MATH 411 EXAM II

This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and then *any four* of the remaining problems. There are problems on *both sides* of the page. Unless indicated, you must show your work to receive credit for a solution. Make sure you answer every part of every problem you do.

When submitting your exam, please indicate which problems you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly four problems from problems 2 and higher (problem 1 is automatically selected and you need not indicate it); any unselected problems will not be graded, and if you select more than four only the first four (in numerical order) will be graded.

(1) Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem.
  (a) (4 pts) If $x \in G$ has order 10, then x^2 has order 10 also.
  (b) (4 pts) Let $H \leq G$ be a subroup and $a, b \in G$. Then if $Ha \cap Hb \neq \emptyset$, then $Ha = Hb$.
  (c) (4 pts) Every element of S_n can be written as a product of cycles.
  (d) (4 pts) If G is a finite group and H is a subgroup, then $|G|/|H|$ is an integer.
  (e) (4 pts) The permutation $(1,2,3,4)(5,6,7) \in S_7$ is in the alternating group A_7.

(2) (20 pts) Let $\sigma \in S_{15}$ be the permutation

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 10 & 1 & 15 & 5 & 8 & 12 & 14 & 9 & 4 & 6 & 7 & 13 & 11 & 2 & 3 \end{pmatrix}$$

  (a) Compute σ^{-1}.
  (b) Compute σ^2.
  (c) Compute a representation of σ as a product of disjoint cycles.
  (d) Compute the order of σ in S_{15}.

(3) (20 pts) Let G be a group and H a subgroup. Define a relation on G by $x \sim y$ if there exists $h_1, h_2 \in H$ such that $y = h_1 x h_2$.
  (a) (12 pts) Show that $\sim$ is an equivalence relation.
  (b) (8 pts) Let $G = S_3$ and let H be the subgroup $\langle (1,2) \rangle$ of order 2. Compute the equivalence classes of $\sim$.

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(4) (20 pts) Let σ and τ be two distinct transpositions in S_n, $n \geq 3$.
   (a) (10 pts) Show that if σ and τ are not disjoint, then the product $\sigma\tau$ can be written as a 3-cycle.
   (b) (10 pts) Show that if σ and τ are disjoint, then the product $\sigma\tau$ can be written as a product of two (not necessarily disjoint) 3-cycles.

(5) (20 pts)
   (a) (10 pts) Show that if $|G|$ is a prime number, then G is cyclic.
   (b) (10 pts) Suppose that G is a group and H and K are subgroups such that $|H| = 39$, $|K| = 65$. Show that the subgroup $H \cap K$ is cyclic.

(6) (20 pts) Let G be a group of order p^2, where p is a prime. Show that G must have a subgroup of order p.

(7) (20 pts) Let $G = \{\pm 1, \pm I, \pm J, \pm K\}$ be the quaternion group,
   (a) (10 pts) Let H be the cyclic subgroup generated by I. Find all right cosets of H in G.
   (b) (10 pts) Let H' be the cyclic subgroup generated by -1. Compute
$$[G : H'].$$

(8) (20 pts) Let H be a normal subgroup of G, and assume that $|H| = 2$. Show that H is contained in the center of G (recall that the center of G is the subgroup $\{g \in G \mid xg = gx \text{ for all } x \in G\}$).

(9) (20 pts)
   (a) (15 pts) Prove that every subgroup of an abelian group is normal.
   (b) (5 pts) Give an example to show that a nonabelian group can have a nonnormal subgroup (you must verify that your example works).

M411 Exam 2 Answers

1 (a) FALSE. x^2 has order 5.
(b) TRUE. Cosets are either disjoint or coincide.
(c) TRUE. We can even make the cycles disjoint
(d) TRUE. $|G|/|H|$ is an integer by Lagrange's theorem.
(e) FALSE. Odd cycles are even, even cycles are odd. This product is an odd permutation.

2 (a) $\sigma =$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 10 & 1 & 15 & 5 & 8 & 12 & 14 & 9 & 4 & 6 & 7 & 13 & 11 & 2 & 3 \end{pmatrix}$$

σ^{-1} (read σ from bottom to top) $=$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 2 & 14 & 15 & 9 & 4 & 10 & 11 & 5 & 8 & 1 & 13 & 6 & 12 & 7 & 3 \end{pmatrix}$$

(b) σ^2. (read what σ does twice. e.g.

