Hi students!

I am putting this old version of my review for the first midterm review, place and time to be announced. Check for updates on the web site as to which sections of the book will actually be covered. **Enjoy!!**
Best, **Bill Meeks**

PS. There are probably errors in some of the solutions presented here and for a few problems you need to complete them or simplify the answers; some questions are left to you the student. Also you might need to add more detailed explanations or justifications on the actual similar problems on your exam. I will keep updating these solutions with better corrected/improved versions.

Problem 1(a) - Fall 2008

Find **parametric equations** for the line **L** which contains $A(1,2,3)$ and $B(4,6,5)$.

Solution:

- To get the **parametric equations** of **L** you need a point through which the line passes and a vector parallel to the line.
- Take the point to be A and the vector to be the $\overrightarrow{AB}$.
- The vector equation of **L** is

$$\mathbf{r}(t) = \overrightarrow{OA} + t\overrightarrow{AB} = \langle 1,2,3\rangle + t\,\langle 3,4,2\rangle = \langle 1+3t, 2+4t, 3+2t\rangle\,,$$

  where O is the origin.
- The **parametric equations** are:

$$\begin{aligned} x &= 1+3t \\ y &= 2+4t, \qquad t \in \mathbb{R}. \\ z &= 3+2t \end{aligned}$$

$\square$

Problem 1(b) - Fall 2008

Find **parametric equations** for the line **L** of intersection of the planes $x - 2y + z = 10$ and $2x + y - z = 0$.

Solution:

- The vector part **v** of the line **L** of intersection is orthogonal to the normal vectors $\langle 1, -2, 1\rangle$ and $\langle 2, 1, -1\rangle$. Hence **v** can be taken to be:
$$\mathbf{v} = \langle 1, -2, 1\rangle \times \langle 2, 1, -1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix} = 1\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}.$$
- Choose $P \in \mathbf{L}$ so the z-coordinate of P is zero. Setting $z = 0$, we obtain:
$$x - 2y = 10$$
$$2x + y = 0.$$
Solving, we find that $x = 2$ and $y = -4$. Hence, $P = \langle 2, -4, 0\rangle$ lies on the line **L**.
- The **parametric equations** are:
$$x = 2 + t$$
$$y = -4 + 3t$$
$$z = 0 + 5t = 5t.$$

$\square$

Problem 2(a) - Fall 2008

Find an **equation of the plane** which contains the points $P(-1, 0, 1)$, $Q(1, -2, 1)$ and $R(2, 0, -1)$.

Solution:

Method 1

- Consider the vectors $\overrightarrow{PQ} = \langle 2, -2, 0 \rangle$ and $\overrightarrow{PR} = \langle 3, 0, -2 \rangle$ which lie parallel to the plane.
- Then consider the normal vector:

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 0 \\ 3 & 0 & -2 \end{vmatrix} = 4\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}.$$

- So the **equation of the plane** is given by:

$$\langle 4, 4, 6 \rangle \cdot \langle x + 1, y, z - 1 \rangle = 4(x + 1) + 4y + 6(z - 1) = 0.$$

□

Problem 2(a) - Fall 2008

Find an **equation of the plane** which contains the points $P(-1,0,1)$, $Q(1,-2,1)$ and $R(2,0,-1)$.

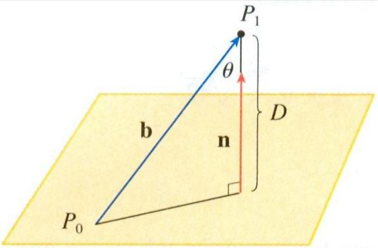
Solution:

Method 2

- The plane consists of all the points $S(x,y,z) \in \mathbb{R}^3$, such that $\overrightarrow{PS}$, $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are in the same plane (coplanar).
- But this happens if and only if their box product is zero.
- So the **equation of the plane** is:

$$\begin{vmatrix} x+1 & y & z-1 \\ 2 & -2 & 0 \\ 3 & 0 & -2 \end{vmatrix} = 4(x+1) + 4y + 6(z-1) = 0.$$

$\square$



Problem 2(b) - Fall 2008

Find the distance **D** from the point $(1, 6, -1)$ to the plane $2x + y - 2z = 19$.

Solution:

- Recall the distance formula $\mathbf{D} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}}$ from a point $P = (x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$.
- In order to apply the formula, rewrite the equation of the plane in standard form: $2x + y - 2z - 19 = 0$.
- So, the distance from $(1, 2, -1)$ to the plane is:

$$\mathbf{D} = \frac{|(2 \cdot 1) + (1 \cdot 6) + (-2 \cdot -1) - 19|}{\sqrt{2^2 + 1^2 + (-2)^2}} = \frac{|-9|}{\sqrt{9}} = 3. \qquad \square$$

Problem 2(c) - Fall 2008

Find the point Q in the plane $2x + y - 2z = 19$ which is closest to the point $(1, 6, -1)$. (Hint: You can use part b) of this problem to help find Q or first find the equation of the line **L** passing through Q and the point $(1, 6, -1)$ and then solve for Q.)

Solution:

- The line **L** in the Hint passes through $(1, 6, -1)$ and is parallel to $\mathbf{n} = \langle 2, 1, -2 \rangle$.
- So, **L** has **parametric equations**:

$$\begin{aligned} x &= 1 + 2t \\ y &= 6 + t, \qquad\qquad t \in \mathbb{R}. \\ z &= -1 - 2t \end{aligned}$$

- **L** intersects the plane $2x + y - 2z = 19$ if and only if

$$2(1 + 2t) + (6 + t) - 2(-1 - 2t) = 19 \iff 9t = 9 \iff t = 1.$$

- Substituting $t = 1$ in the **parametric equations** of **L** gives the point $Q = (3, 7, -3)$. □

Problem 3(a) - Fall 2008

Find the volume **V** of the **parallelepiped** such that the following four points $A = (3, 4, 0)$, $B = (3, 1, -2)$, $C = (4, 5, -3)$, $D = (1, 0, -1)$ are vertices and the vertices B, C, D are all adjacent to the vertex A.

Solution:

The **parallelepiped** is determined by its edges

$$\overrightarrow{AB} = \langle 0, -3, -2 \rangle\,, \quad \overrightarrow{AC} = \langle 1, 1, -3 \rangle\,, \quad \overrightarrow{AD} = \langle -2, -4, -1 \rangle\,.$$

Its volume can be computed as the absolute value of the box product $\overrightarrow{AB} \cdot (\overrightarrow{AC} \times \overrightarrow{AD})$, i.e.,

$$\mathbf{V} = \left\| \begin{matrix} 0 & -3 & -2 \\ 1 & 1 & -3 \\ -2 & -4 & -1 \end{matrix} \right\| = |3(-1-6) - 2(-4+2)| = |-17| = 17.$$

$\square$

Problem 3(b) - Fall 2008

Find the **center** and **radius** of the sphere
$x^2 - 4x + y^2 + 4y + z^2 = 8$.

Solution:

- Completing the square we get

$$x^2-4x+y^2+4y+z^2 = (x^2-4x+4)-4+(y^2+4y+4)-4+(z^2)$$

$$= (x-2)^2 - 4 + (y+2)^2 - 4 + z^2 = 8$$

$$\iff$$

$$(x-2)^2 + (y+2)^2 + z^2 = 16.$$

- This gives:

$$\textbf{Center} = (2, -2, 0) \qquad \textbf{Radius} = 4$$

$\square$

Problem 4(a) - Fall 2008

The position vector of a particle moving in space equals $\mathbf{r}(t) = \; t^2\mathbf{i} - t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$ at any time $t \geq 0$.
a) Find an **equation of the tangent line** to the curve at the point $(4, -4, 2)$.

Solution:

- The parametrized curve passes through the point $(4, -4, 2)$ if and only if
$$t^2 = 4, \;\; -t^2 = -4, \;\; \frac{1}{2}t^2 = 2 \iff t^2 = 4 \iff t = \pm 2.$$
- Since we have that $t \geq 0$, we are left with the choice $t_0 = 2$.
- The **velocity vector field** to the curve is given by
$$\mathbf{r}'(t) = \langle\, 2t, -2t, t\rangle \;\text{ hence }\; \mathbf{r}'(2) = \langle 4, -4, 2\rangle\,.$$
- The **equation of the tangent line** in question is:
$$\begin{aligned} x &= 4 + 4t \\ y &= -4 - 4t, \quad t \geq 0. \\ z &= 2 + 2t \end{aligned}$$

$\square$

Problem 4(b) - Fall 2008

The position vector of a particle moving in space equals $\mathbf{r}(t) = t^2\mathbf{i} - t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$ at any time $t \geq 0$.
(b) Find the length $\mathbf{L}$ of the arc traveled from time $t = 1$ to time $t = 4$.

Solution:

- The velocity field is:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, -2t, t \rangle .$$

- Since $t \geq 0$, the speed is:

$$|\mathbf{r}'(t)| = \sqrt{9t^2} \Longrightarrow |\mathbf{r}'(t)| = 3t.$$

- Therefore, the length is:

$$\mathbf{L} = \int_1^4 3t\,dt = \frac{3}{2}t^2\Big|_1^4 = \frac{3}{2}\cdot 16 - \frac{3}{2}\cdot 1 = \frac{3}{2}\cdot 15 = \frac{45}{2}. \qquad \square$$

Problem 4(c) - Fall 2008

Suppose a particle moving in space has velocity

$$\mathbf{v}(t) = \langle \sin t, 2\cos 2t, 3e^t \rangle$$

and initial position $\mathbf{r}(0) = \langle 1, 2, 0 \rangle$. Find the position vector function $\mathbf{r}(t)$.

Solution:

- One can recover the position, by integrating the velocity:

$$\mathbf{r}(t) = \int_0^t \mathbf{v}(\tau) d\tau + \mathbf{r}(0)$$

- Carrying out this integral yields:

$$\mathbf{r}(t) = \left\langle \; -\cos\tau\Big|_0^t, \;\; \sin 2\tau\Big|_0^t, \;\; 3e^t\Big|_0^t \right\rangle + \langle 1, 2, 0 \rangle$$

$$= \left\langle 2 - \cos t, \; 2 + \sin 2t, \; 3e^t - 3 \right\rangle . \qquad \square$$

Problem 5(a) - Fall 2008

Consider the points $A(2,1,0)$, $B(3,0,2)$ and $C(0,2,1)$. Find the area of the triangle ABC. (Hint: If you know how to find the area of a parallelogram spanned by 2 vectors, then you should be able to solve this problem.)

