts) Let $H \subset G$ be a subgroup of a group G. Let $h \in H$. Then for any $g \in G$, the elements hg and h^2g lie in the same right coset.

(2) (20 pts) Let $\sigma \in S_{10}$ be the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 7 & 4 & 10 & 5 & 8 & 3 & 9 & 2 & 1 & 6 \end{pmatrix}$$

   (a) (5 pts) Compute σ^2.
   (b) (5 pts) Compute σ^{-1}.
   (c) (5 pts) Write σ as a product of disjoint cycles.
   (d) (5 pts) Is $\sigma \in A_{10}$? Why or why not?

(3) (a) (10 pts) Let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/4\mathbb{Z}$. Let H be the cyclic subgroup generated by $(1,1)$. Find all the right cosets of H in G.
   (b) (10 pts) Now let $G = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/5\mathbb{Z}$ and again let H be the cyclic subgroup generated by $(1,1)$. Find all the right cosets of H in G.

(4) (20 pts) Let $\tau = (x_1, x_2, \ldots, x_r)$ be an r-cycle in a symmetric group. Show that

$$\tau = (x_1, x_2)(x_2, x_3) \cdots (x_{r-1}, x_r).$$

(5) Let $G = S_5$ and let H be the subset of permutations that fix $5 \in \{1, 2, 3, 4, 5\}$.

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(a) (10 pts) Show that H is a subgroup of G.
(b) (10 pts) What is the order of H? Why?

(6) (a) (12 pts) Let G be a group and define a relation on G by $x \sim y$ if and only if there exists a $g \in G$ with $x = gyg^{-1}$. Show that $x \sim y$ is an equivalence relation.
(b) (8 pts) Suppose that G is *abelian*. What are the equivalence classes of $x \sim y$ in G?

(7) Let G be the group $GL_2(\mathbb{R})$ and let H be the subgroup of matrices of the form $\left(\begin{smallmatrix} 1 & m \\ 0 & 1 \end{smallmatrix}\right)$, where $m \in \mathbb{Z}$.
(a) (10 pts) Show that if two matrices $A, B \in G$ are in the same right coset of H, then A, B have the same bottom rows.
(b) (10 pts) Show that the converse of the previous statement is false by finding two matrices $A, B \in G$ with the same bottom rows that are in different right cosets of H.