$1 \to 10 \to 6$ so $1 \to 6$

$2 \to 1 \to 10$ so $2 \to 10$ etc.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \\ 6 & 10 & 3 & 8 & 9 & 13 & 2 & 4 & 5 & 12 & 14 & 11 & 7 & 1 & 15 \end{pmatrix}$$

2c) use algorithm from class:
$\sigma = (1,10,6,12,13,11,7,14,2)(3,15)(4,5,8,9)$

d) ~~Then~~ The order is the LCM of the cycle lengths from (c). $|\sigma| = LCM(9,2,4) = \boxed{36}$

3) a) $x \sim x$: take $h_1 = h_2 = e$

$x \sim y \Rightarrow y \sim x$: If $\cancel{x}y = h_1 x h_2$ then
$x = h_1^{-1} \cancel{x} h_2^{-1}$. y

$x \sim y, y \sim z \Rightarrow x \sim z$: If $y = h_1 x h_2$, $z = h_1' y h_2'$, then $z = (h_1' h_1) x (h_2 h_2')$.

b) $H = \{e, (12)\}$, so any $x \in S_3$ is equivalent to at most 4 different things: $x \sim exe, exh, hxe, hxh$ where we have written $h = (12)$. These four products don't have to be distinct, though. Some may be the same.

Equivalence class containing e:
$\{eee = e, eeh = hee = h, heh = e\}$
so we get $\{e, (12)\}$ for this one.

Equivalence class containing $(123) =: g$:

$ege = (123)$
$hge = (12)(123) = (23)$
$egh = (123)(12) = (13)$
$hgh = (12)(123)(12) = (132)$

3b) so the other equivalence class is the four remaining elts.

Ans: $\{e, (12)\}$, $\{(13), (23), (123), (132)\}$.

4a) write $\sigma = (ij)$, $\tau = (jk)$. Then
$\sigma\tau = (ij)(jk) = (ijk)$, which is a 3-cycle.

b) write $\sigma = (ij)$ $\tau = (kl)$. We only have 4 symbols i, j, k, l to use, so if we are looking for 2 3-cycles we might expect them to have 2 symbols in common. We try $(ijk)(jkl) = (ij)(kl)$ which works. (If you guessed something else like this to try, you can fiddle with the symbols to get it to work out.)

5a) Let $x \in G$, $x \neq e$. Consider $H = \langle x \rangle$. We know $H \neq \{e\}$ and $|H| \,\big|\, |G|$. But then $|H| = p = |G|$ so $H = G$ and G is cyclic.

5b) $H \cap K \leq H$ and $H \cap K \leq K$ so $|H \cap K|$ divides both $|H|$ and $|K|$. Since $|H| = 39 = 3 \cdot 13$ and $|K| = 65 = 5 \cdot 13$, the only common divisors are $\underline{1}$ and 13. If $|H \cap K| = 1$ then $H = \langle e \rangle$ and is cyclic. If $|H \cap K| = 13$ then by (a) H is cyclic.

6) Suppose $x \neq e$ is an element of G. Consider $H = \langle x \rangle$. There are two possibilities: (1) $H = G$. Then G is cyclic and has subgroups of orders the different divisors of $|G| = p^2$. Thus G has a subgp of order p.
(2) G is not equal to H. Then H is a proper nontrivial subgp of G and thus must have order a proper nontrivial divisor of $|G| = p^2$. Therefore $|H| = p$.

7) (a) $H = \{1, I, -1, -I\}$.
There are 2 cosets, this one and everybody else: $\{J, -J, K, -K\}$

(b) $H' = \{1, -1\}$ so $|H'| = 2$.
Then $[G:H'] = |G|/|H| = 4$.

8 Let $H = \{e, h\}$. Clearly $e \in Z(G)$ (center of G) so we must show $h \in Z(G)$. Since H is normal, we have $ghg^{-1} \in H$ for all $g \in G$. Now we must have either $ghg^{-1} = e$ or $ghg^{-1} = h$. The first means $gh = g$ or $h = e$, but we picked $h \neq e$ so this can't happen. Therefore it must be true that $ghg^{-1} = h$ for all $g \in G$, or $gh = hg$ for all $g \in G$. Thus $h \in Z(G)$.

9 (a) Let $H \leq G$. We must show $ghg^{-1} \in H$ for all $g \in G$, $h \in H$. But since G is abelian we have $ghg^{-1} = h \in H$. So H is normal.

(b) Take $G = S_3$, $H = \{e, (12)\}$. Then $(13)(12)(13) = (23)$ and since $(12) \neq (23)$ H isn't normal.