Solution:

- Recall that:

$$\textbf{Area} = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|.$$

- Since $\overrightarrow{AB} = \langle 1, -1, 2\rangle$ and $\overrightarrow{AC} = \langle -2, 1, 1\rangle$,

$$\textbf{Area} = \frac{1}{2}\left\|\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 2 \\ -2 & 1 & 1 \end{vmatrix}\right\| = \frac{1}{2}\left|\langle -3, -5, -1\rangle\right| = \frac{1}{2}\sqrt{35}.$$

$\square$

Problem 5(b) - Fall 2008

Three of the four vertices of a parallelogram are $P(0, -1, 1)$, $Q(0, 1, 0)$ and $R(2, 1, 1)$. Two of the sides are PQ and PR. Find the coordinates of the fourth vertex.

Solution:

Denote the fourth vertex by **S**. Then

$$\overrightarrow{O\mathbf{S}} = \overrightarrow{OQ} + \overrightarrow{PR} = \langle 0, 1, 0 \rangle + \langle 2, 2, 0 \rangle = \langle 2, 3, 0 \rangle ,$$

where O is the origin. That is,

$$\mathbf{S} = (2, 3, 0).$$

$\square$

Problem 6(a) - Spring 2008

Find an **equation of the plane** through the points $A = (1, 2, 3)$, $B = (0, 1, 3)$, and $C = (2, 1, 4)$.

Solution:

Since a plane is determined by its normal vector **n** and a point on it, say the point A, it suffices to find **n**. Note that:

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 0 \\ 1 & -1 & 1 \end{vmatrix} = \langle -1, 1, 2 \rangle.$$

So the **equation of the plane** is:

$$-(x-1) + (y-2) + 2(z-3) = 0.$$

□

Problem 6(b) - Spring 2008

Find the area of the triangle $\mathbf{\Delta}$ with vertices at the points $A = (1, 2, 3)$, $B = (0, 1, 3)$, and $C = (2, 1, 4)$.
(*Hint: the area of this triangle is related to the area of a certain parallelogram*)

Solution:

Consider the points $A = (1, 2, 3)$, $B = (0, 1, 3)$ and $C = (2, 1, 4)$. Then the area of the triangle $\mathbf{\Delta}$ with these vertices can be found by taking the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and dividing by 2. Thus:

$$\textbf{Area}(\mathbf{\Delta}) = \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2} = \frac{1}{2}\left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 0 \\ 1 & -1 & 1 \end{vmatrix} \right\|$$

$$= \frac{1}{2}|\langle -1, 1, 2\rangle| = \frac{1}{2}\sqrt{1+1+4} = \frac{1}{2}\sqrt{6}.$$

$\square$

Problem 7(a) - Spring 2008

Find the **parametric equations** of the line passing through the point $(2, 4, 1)$ that is perpendicular to the plane $3x - y + 5z = 77$.

Solution:

- The vector part of the line **L** is the normal vector $\mathbf{n} = \langle 3, -1, 5 \rangle$ to the plane.
- The **vector equation** of **L** is:

$$\mathbf{r}(t) = \langle 2, 4, 1 \rangle + t\mathbf{n}$$

$$= \langle 2, 4, 1 \rangle + t\langle 3, -1, 5 \rangle = \langle 2 + 3t, 4 - t, 1 + 5t \rangle.$$

- The **parametric equations** are:

$$\begin{aligned} x &= 2 + 3t \\ y &= 4 - t \\ z &= 1 + 5t. \end{aligned}$$

$\square$

Problem 7(b) - Spring 2008

Find the intersection point of the line $\mathbf{L}(t) = \langle 2+3t, 4-t, 1+5t \rangle$ in part (a) and the plane $3x - y + 5z = 77$.

Solution:

- By part (a), we have $\mathbf{L}$ has **parametric equations**:

$$x = 2 + 3t$$
$$y = 4 - t$$
$$z = 1 + 5t.$$

- Plug these t-values into equation of plane and solve for t:

$$3(2+3t) - (4-t) + 5(1+5t) = 77,$$

$$6 + 9t - 4 + t + 5 + 25t = 77,$$

$$35t = 70; \quad \Longrightarrow t = 2.$$

  So $\mathbf{L}$ intersects the plane at time $t = 2$.

- At $t = 2$, the **parametric equations** give the point:

$$\langle 2 + 3 \cdot 2, 4 - 2, 1 + 5 \cdot 2 \rangle = \langle 8, 2, 11 \rangle.$$

$\square$

Problem 8(a) - Spring 2008

A *plane* curve is given by the graph of the vector function

$$\mathbf{u}(t) = \langle 1 + \cos t, \sin t \rangle, \quad 0 \le t \le 2\pi.$$

Find a single equation for the curve in terms of x and y by eliminating t.

Solution:

- Rewriting $\mathbf{u}$, we get:

$$\mathbf{u}(t) = \langle 1 + \cos t, \sin t \rangle = \langle 1, 0 \rangle + \langle \cos t, \sin t \rangle.$$

- Since $\langle \cos t, \sin t \rangle$ is the **parametrization** of the circle of radius 1 centered at the origin, then $\mathbf{u}$ is a circle of radius $\mathbf{r} = 1$ centered at $(1, 0)$.
- So the answer is:

$$(x-1)^2 + (y-0)^2 = 1^2$$

or

$$(x-1)^2 + y^2 = 1.$$

□

Problem 8(b) - Spring 2008

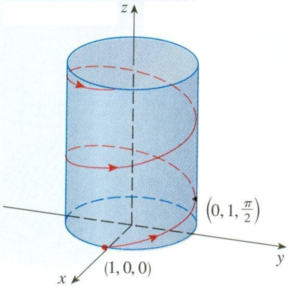
Consider the *space* curve given by the graph of the vector function

$$\mathbf{r}(t) = \langle 1 + \cos t, \sin t, t \rangle, \quad 0 \leq t \leq 2\pi.$$

Sketch the curve and indicate the direction of increasing t in your graph.

Solution:

The sketch would be the following one translated 1 unit along the x-axis.



Problem 8(c) - Spring 2008

Determine **parametric equations** for the line $\mathbf{T}$ tangent to the graph of the *space* curve for $\mathbf{r}(t) = \langle 1+\cos t, \sin t, t\rangle$ at $t = \pi/3$, and sketch $\mathbf{T}$ in the graph obtained in part (b).

Solution:

- First find the velocity vector $\mathbf{r}'(t)$:
$$\mathbf{r}'(t) = \langle (1+\cos t)', (\sin t)', 1\rangle = \langle -\sin t, \cos t, 1\rangle.$$
- At $t = \frac{\pi}{3}$,
$$\mathbf{r}(\frac{\pi}{3}) = \langle 1+\cos\frac{\pi}{3}, \sin\frac{\pi}{3}, \frac{\pi}{3}\rangle = \langle \frac{3}{2}, \frac{\sqrt{3}}{2}, \frac{\pi}{3}\rangle,$$
$$\mathbf{r}'(\frac{\pi}{3}) = \langle -\sin\frac{\pi}{3}, \cos\frac{\pi}{3}, 1\rangle = \langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, 1\rangle.$$
- The vector part of tangent line $\mathbf{T}$ is $\mathbf{r}'(\frac{\pi}{3})$ and a point on line is $\mathbf{r}(\frac{\pi}{3})$.
- The **vector equation** is: $\mathbf{T}(t) = \mathbf{r}(\frac{\pi}{3}) + t\mathbf{r}'(\frac{\pi}{3})$.
- The **parametric equations** are:
$$x = \frac{3}{2} - \frac{\sqrt{3}}{2}t$$
$$y = \frac{\sqrt{3}}{2} + \frac{1}{2}t$$
$$z = \frac{\pi}{3} + t.$$

□

Problem 9(a) - Spring 2008

Suppose that $\mathbf{r}(t)$ has derivative $\mathbf{r}'(t) = \langle -\sin 2t, \cos 2t, 0\rangle$ on the interval $0 \le t \le 1$. Suppose we know that $\mathbf{r}(0) = \langle \frac{1}{2}, 0, 1\rangle$. Determine $\mathbf{r}(t)$ for all t.

Solution:

- Find $\mathbf{r}(t)$ by integration:

$$\mathbf{r}(t) = \int \mathbf{r}'(t)\, dt = \int \langle -\sin 2t, \cos 2t, 0\rangle\, dt$$

$$= \langle \frac{1}{2}\cos(2t) + x_0, \frac{1}{2}\sin(2t) + y_0, z_0\rangle.$$

- Now solve for the point (x_0, y_0, z_0) using $\mathbf{r}(0) = \langle \frac{1}{2}, 0, 1\rangle$:
  $(\frac{1}{2}\cos(0) + x_0, \frac{1}{2}\sin(0) + y_0, z_0) = (\frac{1}{2} + x_0, y_0, z_0) = (\frac{1}{2}, 0, 1)$.
  So $x_0 = 0,\ y_0 = 0,\ z_0 = 1$.
- Thus,

$$\mathbf{r}(t) = \langle \frac{1}{2}\cos 2t, \frac{1}{2}\sin 2t, 1\rangle.$$

$\square$

Problem 9(b) - Spring 2008

Suppose that $\mathbf{r}(t)$ has derivative $\mathbf{r}'(t) = \langle -\sin 2t, \cos 2t, 0\rangle$ on the interval $0 \le t \le 1$. Suppose we know that $\mathbf{r}(0) = \langle \frac{1}{2}, 0, 1\rangle$. Show that $\mathbf{r}(t)$ is **orthogonal** to $\mathbf{r}'(t)$ for all t.

Solution:

- By part (a),
$$\mathbf{r}(t) = \langle \frac{1}{2}\cos 2t, \frac{1}{2}\sin 2t, 1\rangle,$$
- Taking dot products, we get:
$$\mathbf{r}(t)\cdot\mathbf{r}'(t) = -\frac{1}{2}\cos 2t\sin 2t + \frac{1}{2}\sin 2t\cos 2t + 0 = 0.$$
- Since the dot product is zero, then for each t, $\mathbf{r}(t)$ is **orthogonal** to $\mathbf{r}'(t)$.

$\square$

Problem 9(c) - Spring 2008

Suppose that $\mathbf{r}(t)$ has derivative $\mathbf{r}'(t) = \langle -\sin 2t, \cos 2t, 0 \rangle$ on the interval $0 \leq t \leq 1$. Suppose we know that $\mathbf{r}(0) = \langle \frac{1}{2}, 0, 1 \rangle$. Find the arclength $\mathbf{L}$ of the graph of the vector function $\mathbf{r}(t)$ on the interval $0 \leq t \leq 1$.

Solution:

- Recall that the length of $\mathbf{r}(t)$ on the interval $[0, 1]$ is gotten by integrating the speed $|\mathbf{r}'(t)|$.
- Calculating, we get:

$$\mathbf{L} = \int_0^1 |\mathbf{r}'(t)|\, dt = \int_0^1 |\langle -\sin 2t, \cos 2t, 0 \rangle| \; dt$$

$$= \int_0^1 \sqrt{\sin^2 2t + \cos^2 2t}\; dt = \int_0^1 |1|\, dt = \; t\Big|_0^1 = 1.$$

- Thus

$$\mathbf{L} = 1.$$

$\square$

Problem 10(a) - Spring 2008

If $\mathbf{r}(t) = (2t)\mathbf{i} + (t^2 - 6)\mathbf{j} - (\frac{1}{3}t^3)\mathbf{k}$ represents the position vector of a moving object (where $t \geq 0$ is measured in seconds and distance is measured in feet),
Find the speed $\mathbf{s}(t)$ and the velocity $\mathbf{v}(t)$ of the object at time t.

Solution:

- Recall that the velocity $\mathbf{v}(t)$ vector of $\mathbf{r}(t)$ at time t is $\mathbf{r}'(t)$ and the speed $\mathbf{s}(t)$ is its length $|\mathbf{r}'(t)|$.
- Calculating with $\mathbf{r}(t) = \langle 2t, t^2 - 6, -\frac{1}{3}t^3 \rangle$:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2, 2t, -t^2 \rangle,$$

$$\mathbf{s}(t) = |\mathbf{r}'(t)| = \sqrt{2^2 + (2t)^2 + (-t^2)^2} = \sqrt{4 + 4t^2 + t^4}.$$

$\square$

Problem 10(b) - Spring 2008

If $\mathbf{r}(t) = (2t)\mathbf{i} + (t^2 - 6)\mathbf{j} - (\frac{1}{3}t^3)\mathbf{k}$ represents the position vector of a moving object (where $t \geq 0$ is measured in seconds and distance is measured in feet.)

If a second object travels along a path given defined by the graph of the vector function $\mathbf{w}(s) = \langle 2, 5, 1\rangle + s\langle 2, -1, -5\rangle$, show that the paths of the two objects **intersect** at a common point P.

Solution:

- Note that $\mathbf{w}(s) = \langle 2 + 2s, 5 - s, 1 - 5s\rangle$ and $\mathbf{r}(t) = \langle 2t, t^2 - 6, -\frac{1}{3}t^3\rangle$.
- Setting the x and y-coordinates of $\mathbf{w}(s)$ and $\mathbf{r}(t)$ equal, we obtain:

$$x = 2t = 2 + 2s \Longrightarrow t = s + 1$$

$$y = t^2 - 6 = 5 - s \Longrightarrow (s+1)^2 - 6 = s^2 + 2s - 5 = 5 - s$$

$$\Longrightarrow s^2 + 3s - 10 = 0 \Longrightarrow (s+5)(s-2) = 0.$$

- So, ($s = 2$ and $t = 3$) or ($s = -5$ and $t = -4$).
- Since

$$\mathbf{r}(3) = \langle 6, 3, -9\rangle = \mathbf{w}(2),$$

the paths **intersect** at $P = (6, 3, -9)$. $\square$

Problem 10(c) - Spring 2008

If $\mathbf{r}(t) = (2t)\mathbf{i} + (t^2 - 6)\mathbf{j} - (\frac{1}{3}t^3)\mathbf{k}$ represents the position vector of a moving object (where $t \geq 0$ is measured in seconds and distance is measured in feet),
If $s = t$ in part **(b)**, (i.e. the position of the second object is $\mathbf{w}(t)$ when the first object is at position $\mathbf{r}(t)$), do the two objects ever collide?

Solution:

- Set $t = s$ in part **(b)**.
- Then the x-coordinate of $\mathbf{r}(t)$ is $2t$ and the x-coordinate of $\mathbf{w}(t) = \langle 2 + 2t, 5 - t, 1 - 5t \rangle$ is $2 + 2t$, and $2t \neq 2 + 2t$ for all t.
- Since $\mathbf{r}(t)$ and $\mathbf{w}(t)$ have different x-coordinates for all values of t, then they **never collide**.

$\square$

Problem 11(a) - Spring 2007

Find **parametric equations** for the line **L** which contains $A(7,6,4)$ and $B(4,6,5)$.

Solution:

- A vector parallel to the line **L** is:

$$\mathbf{v} = \overrightarrow{AB} = \langle 4-7, 6-6, 5-4, \rangle = \langle -3, 0, 1 \rangle.$$

- A point on the line is $A(7,6,4)$.
- Therefore **parametric equations** for the line **L** are:

$$\begin{aligned} x &= 7 - 3t \\ y &= 6 \\ z &= 4 + t. \end{aligned}$$

$\square$

Problem 11(b) - Spring 2007

Find the **parametric equations** for the **line L of intersection** of the planes $x - 2y + z = 5$ and $2x + y - z = 0$.

Solution:

- A vector **v** parallel to the line is the cross product of the normal vectors of the planes:

$$\mathbf{v} = \langle 1, -2, 1\rangle \times \langle 2, 1, -1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 1 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & -1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -2 \\ 2 & 1 \end{vmatrix}\mathbf{k} = \langle 1, 3, 5\rangle.$$

- A point on **L** is any (x_0, y_0, z_0) that satisfies **both** of the plane equations. Setting $z = 0$, we obtain the equations $x - 2y = 5$ and $2x + y = 0$ and find such a point $(1, -2, 0)$.
- Therefore **parametric equations** for **L** are:

$$\begin{aligned} x &= 1 + t \\ y &= -2 + 3t \\ z &= 5t. \end{aligned}$$

□

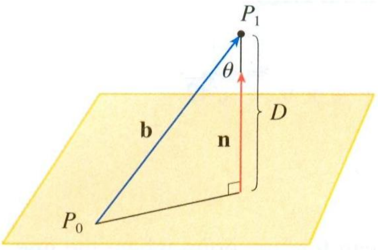
Problem 12(a) - Spring 2007

Find an **equation of the plane** which contains the points $P(-1,0,2)$, $Q(1,-2,1)$ and $R(2,0,-1)$.

Solution:

- A normal vector to the plane can be found by taking the cross product of *any* two vectors that lie **in** the plane. Two vectors that lie in the plane are $\overrightarrow{PQ} = \langle 2,-2,-1\rangle$ and $\overrightarrow{PR} = \langle 3,0,-3\rangle$.
- So the normal vector is
$$\mathbf{n} = \langle 2,-2,-1\rangle \times \langle 3,0,-3\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & -1 \\ 3 & 0 & -3 \end{vmatrix} =$$
$$\begin{vmatrix} -2 & -1 \\ 0 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ 3 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & -2 \\ 3 & 0 \end{vmatrix}\mathbf{k} = \langle 6,3,6\rangle.$$
- A point on the plane is $P(-1,0,2)$. Therefore,
$$6(x-(-1)) + 3(y-0) + 6(z-2) = 0,$$
or simplified, $$6x + 3y + 6z - 6 = 0.$$

$\square$



Problem 12(b) - Spring 2007

Find the distance **D** from the point $P_1 = (1, 0, -1)$ to the plane $2x + y - 2z = 1$.

Solution:

The normal to the plane is $\mathbf{n} = \langle 2, 1, -2\rangle$ and the point $P_0 = (0, 1, 0)$ lies on this plane. Consider the vector from P_0 to $P_1 = (1, 0, -1)$ which is $\mathbf{b} = \langle 1, -1, -1\rangle$. The distance **D** from $(1, 0, -1)$ to the plane is equal to:

$$|\mathbf{comp}_{\mathbf{n}}\ \mathbf{b}| = \left|\mathbf{b} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = |\langle 1, -1, -1\rangle \cdot \frac{1}{3}\langle 2, 1, -2\rangle| = 1.$$

□

Problem 12(c) - Spring 2007

Find the point P in the plane $2x + y - 2z = 1$ which is closest to the point $(1, 0, -1)$. (Hint: You can use part (b) of this problem to help find P or first find the equation of the line passing through P and the point $(1, 0, -1)$ and then solve for P.)

Solution:

- First find the **parametric equations** of the line that goes through the point $(1, 0, -1)$ that is normal to the plane: $x = 1 + 2t$, $y = t$, $z = -1 - 2t$; here $\mathbf{n} = \langle 2, 1, -2 \rangle$ is a normal to the plane.
- The point P in the plane closest to $(1, 0, -1)$ is the intersection of this line and the plane.
- Substitute the **parametric equations** of the line into the plane equation:
$$2(1 + 2t) + (t) - 2(-1 - 2t) = 1.$$
Simplifying and solving for t,
$$9t + 4 = 1 \Longrightarrow t = -\frac{1}{3}.$$
- Plugging this t-value into the **parametric equations**, we get the coordinates of the point of intersection: $x = 1 + 2(-\frac{1}{3}) = \frac{1}{3}$, $y = -\frac{1}{3}$, $z = -1 - 2(-\frac{1}{3}) = -\frac{1}{3}$.
- So the point on the plane closest to $(1, 0, -1)$ is $P = (\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3})$.

Problem 13(a) - Spring 2007

Consider the two space curves
$\mathbf{r}_1(t) = \langle \cos(t-1), t^2-1, 2t^4 \rangle, \quad \mathbf{r}_2(s) = \langle 1+\ln s, s^2-2s+1, 2s^2 \rangle,$
where t and s are two independent real parameters. Find the cosine of the angle θ between the tangent vectors of the two curves at the intersection point $(1,0,2)$.

Solution:

- The point $(1,0,2)$ corresponds to the t-value $t=1$ for $\mathbf{r}_1$ and s-value $s=1$ for $\mathbf{r}_2$.
- $\mathbf{r}_1'(t) = \langle -\sin(t-1), 2t, 8t^3 \rangle$ is the tangent vector to $\mathbf{r}_1(t)$.
- At $t=1$, $\mathbf{r}_1'(1) = \langle -\sin(1-1), 2(1), 8(1^3) \rangle = \langle 0,2,8 \rangle$.
- $\mathbf{r}_2'(s) = \langle \frac{1}{s}, 2s-2, 4s \rangle$ is the tangent vector to $\mathbf{r}_2(s)$.
- At $s=1$, $\mathbf{r}_2'(1) = \langle \frac{1}{1}, 2(1)-2, 4(1) \rangle = \langle 1,0,4 \rangle$.
- Therefore,

$$\cos(\theta) = \frac{\langle 0,2,8 \rangle \cdot \langle 1,0,4 \rangle}{|\langle 0,2,8 \rangle||\langle 1,0,4 \rangle|} = \frac{32}{\sqrt{68}\sqrt{17}}.$$

□

Problem 13(b) - Spring 2007

Find the **center** and **radius** of the sphere

$$x^2 + y^2 + 2y + z^2 + 4z = 20.$$

Solution:

- Completing the square in the y and z variables, we get

$$x^2 + (y^2 + 2y + 1) + (z^2 + 4z + 4) = 20 + 1 + 4.$$

- Rewriting, we have

$$x^2 + (y + 1)^2 + (z + 2)^2 = 25 = 5^2.$$

- Hence, the **center** is $\mathbf{C} = (0, -1, -2)$ and the **radius** is $r = 5$.

$\square$

Problem 14(a) - Spring 2007

The velocity vector of a particle moving in space equals $\mathbf{v}(t) = 2t\mathbf{i} - 2t\mathbf{j} + t\mathbf{k}$ at any time $t \geq 0$.
At the time $t = 4$, this particle is at the point $(0, 5, 4)$. Find an **equation of the tangent line T** to the position curve $\mathbf{r}(t)$at the time $t = 4$.

Solution:

- This line goes through the point $(0, 5, 4)$ and has vector part parallel to the tangent vector $\mathbf{v}(4) = \langle 8, -8, 4\rangle$.
- The **vector equation** is: $\mathbf{T}(t) = \langle 0, 5, 4\rangle + t\langle 8, -8, 4\rangle$
- So the line **T** has the **parametric equations**:

$$\begin{aligned} x &= 8t \\ y &= 5 - 8t \\ z &= 4 + 4t. \end{aligned}$$

$\square$

Problem 14(b) - Spring 2007

The velocity vector of a particle moving in space equals $\mathbf{v}(t) = 2t\mathbf{i} - 2t\mathbf{j} + t\mathbf{k}$ at any time $t \geq 0$.
Find the length $\mathbf{L}$ of the arc traveled from time $t = 2$ to time $t = 4$.

Solution:

Using the arclength formula,

$$\mathbf{L} = \int_2^4 |\mathbf{v}(t)|\, dt = \int_2^4 \sqrt{(2t)^2 + (-2t)^2 + t^2}\, dt$$

$$= \int_2^4 \sqrt{9t^2}\, dt = \int_2^4 3t\, dt$$

$$= \frac{3}{2}t^2\Big|_2^4 = \frac{3}{2}(16 - 4) = 18.$$

$\square$

Problem 14(c) - Spring 2007

Find a vector function $\mathbf{r}(t)$ which represents the **curve of intersection** of the cylinder $x^2 + y^2 = 1$ and the plane $x + 2y + z = 4$.

Solution:

- Since the first equation is the equation of a circular cylinder, parametrize the x and y coordinates by setting $x = \cos(t)$ and $y = \sin(t)$.
- Next use the second equation $z = 4 - x - 2y$ to solve for z in terms of t:

$$z = 4 - x - 2y = 4 - \cos(t) - 2\sin(t).$$

- Therefore,

$$\mathbf{r}(t) = \langle \cos(t), \sin(t), 4 - \cos(t) - 2\sin(t) \rangle.$$

□

Problem 15(a) - Spring 2008

Consider the points $A(2,1,0)$, $B(1,0,2)$ and $C(0,2,1)$. Find the area $\mathbf{A}$ of the triangle ABC. (Hint: If you know how to find the area of a parallelogram spanned by 2 vectors, then you should be able to solve this problem.)

Solution:

- The area of the parallelogram is

$$|\overrightarrow{AB} \times \overrightarrow{AC}| = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 2 \\ -2 & 1 & 1 \end{array} \right|$$

$$= \left| \left| \begin{array}{cc} -1 & 2 \\ 1 & 1 \end{array} \right| \mathbf{i} - \left| \begin{array}{cc} -1 & 2 \\ -2 & 1 \end{array} \right| \mathbf{j} + \left| \begin{array}{cc} -1 & -1 \\ -2 & 1 \end{array} \right| \mathbf{k} \right|$$

$$= |\langle -3, -3, -3 \rangle| = \sqrt{27}.$$

- So the area of the triangle ABC is

$$\mathbf{A} = \frac{\sqrt{27}}{2}.$$

$\square$

Problem 15(b) - Spring 2008

Suppose a particle moving in space has velocity

$$\mathbf{v}(t) = \langle \sin t, \cos 2t, e^t \rangle$$

and initial position $\mathbf{r}(0) = \langle 1, 2, 0 \rangle$. Find the position vector function $\mathbf{r}(t)$.

Solution:

- We find $\mathbf{r}(t)$ by integrating $\mathbf{r}'(t) = \mathbf{v}(t)$:

$$\begin{aligned}\mathbf{r}(t) = \int_0^t \mathbf{v}(t)\,dt + \mathbf{r}(0) &= \langle -\cos t, \frac{1}{2}\sin 2t, e^t \rangle \Big|_0^t + \langle 1, 2, 0 \rangle \\ &= \langle -\cos t, \frac{1}{2}\sin 2t, e^t \rangle - \langle -\cos 0, \frac{1}{2}\sin 0, e^0 \rangle + \langle 1, 2, 0 \rangle \\ &= \langle -\cos t, \frac{1}{2}\sin 2t, e^t \rangle - \langle -1, 0, 1 \rangle + \langle 1, 2, 0 \rangle \\ &= \langle -\cos t, \frac{1}{2}\sin 2t, e^t \rangle + \langle 2, 2, -1 \rangle.\end{aligned}$$

- So, $$\mathbf{r}(t) = \langle 2 - \cos t,\ 2 + \frac{1}{2}\sin 2t,\ -1 + e^t \rangle.$$

□

Problem 16 - Fall 2007

Find the **equation of the plane** containing the lines

$$x = 4 - 4t, \quad y = 3 - t, \quad z = 1 + 5t \quad \text{and}$$

$$x = 4 - t, \quad y = 3 + 2t, \quad z = 1$$

Solution:

- To find the equation of a plane, we need to find its normal $\mathbf{n}$ and a point on it. Setting $t = 0$, we find the point $(4, 3, 1)$ on the first line.
- The part vector $\mathbf{v}_1$ of the first line is $\langle -4, -1, 5 \rangle$ and the vector part $\mathbf{v}_2$ of the second line is $\langle -1, 2, 0 \rangle$.
- Since the vector

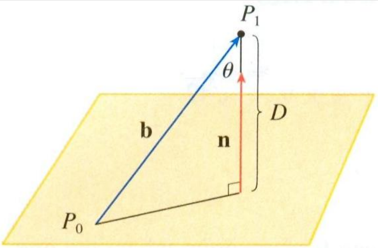
$$\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -4 & -1 & 5 \\ -1 & 2 & 0 \end{vmatrix} = \langle -10, -5, -9 \rangle,$$

  is orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, it is the normal to the plane.
- The **equation of the plane** is:

$$\langle -10, -5, -9 \rangle \cdot \langle x - 4, y - 3, z - 1 \rangle$$

$$= -10(x - 4) - 5(y - 3) - 9(z - 1) = 0.$$

$\square$



Problem 17 - Fall 2007

Find the distance **D** from the point $P_1 = (3, -2, 7)$ and the plane $4x - 6y - z = 5$.

Solution:

- Recall the distance formula $\mathbf{D} = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}}$ from a point $P = (x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$.
- In order to apply the formula, rewrite the equation of the plane in standard form: $4x - 6y - z - 5 = 0$.
- So, the distance from $(3, -2, 7)$ to the plane is:

$$\mathbf{D} = \frac{|(4 \cdot 3) + (-6 \cdot -2) + (-1 \cdot 7) - 5|}{\sqrt{4^2 + (-6)^2 + (-1)^2}} = \frac{12}{\sqrt{53}}.$$

□

Problem 18 - Fall 2007

Determine whether the lines $\mathbf{L}_1$ and $\mathbf{L}_2$ given below are **parallel**, **skew** or **intersecting**. If they intersect, find the point of intersection.

$$\mathbf{L}_1 : \frac{x}{1} = \frac{y-1}{2} = \frac{z-2}{3}$$
$$\mathbf{L}_2 : \frac{x-3}{-4} = \frac{y-2}{-3} = \frac{z-1}{2}$$

Solution:

- Rewrite these lines as vector equations:
$$\mathbf{L}_1(t) = \langle t, 2t+1, 3t+2 \rangle$$
$$\mathbf{L}_2(s) = \langle -4s+3, -3s+2, 2s+1 \rangle$$
- Equating x and y-coordinates:
$$x = t = -4s + 3$$
$$y = 2t + 1 = -3s + 2.$$
- Solving gives $s = 1$ and $t = -1$.
- $\mathbf{L}_1(-1) = \langle -1, -1, -1 \rangle \neq \langle -1, -1, 3 \rangle = \mathbf{L}_2(1)$. So these lines do **not intersect**.
- Since the lines are clearly **not parallel** (the direction vectors $\langle 1, 2, 3 \rangle$ and $\langle -4, -3, 2 \rangle$ are **not parallel**), the lines are **skew**. $\square$

Problem 19(a) - Fall 2007

Suppose a particle moving in space has the velocity

$$\mathbf{v}(t) = \langle 3t^2, 2\sin(2t), e^t \rangle.$$

Find the **acceleration** of the particle. Write down a formula for the **speed** of the particle (you do not need to simplify the expression algebraically).

Solution:

- Recall the **acceleration vector** $\mathbf{a}(t) = \mathbf{v}'(t)$. Hence,

$$\mathbf{a}(t) = \langle 6t, 4\cos(2t), e^t \rangle.$$

- Recall that the **speed**(t) is the length of the velocity vector. Hence,

$$\textbf{speed}(t) = \sqrt{9t^4 + 4\sin^2(2t) + e^{2t}}.$$

□

Problem 19(b) - Fall 2007

Suppose a particle moving in space has the velocity

$$\mathbf{v}(t) = \langle 3t^2, 2\sin(2t), e^t \rangle.$$

If initially the particle has the position $\mathbf{r}(0) = \langle 0, -1, 2 \rangle$, what is the position at time t?

Solution:

- To find the position $\mathbf{r}(t)$, we first integrate the velocity $\mathbf{v}(t)$ and second use the initial position value $\mathbf{r}(0) = \langle 0, -1, 2 \rangle$ to solve for the **constants of integration**.

  $$\mathbf{r}(t) = \int \langle 3t^2, 2\sin 2t, e^t \rangle \, dt = \langle t^3 + x_0, -\cos(2t) + y_0, e^t + z_0 \rangle.$$

- Plugging in the position at $t = 0$, we get:

  $$\langle 0^3 + x_0, -\cos(0) + y_0, e^0 + z_0 \rangle = \langle x_0, -1 + y_0, 1 + z_0 \rangle = \langle 0, -1, 2 \rangle.$$

  Thus, $x_0 = 0$, $y_0 = 0$ and $z_0 = 1$.

- Hence,

  $$\mathbf{r}(t) = \langle t^3, -\cos 2t, e^t + 1 \rangle.$$ $\square$

Problem 20(a) - Fall 2007

Three of the four vertices of a parallelogram Δ are $P(0,-1,1)$, $Q(0,1,0)$ and $R(3,1,1)$. Two of the sides are PQ and PR. Find the area of the parallelogram.

Solution:

Consider the vectors $\overrightarrow{PQ} = \langle 0,2,-1\rangle$ and $\overrightarrow{PR} = \langle 3,2,0\rangle$. Then the area of the parallelogram Δ spanned by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is:

$$\textbf{Area}(\Delta) = |\overrightarrow{PQ} \times \overrightarrow{PR}| = \left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 2 & -1 \\ 3 & 2 & 0 \end{vmatrix} \right\|$$

$$= |\langle 2,-3,-6\rangle| = \sqrt{4+9+36} = 7$$

$\square$

Problem 20(b) - Fall 2007

Three of the four vertices of a parallelogram are $P(0,-1,1)$, $Q(0,1,0)$ and $R(3,1,1)$. Two of the sides are PQ and PR. Find the cosine of the angle between the vector PQ and PR.

Solution:

- Note that:

$$\overrightarrow{PQ} = \langle 0,2,-1\rangle \qquad \overrightarrow{PR} = \langle 3,2,0\rangle.$$

- By our formula for dot products:

$$\cos\theta = \frac{\overrightarrow{PQ}\cdot\overrightarrow{PR}}{|\overrightarrow{PQ}||\overrightarrow{PR}|} = \frac{\langle 0,2,-1\rangle\cdot\langle 3,2,0\rangle}{\sqrt{5}\sqrt{13}} = \frac{4}{\sqrt{5}\sqrt{13}}.$$

$\square$

Problem 20(c) - Fall 2007

Three of the four vertices of a parallelogram are $P(0,-1,1)$, $Q(0,1,0)$ and $R(3,1,1)$. Two of the sides are PQ and PR. Find the coordinates of the fourth vertex.

Solution:

Denote the fourth vertex by $\mathbf{S}$. Then

$$\overrightarrow{O\mathbf{S}} = \overrightarrow{OQ} + \overrightarrow{PR} = \langle 0,1,0\rangle + \langle 3,2,0\rangle = \langle 3,3,0\rangle\,,$$

where O is the origin. That is,

$$\mathbf{S} = (3,3,0).$$

$\square$

Problem 21(a) - Fall 2007

Let **C** be the parametric curve

$$x = 2 - t^2, \;\; y = 2t - 1, \;\; z = \ln t.$$

Determine the point(s) of **intersection** of **C** with the *xz*-plane.

Solution:

- The points of intersection of **C** with the *xz*-plane correspond to the points where the *y*-coordinate of **C** is 0.
- When $y = 0$, then $0 = 2t - 1$ or $t = \frac{1}{2}$.
- Hence,

$$\langle 2 - (\frac{1}{2})^2, 2 \cdot \frac{1}{2} - 1, \ln \frac{1}{2} \rangle = \langle 1\frac{3}{4}, 0, -\ln 2 \rangle$$

is the unique point of the **intersection** of **C** with *xz*-plane.

$\square$

Problem 21(b) - Fall 2007

Let **C** be the parametric curve
$$x = 2 - t^2, \;\; y = 2t - 1, \;\; z = \ln t.$$
Determine **parametric equations** of tangent line to **C** at $(1, 1, 0)$.

Solution:

- Using the y-coordinate of **C**, note that $t = 1$ when $(1, 1, 0) \in$ **C**.
- The velocity vector to
$$\mathbf{C}(t) = \langle 2 - t^2, 2t - 1, \ln t\rangle$$
is:
$$\mathbf{C}'(t) = \langle -2t, 2, \frac{1}{t}\rangle.$$
- Thus,
$$\mathbf{C}'(1) = \langle -2, 2, 1\rangle$$
is the vector part of the tangent line to **C** at $(1, 1, 0)$.
- The **parametric equations** are:
$$x = 1 - 2t$$
$$y = 1 + 2t$$
$$z = t.$$

□

Problem 21(c) - Fall 2007

Let **C** be the parametric curve

$$x = 2 - t^2, \;\; y = 2t - 1, \;\; z = \ln t.$$

Set up, but not solve, a formula that will determine the length **L** of **C** for $1 \le t \le 2$.

Solution:

- The vector equation of **C** is $\mathbf{r}(t) = \langle 2 - t^2, 2t - 1, \ln t \rangle$ with velocity vector

$$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -2t, 2, \frac{1}{t} \rangle.$$

- Since the length of **L** is the integral of the speed $|\mathbf{r}'(t)|$,

$$\mathbf{L} = \int_1^2 |\langle -2t, 2, \frac{1}{t} \rangle| \, dt = \int_1^2 \sqrt{4t^2 + 4 + \frac{1}{t^2}} \, dt.$$

□

Problem 22(a) - Fall 2006

Find **parametric equations** for the line **r** which contains $A(2, 0, 1)$ and $B(-1, 1, -1)$.

Solution:

- Note that $\overrightarrow{AB} = \langle -3, 1, -2 \rangle$ and the **vector equation** is:

$$\mathbf{r}(t) = \vec{A} + t\overrightarrow{AB} = \langle 2, 0, 1 \rangle + t\langle -3, 1, -2 \rangle = \langle 2 - 3t, t, 1 - 2t \rangle.$$

- The **parametric equations** are:

$$x = 2 - 3t$$
$$y = t$$
$$z = 1 - 2t.$$

$\square$

Problem 22(b) - Fall 2006

Determine whether the lines $\mathbf{L}_1 : x = 1 + 2t,\ y = 3t,\ z = 2 - t$ and $\mathbf{L}_2 : x = -1 + s,\ y = 4 + s,\ z = 1 + 3s$ are **parallel**, **skew** or **intersecting**.

Solution:

- Vector part of line $\mathbf{L}_1$ is $\mathbf{v}_1 = \langle 2, 3, -1 \rangle$ and for line $\mathbf{L}_2$ is $\mathbf{v}_2 = \langle 1, 1, 3 \rangle$. Clearly, $\mathbf{v}_1$ is not a scalar multiple of $\mathbf{v}_2$ and so these lines are **not parallel**.
- If these lines intersect, then for some values of t and s:

$$x = 1 + 2t = -1 + s \implies 2t = -2 + s,$$

$$y = 3t = 4 + s \implies 3t = 4 + s.$$

Solving yields: $$t = 6 \ \textbf{ and } \ s = 14.$$

Plugging these values into $z = 2 - t = 1 + 3s$ yields the inequality $-4 \neq 43$, which means the z-coordinates are never equal and the lines do **not intersect**.

- Thus, the lines are **skew**.

$\square$

Problem 23(a) - Fall 2006

Find an **equation of the plane** which contains the points $P(-1,2,1)$, $Q(1,-2,1)$ and $R(1,1,-1)$.

Solution:

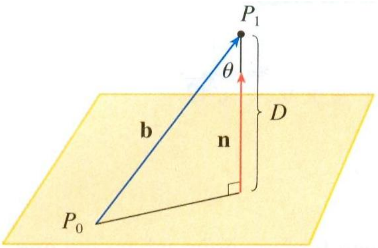
- Consider the vectors $\overrightarrow{PQ} = \langle 2,-4,0\rangle$ and $\overrightarrow{PR} = \langle 2,-1,-2\rangle$ which are parallel to the plane.
- The normal vector to the plane is:

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -4 & 0 \\ 2 & -1 & -2 \end{vmatrix} = 8\mathbf{i} + 4\mathbf{j} + 6\mathbf{k}.$$

- Since $P(-1,2,1)$ lies on the plane, the **equation of the plane** is:

$$\langle 8,4,6\rangle \cdot \langle x+1, y-2, z-1\rangle = 8(x+1) + 4(y-2) + 6(z-1) = 0.$$

□



Problem 23(b) - Fall 2006

Find the distance **D** from the point $(1, 2, -1)$ to the plane $2x + y - 2z = 1$.

Solution:

The normal to the plane is $\mathbf{n} = \langle 2, 1, -2\rangle$ and the point $P_0 = (0, 1, 0)$ lies on this plane. Consider the vector from P_0 to $P_1 = (1, 2, -1)$ which is $\mathbf{b} = \langle 1, 1, -1\rangle$. The distance **D** from $(1, 2, -1)$ to the plane is equal to:

$$|\mathbf{comp_n}\ \mathbf{b}| = \left|\mathbf{b} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right| = |\langle 1, 1, -1\rangle \cdot \frac{1}{3}\langle 2, 1, -2\rangle| = \frac{5}{3}.$$

□

Problem 24(a) - Fall 2006

Let two space curves
$\mathbf{r}_1(t) = \langle \cos(t-1), t^2-1, t^4\rangle, \quad \mathbf{r}_2(s) = \langle 1+\ln s, s^2-2s+1, s^2\rangle,$
be given where t and s are two independent real parameters. Find the cosine of the angle between the tangent vectors of the two curves at the intersection point $(1, 0, 1)$.

Solution:

- When $\mathbf{r}_1(t) = \langle 1, 0, 1\rangle$, then $t = 1$.
- When $\mathbf{r}_2(s) = \langle 1, 0, 1\rangle$, then $s = 1$.
- Calculating derivatives, we obtain:
  $\mathbf{r}_1'(t) = \langle -\sin(t-1), 2t, 4t^3\rangle$
  $\mathbf{r}_1'(1) = \langle 0, 2, 4\rangle$
  $\mathbf{r}_2'(s) = \langle \frac{1}{s}, 2s-2, 2s\rangle$
  $\mathbf{r}_2'(1) = \langle 1, 0, 2\rangle$.
- Hence,

$$\cos\theta = \frac{\mathbf{r}_1'(1)\cdot\mathbf{r}_2'(1)}{|\mathbf{r}_1'(1)||\mathbf{r}_2'(1)|} = \frac{\langle 0,2,4\rangle\cdot\langle 1,0,2\rangle}{\sqrt{20}\sqrt{5}}$$
$$= \frac{1}{\sqrt{100}}(0\cdot 1 + 2\cdot 0 + 4\cdot 2) = \frac{8}{10} = \frac{4}{5}.$$

□

Problem 24(b) - Fall 2006

Suppose a particle moving in space has velocity

$$\mathbf{v}(t) = \langle \sin t, \cos 2t, e^t \rangle$$

and initial position $\mathbf{r}(0) = \langle 1, 2, 0 \rangle$. Find the position vector function $\mathbf{r}(t)$.

Solution:

- The **position vector function** $\mathbf{r}(t)$ is the integral of its derivative $\mathbf{r}'(t) = \mathbf{v}(t)$:

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt$$

$$= \int \langle \sin t, \cos 2t, e^t \rangle\, dt = \langle -\cos(t) + x_0, \frac{1}{2}\sin(2t) + y_0, e^t + z_0 \rangle.$$

- Now use the initial position $\mathbf{r}(0) = \langle 1, 2, 0 \rangle$ to solve for x_0, y_0, z_0.

$$-\cos(0) + x_0 = -1 + x_0 = 1 \Longrightarrow x_0 = 2.$$

$$\frac{1}{2}\sin(0) + y_0 = 0 + y_0 = 2 \Longrightarrow y_0 = 2.$$

$$e^0 + z_0 = 1 + z_0 = 0 \Longrightarrow z_0 = -1.$$

- Hence, $$\mathbf{r}(t) = \langle -\cos(t) + 2, \frac{1}{2}\sin(2t) + 2, e^t - 1. \rangle$$

$\square$

Problem 25(a) - Fall 2006

Let $f(x, y) = e^{x^2-y} + x\sqrt{4-y^2}$. Find partial derivatives f_x, f_y and f_{xy}.

Problem 25(b) - Fall 2006

Find an equation for the tangent plane of the graph of

$$f(x, y) = \sin(2x + y) + 1$$

at the point $(0, 0, 1)$.

Problem 26(a) - Fall 2006

Let $g(x, y) = ye^x$. Estimate $g(0.1, 1.9)$ using the linear approximation of $g(x, y)$ at $(x, y) = (0, 2)$.

Solutions to these problems:

These types of problems might not be on this exam (check web site). □

Problem 26(b) - Fall 2006

Find the **center** and **radius** of the sphere $x^2 + y^2 + z^2 + 6z = 16$.

Solution:

- Complete the square in order to put the equation in the form:

$$(x - x_0)^2 + (y - y_0) + (z - z_0)^2 = r^2.$$

- We get:

$$x^2 + y^2 + (z^2 + 6z) = x^2 + y^2 + (z^2 + 6z + 9) - 9 = 16.$$

- This gives the equation

$$(x - 0)^2 + (y - 0)^2 + (z + 3)^2 = 25 = 5^2.$$

Hence, the **center** is $\mathbf{C} = (0, 0, -3)$ and the **radius** is $r = 5$.

□

Problem 26(c) - Fall 2006

Let $f(x, y) = \sqrt{16 - x^2 - y^2}$. Draw a contour map of level curves $f(x, y) = k$ with $k = 1,\ 2,\ 3.$ Label the level curves by the corresponding values of k.

Solution:

A problem of this type might not be on this exam (check web site). $\square$

Problem 27

Consider the line **L** through points $A = (2, 1, -1)$ and $B = (5, 3, -2)$. Find the **intersection** of the line **L** and the plane given by $2x - 3y + 4z = 13$.

Solution:

- The vector part of **L** is $\overrightarrow{AB} = \langle 3, 2, -1 \rangle$ and the point A is on the line.
- The **vector equation** of **L** is:

  $$\mathbf{L} = \vec{A} + t\overrightarrow{AB} = \langle 2, 1, -1 \rangle + t\langle 3, 2, -1 \rangle = \langle 2+3t, 1+2t, -1-t \rangle.$$

- Plugging $x = 2 + 3t$, $y = 1 + 2t$ and $z = -1 - t$ into the **equation of the plane** gives:

  $$2(2 + 3t) - 3(1 + 2t) + 4(-1 - t) = -4t - 3 = 13$$

  $$\implies -4t = 16 \implies t = -4.$$

- So, the **point of intersection** is:

  $$\mathbf{L}(-4) = \langle 2 - 12, 1 - 8, -1 - (-4) \rangle = \langle -10, -7, 3 \rangle. \quad \square$$

Problem 28(a)

Two masses travel through space along space curve described by the two vector functions

$$\mathbf{r}_1(t) = \langle t, 1-t, 3+t^2 \rangle, \mathbf{r}_2(s) = \langle 3-s, s-2, s^2 \rangle$$

where t and s are two independent real parameters.
Show that the two space curves **intersect** by finding the point of intersection and the **parameter values** where this occurs.

Solution:

- Equate the x and z-coordinates:

$$x = t = 3 - s$$

$$z = 3 + t^2 = 3 + (3-s)^2 = 3 + 9 - 6s + s^2 = s^2$$

- Thus, the **parameter values** are:

$$12 - 6s = 0 \Longrightarrow (s = 2 \text{ and } t = 1).$$

- So, $\mathbf{r}_1(1) = \langle 1, 0, 4 \rangle = \mathbf{r}_2(2)$ is the desired **intersection point**. $\square$

Problem 28(b)

Two masses travel through space along space curve described by the two vector functions

$$\mathbf{r}_1(t) = \langle t, 1-t, 3+t^2 \rangle, \qquad \mathbf{r}_2(s) = \langle 3-s, s-2, s^2 \rangle$$

where t and s are two independent real parameters.

Find **parametric equation** for the tangent line to the space curve $\mathbf{r}_1(t)$ at the intersection point. (Use the value $t = 1$ in part **(a)**).

Solution:

- The velocity vector of $\mathbf{r}_1(t)$ at the intersection point is $\mathbf{r}_1'(1)$.
- Since
$$\mathbf{r}_1'(t) = \langle 1, -1, 2t \rangle,$$
$$\mathbf{r}_1'(1) = \langle 1, -1, 2 \rangle.$$
- The **vector equation** of the tangent line is:
$\mathbf{T}(t) = \mathbf{r}_1(1) + t\langle 1, -1, 2 \rangle = \langle 1, 0, 4 \rangle + t\langle 1, -1, 2 \rangle = \langle 1+t, -t, 4+2t \rangle.$
- The **parametric equations** are:
$$\begin{aligned} x &= 1+t \\ y &= -t \\ z &= 4+2t \end{aligned} \quad .$$

$\square$

Problem 29

Consider the parallelogram with vertices A, B, C, D such that B and C are adjacent to A. If $A = (2,5,1)$, $B = (3,1,4)$, $D = (5,2,-3)$, find the point C.

Solution:

After drawing a picture, the point C is easily seen to be:

$$\overrightarrow{OA} + \overrightarrow{BD} = \langle 2,5,1 \rangle + \langle 2,1,-7 \rangle = \langle 4,6,-6 \rangle,$$

where O is the origin. $\square$

Problem 30(a)

Consider the points $A = (2, 1, 0)$, $B = (1, 0, 2)$ and $C = (0, 2, 1)$. Find the **orthogonal projection** $\mathbf{proj}_{\vec{AB}}(\overrightarrow{AC})$ of the vector $\overrightarrow{AC}$ onto the vector $\overrightarrow{AB}$.

Solution:

- We just plug in the vectors $\mathbf{a} = \overrightarrow{AB} = \langle -1, -1, 2\rangle$ and $\mathbf{b} = \overrightarrow{AC} = \langle -2, 1, 1\rangle$ into the formula:

$$\mathbf{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\,\mathbf{a}.$$

- Plugging in, we get:

$$\mathbf{proj}_{\overrightarrow{AB}}(\overrightarrow{AC}) = \frac{\langle -1, -1, 2\rangle \cdot \langle -2, 1, 1\rangle}{\langle -1, -1, 2\rangle \cdot \langle -1, -1, 2\rangle}\,\langle -1, -1, 2\rangle = \frac{1}{2}\langle -1, -1, 2\rangle.$$

$\square$

Problem 30(b)

Consider the points $A = (2, 1, 0)$, $B = (1, 0, 2)$ and $C = (0, 2, 1)$. Find the area of triangle ABC.

Solution:

- Consider the points $A = (2, 1, 0)$, $B = (1, 0, 2)$ and $C = (0, 2, 1)$.
- Then the area of the triangle $\boldsymbol{\Delta}$ with these vertices can be found by taking the area of the parallelogram spanned by $\overrightarrow{AB}$ and $\overrightarrow{AC}$ and dividing by 2.
- Thus:

$$\textbf{Area}(\boldsymbol{\Delta}) = \frac{|\overrightarrow{AB} \times \overrightarrow{AC}|}{2} = \frac{1}{2}\left\| \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & -1 & 2 \\ -2 & 1 & 1 \end{vmatrix} \right\|$$

$$= \frac{1}{2}|\langle -3, -3, -3 \rangle| = \frac{1}{2}\sqrt{9+9+9} = \frac{1}{2}\sqrt{27}.$$

$\square$

Problem 30(c)

Consider the points $A = (2, 1, 0)$, $B = (1, 0, 2)$ and $C = (0, 2, 1)$. Find the distance **d** from the point C to the line **L** that contains points A and B.

Solution:

- From the figure drawn on the blackboard, we see that the distance **d** from C to **L** is the absolute value of the scalar projection of $\overrightarrow{AC}$ in the direction

$$\mathbf{v} = \overrightarrow{AC} - \mathbf{proj}_{\overrightarrow{AB}}\,\overrightarrow{AC}.$$

- The vector **v** lies in the plane containing A, B, C and is perpendicular to $\overrightarrow{AB}$.
- Hence,

$$\mathbf{d} = |\mathbf{v}| = \sqrt{|\overrightarrow{AC}|^2 - |\mathbf{proj}_{\overrightarrow{AB}}\,\overrightarrow{AC}|^2}.$$

- Next, you the student, do the algebraic calculation of **d**. □

Problem 31

Find **parametric equations** for the line **L** of intersection of the planes $x - 2y + z = 1$ and $2x + y + z = 1$.

Solution:

- The vector part $\mathbf{v}$ of the line **L** of intersection is orthogonal to the normal vectors $\langle 1, -2, 1\rangle$ and $\langle 2, 1, 1\rangle$. Hence $\mathbf{v}$ can be taken to be:

$$\mathbf{v} = \langle 1, -2, 1\rangle \times \langle 2, 1, 1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 1 & 1 \end{vmatrix} = -3\mathbf{i} + \mathbf{j} + 5\mathbf{k}.$$

- Choose $\mathbf{P} \in \mathbf{L}$ so the z-coordinate of $\mathbf{P}$ is zero. Setting $z = 0$, we get:

$$\begin{aligned} x - 2y &= 1 \\ 2x + y &= 1. \end{aligned}$$

  Solving, we find that $x = \frac{3}{5}$ and $y = -\frac{1}{5}$. Hence, $\mathbf{P} = \langle \frac{3}{5}, -\frac{1}{5}, 0\rangle$ lies on the line **L**.

- The **parametric equations** are:

$$\begin{aligned} x &= \tfrac{3}{5} - 3t \\ y &= -\tfrac{1}{5} + t \\ z &= 5t. \end{aligned}$$

$\square$

Problem 32

Let $\mathbf{L}_1$ denote the line through the points $(1, 0, 1)$ and $(-1, 4, 1)$ and let $\mathbf{L}_2$ denote the line through the points $(2, 3, -1)$ and $(4, 4, -3)$. Do the lines $\mathbf{L}_1$ and $\mathbf{L}_2$ **intersect**? If not, are they **skew** or **parallel**?

Solution:

- The vector equations of the lines are:

$$\mathbf{L}_1(t) = \langle 1, 0, 1\rangle + t\langle -2, 4, 0\rangle = \langle 1 - 2t, 4t, 1\rangle$$
$$\mathbf{L}_2(s) = \langle 2, 3, -1\rangle + s\langle 2, 1, -2\rangle = \langle 2 + 2s, 3 + s, -1 - 2s\rangle$$

- Equating z-coordinates, we find $1 = -1 - 2s \implies s = -1$.
- Equating y-coordinates with $s = -1$, we find $4t = 3 - 1 \implies t = \frac{1}{2}$.
- Equating x-coordinates with $s = -1$ and $t = \frac{1}{2}$, we find:

$$\mathbf{L}_1(\frac{1}{2}) = \langle 0, 2, 1\rangle = \mathbf{L}_2(-1).$$

- Hence, the lines **intersect**.

$\square$

Problem 33(a)

Find the volume $\mathbf{V}$ of the **parallelepiped** such that the following four points $A = (1, 4, 2)$, $B = (3, 1, -2)$, $C = (4, 3, -3)$, $D = (1, 0, -1)$ are vertices and the vertices B, C, D are all adjacent to the vertex A.

Solution:

The volume $\mathbf{V}$ is equal to the absolute value of the determinant of the matrix with rows $\overrightarrow{AB} = \langle 2, -3, -4 \rangle$, $\overrightarrow{AC} = \langle 3, -1, -5 \rangle$, $\overrightarrow{AD} = \langle 0, -4, -3 \rangle$.

$$\mathbf{V} = \left\| \begin{array}{ccc} 2 & -3 & -4 \\ 3 & -1 & -5 \\ 0 & -4 & -3 \end{array} \right\|$$

$$= |2 \cdot (-17) + -(-3) \cdot (-9) + (-4) \cdot (-12)| = |-13| = 13.$$

□

Problem 33(b)

Find an **equation of the plane** through $A = (1, 4, 2), B = (3, 1, -2), C = (4, 3, -3)$.

Solution:

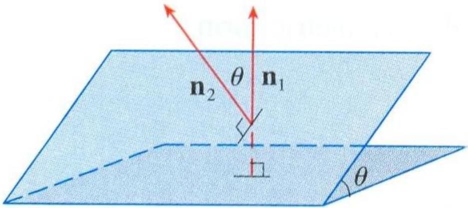
- Consider the vectors $\overrightarrow{AB} = \langle 2, -3, -4 \rangle$ and $\overrightarrow{AC} = \langle 3, -1, -5 \rangle$ which lie parallel to the plane.
- The normal vector is:

$$\mathbf{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & -4 \\ 3 & -1 & -5 \end{vmatrix} = 11\mathbf{i} - 2\mathbf{j} + 7\mathbf{k}.$$

- Since $A = (1, 4, 2)$, is on the plane, then the **equation of the plane** is given by:

$$11(x-1) - 2(y-4) + 7(z-2) = 0.$$

□



Problem 33(c)

Find the angle between the plane through
$A = (1, 4, 2)$, $B = (3, 1, -2)$, $C = (4, 3 - 3)$ and the xy-plane.

Solution:

- The normal vectors of these planes are $\mathbf{n}_1 = \langle 0, 0, 1 \rangle$, $\mathbf{n}_2 = \langle 11, -2, 7 \rangle$.
- If θ is the angle between the planes, then:
$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{7}{\sqrt{11^2 + (-2)^2 + 7^2}} = \frac{7}{\sqrt{174}}.$$
- $$\theta = \cos^{-1}\left(\frac{1}{\sqrt{174}}\right).$$

$\square$

Problem 34(a)

The velocity vector of a particle moving in space equals $\mathbf{v}(t) = 2t\mathbf{i} + 2t^{1/2}\mathbf{j} + \mathbf{k}$ at any time $t \geq 0$. At the time $t = 0$ this particle is at the point $(-1, 5, 4)$. Find the position vector $\mathbf{r}(t)$ of the particle at the time $t = 4$.

Solution:

- To find the position $\mathbf{r}(t)$, integrate the velocity vector field $\mathbf{r}'(t) = \mathbf{v}(t)$.

$$\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = \int \langle 2t, 2t^{\frac{1}{2}}, 1\rangle\, dt$$

$$= \langle t^2 + x_0, \frac{4}{3}t^{\frac{3}{2}} + y_0, t + z_0\rangle.$$

- Now use the initial position $\mathbf{r}(0) = \langle -1, 5, 4\rangle$ to find $x_0 = -1;\ y_0 = 5;\ z_0 = 4.$
- Thus,

$$\mathbf{r}(t) = \langle t^2 - 1, \frac{4}{3}t^{\frac{3}{2}} + 5, t + 4\rangle$$

$$\mathbf{r}(4) = \langle 15, \frac{32}{3} + 5, 8\rangle.$$

$\square$

Problem 34(b)

The velocity vector of a particle moving in space equals $\mathbf{v}(t) = 2t\mathbf{i} + 2t^{1/2}\mathbf{j} + \mathbf{k}$ at any time $t \geq 0$.
Find an **equation of the tangent line** $\mathbf{T}$ to the curve at the time $t = 4$.

Solution:

- Vector equation of the tangent line $\mathbf{T}$ to $\mathbf{r}(t)$ at $t = 4$ is:

$$\mathbf{T}(s) = \mathbf{r}(4) + s\mathbf{r}'(4) = \mathbf{r}(4) + s\mathbf{v}(4).$$

- By part (a), $\mathbf{r}(4) = \langle 15, \frac{32}{3} + 5, 8 \rangle$.
- Since

$$\mathbf{v}(4) = 8\mathbf{i} + 4\mathbf{j} + \mathbf{k} = \langle 8, 4, 1 \rangle,$$

then

$$\mathbf{T}(s) = \langle 15, \frac{32}{3} + 5, 8 \rangle + s\langle 8, 4, 1 \rangle.$$

$\square$

Problem 34(c)

The velocity vector of a particle moving in space equals $\mathbf{v}(t) = \ 2t\mathbf{i} + 2t^{1/2}\mathbf{j} + \mathbf{k}$ at any time $t \geq 0$.
Does the particle ever pass through the point $P = (80, 41, 13)$?

Solution:

- From part **(a)**, we have

$$\mathbf{r}(t) = \langle t^2 - 1, \frac{4}{3}t^{\frac{3}{2}} + 5, t + 4\rangle.$$

- If $\mathbf{r}(t) = \langle 80, 41, 13\rangle$, then $\quad t + 4 = 13 \implies t = 9.$
- Hence the point

$$\mathbf{r}(9) = \langle 80, 41, 13\rangle$$

is on the curve $\mathbf{r}(t)$.

$\square$

Problem 34(d)

The velocity vector of a particle moving in space equals
$\mathbf{v}(t) = 2t\mathbf{i} + 2t^{1/2}\mathbf{j} + \mathbf{k}$ at any time $t \geq 0$.
Find the length of the arc traveled from time $t = 1$ to time $t = 2$.

Solution:

$$\textbf{Length} = \int_1^2 |\mathbf{v}(t)| \, dt = \int_1^2 \sqrt{4t^2 + 4t + 1} \, dt.$$

Since we are not using calculators on our exam, then this is the final answer. $\square$

Problem 35(a)

Consider the surface $x^2 + 3y^2 - 2z^2 = 1$.
What are the traces in $x = k, y = k, z = k$? Sketch a few.

Solution:

- For $x = k \neq 1$, we get the hyperbolas $3y^2 - 2z^2 = k$.
- For $x = 1$, we get the 2 lines $y = \pm\frac{3}{2}z$.
- For $z = 0$, we get the ellipse $x^2 + 3y^2 = 1$.
- For $z = 1$, we get the ellipse $x^2 + 3y^2 = 3$.
- I am leaving it to you to do the sketches!

$\square$

Problem 35(b)

Consider the surface $x^2 + 3y^2 - 2z^2 = 1$.
Sketch the surface in the space.

Solution:

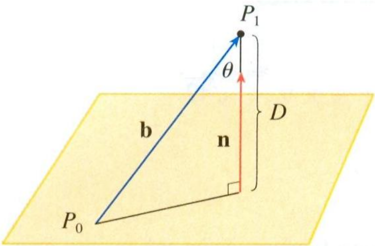
Sorry, you need to do the sketch. $\square$

Problem 36

Find an equation for the tangent plane to the graph of $f(x, y) = y \ln x$ at $(1, 4, 0)$.

Solution:

A problem of this type might not be on this exam (check web site). $\square$



Problem 37

Find the distance **D** between the given parallel planes

$$z = 2x + y - 1, \qquad -4x - 2y + 2z = 3.$$

Solution:

The normal to the first plane is $\mathbf{n} = \langle 2, 1, -1 \rangle$ and the point $P_0 = (0, 0, -1)$ lies on this plane. The point $P_1 = \langle 0, 0, \frac{3}{2} \rangle$ lies on the second plane. Consider the vector from P_0 to P_1 which is $\mathbf{b} = \langle 0, 0, \frac{5}{2} \rangle$. The distance **D** from P_1 to the first plane is equal to:

$$|\mathbf{comp}_{\mathbf{n}}\ \mathbf{b}| = \left| \mathbf{b} \cdot \frac{\mathbf{n}}{|\mathbf{n}|} \right| = |\langle 0, 0, \frac{5}{2} \rangle \cdot \frac{1}{\sqrt{6}} \langle 2, 1, -1 \rangle| = \frac{5}{2\sqrt{6}}.$$

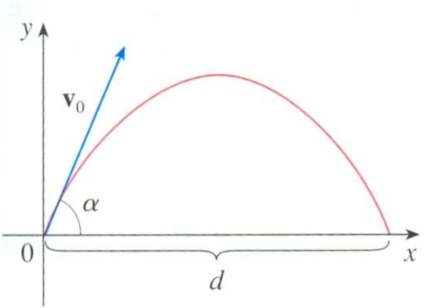
□

Problem 38

Identify the surface given by the equation $4x^2 + 4y^2 - 8y - z^2 = 0$. Draw the traces and sketch the curve.

Solution:

Sorry, no sketch given. $\square$



Problem 39(a)

A projectile is fired from a point 5 m above the ground at an angle of 30 degrees and an initial speed of 100 m/s.
Write an equation for the acceleration vector.

Solution:

Since the force due to gravity acts downward, we have

$$\mathbf{F} = m\mathbf{a} = -mg\mathbf{j},$$

where $g = |a| \approx 9.8\ \mathbf{m}/\mathbf{s}^2$. Thus $\mathbf{a} = -g\mathbf{j}$. $\square$

Problem 39(b) and 34(c)

A projectile is fired from a point 5 m above the ground at an angle of 30 degrees and an initial speed of 100 m/s.
(b) Write a vector for initial velocity $\mathbf{v}(0)$.
(c) Write a vector for the initial position $\mathbf{r}(0)$

Solution:

- Initial velocity is:

$$\mathbf{v}(0) = 100(\cos 30^\circ \mathbf{i} + \sin 30^\circ \mathbf{j}) = 50\sqrt{3}\mathbf{i} + 50\mathbf{j},$$

  in units of m/s.
- The initial position is:

$$\mathbf{r}(0) = 5\mathbf{j},$$

  in units of meters m.

$\square$

Problem 39(d)

A projectile is fired from a point 5 m above the ground at an angle of 30 degrees and an initial speed of 100 m/s.
At what time does the projectile hit the ground?

Solution:

- We first find the velocity $\mathbf{r}(t)$ and position $\mathbf{r}(t)$ functions.
$$\mathbf{r}'(t) = \mathbf{v}(t) = -gt\mathbf{j} + \mathbf{v}(0)$$
$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}(0) + \mathbf{D}.$$
Since $\mathbf{D} = \mathbf{r}(0) = 5\mathbf{j}$, then $\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}(0) + 5\mathbf{j}$.
- Hence,
$$\mathbf{r}(t) = 50\sqrt{3}t\mathbf{i} + [50t - \frac{1}{2}gt^2 + 5]\mathbf{j}.$$
- The projectile hits the ground when $50t - \frac{1}{2}gt^2 + 5 = 0$. Applying the quadratic formula, we find
$$t = \frac{100 + \sqrt{100^2 + 40g}}{2g}.$$

$\square$

Problem 39(e)

A projectile is fired from a point 5 m above the ground at an angle of 30 degrees and an initial speed of 100 m/s.
How far did it travel, horizontally, before it hit the ground?

Solution:

- Recall $\mathbf{r}(t) = 50\sqrt{3}t\mathbf{i} + [50t - \frac{1}{2}gt^2 + 5]\mathbf{j}$ and the projectile hits the ground when $t = \frac{100+\sqrt{100^2+40g}}{2g}$.
- The horizontal distance $\mathbf{d}$ traveled is the value of the x-coordinate of $\mathbf{r}(t)$ at $t = \frac{100+\sqrt{100^2+40g}}{2g}$:

$$\mathbf{d} = 50\sqrt{3}\left(\frac{100 + \sqrt{100^2 + 40g}}{2g}\right).$$

$\square$

Problem 40

Explain why the limit of $f(x, y) = (3x^2y^2)/(2x^4 + y^4)$ does not exist as (x, y) approaches $(0, 0)$.

Solution:

A problem of this type might not be on this exam (check web site). $\square$

Problem 41

Find an **equation of the plane** that passes through the point $P(1,1,0)$ and contains the line given by **parametric equations** $x = 2 + 3t$, $y = 1 - t$, $z = 2 + 2t$.

Solution:

- The direction vector $\mathbf{a} = \langle 3, -1, 2 \rangle$ of the line is parallel to the plane.
- For $t = 0$, the point $Q = \langle 2, 1, 2 \rangle$ on the line and the plane.
- So $\mathbf{b} = \overrightarrow{PQ} = \langle 1, 0, 2 \rangle$ is also parallel to the plane.
- To find a normal vector to the plane, take cross products:

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2 \\ 1 & 0 & 2 \end{vmatrix} = \langle -2, -4, 1 \rangle.$$

- Since $(1,1,0)$ is on the plane, the **equation of the plane** is:

$$\langle -2, -4, 1 \rangle \cdot \langle x - 1, y - 1, z \rangle = -2(x-1) - 4(y-1) + z = 0.$$

□

Problem 42(a)

Find all of the first order and second order partial derivatives of the function. $f(x,y) = x^3 - xy^2 + y$

Solution:

There is no problem of this type on this exam. $\square$

Problem 42(b)

Find all of the first order and second order partial derivatives of the function. $f(x,y) = \ln(x + \sqrt{x^2 + y^2})$

Solution:

There is no problem of this type on this exam. $\square$

Problem 43

Find the linear approximation of the function $f(x,y) = xye^x$ at $(x,y) = (1,1)$, and use it to estimate $f(1.1, 0.9)$.

Solution:

There is no problem of this type on this exam. $\square$

Problem 44

Find a **vector function** $\mathbf{r}(t)$ which represents the **curve of intersection** of the paraboloid $z = 2x^2 + y^2$ and the parabolic cylinder $y = x^2$.

Solution:

- Set $t = x$.
- Since $y = x^2 = t^2$, we get from the equation of the paraboloid a **vector function** $\mathbf{r}(t)$ which represents the **curve of intersection**:

$$\mathbf{r}(t) = \langle t, t^2, 2t^2 + (t^2)^2 \rangle = \langle t, t^2, 2t^2 + t^4 \rangle.$$

$\square$

Problem 1(a) - Spring 2009

Given $\mathbf{a} = \langle 3, 6, -2 \rangle$, $\qquad \mathbf{b} = \langle 1, 2, 3 \rangle$.
Write down the vector projection of $\mathbf{b}$ along $\mathbf{a}$. (Hint: Use projections.)

Solution:

- We have $|\mathbf{a}| = \sqrt{9 + 36 + 4} = \sqrt{49} = 7$.
- Then

$$\mathbf{n} = \frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{7}\mathbf{a} = \text{ unit vector parallel to } \mathbf{a}.$$

- So,

$$\mathbf{proj_a b} = (\mathbf{b} \cdot \mathbf{n})\mathbf{n} = \frac{\mathbf{b} \cdot \mathbf{a}}{|\mathbf{a}|^2}\mathbf{a} =$$

$$\frac{1}{49}\langle 1, 2, 3 \rangle \cdot \langle 3, 6, -2 \rangle \langle 3, 6, -2 \rangle = \frac{9}{49}\langle 3, 6, -2 \rangle.$$

$\square$

Problem 45(b) - Spring 2009

Given $\mathbf{a} = \langle 3, 6, -2 \rangle$, $\mathbf{b} = \langle 1, 2, 3 \rangle$.
Write $\mathbf{b}$ as a sum of a vector parallel to $\mathbf{a}$ and a vector orthogonal to $\mathbf{a}$. (Hint: Use projections.)

Solution:

- We have

$$\begin{aligned} \mathbf{b} = \langle 1, 2, 3 \rangle &= \langle 1, 2, 3 \rangle - \frac{9}{49}\langle 3, 6, -2 \rangle + \frac{9}{49}\langle 3, 6, -2 \rangle \\ &= \frac{1}{49}\langle 22, 44, 165 \rangle + \frac{9}{49}\langle 3, 6, -2 \rangle. \end{aligned}$$

- Here
$$\frac{9}{49}\langle 3, 6, -2 \rangle \textbf{ parallel to } \mathbf{a} = \langle 3, 6, -2 \rangle$$
and
$$\frac{1}{49}\langle 22, 44, 165 \rangle \textbf{ orthogonal to } \mathbf{a} = \langle 3, 6, -2 \rangle.$$

□

Problem 45(b) Continuation - Spring 2009

Given $\mathbf{a} = \langle 3, 6, -2 \rangle$, $\quad \mathbf{b} = \langle 1, 2, 3 \rangle$.
Write $\mathbf{b}$ as a sum of a vector parallel to $\mathbf{a}$ and a vector orthogonal to $\mathbf{a}$. (Hint: Use projections.)

Solution:

- Why so? All we did was to write

$$\mathbf{b} = \mathbf{b} - (\mathbf{b} \cdot \mathbf{n})\mathbf{n} + (\mathbf{b} \cdot \mathbf{n})\mathbf{n}$$

where $\mathbf{n} = \frac{\mathbf{a}}{7}$, $\mathbf{n}^2 = 1$.

- Of course this is the same as

$$\mathbf{b} = (\mathbf{b} - \mathbf{proj_a b}) + \mathbf{proj_a b}.$$

That is, we write $\mathbf{b}$ as $\mathbf{proj_a b}$ plus "the rest". But "the rest" is orthogonal to $\mathbf{n}$ (and to $\mathbf{a}$), since

$$(\mathbf{b} - (\mathbf{b} \cdot \mathbf{n})\mathbf{n}) \cdot \mathbf{n} = \mathbf{b} \cdot \mathbf{n} - (\mathbf{b} \cdot \mathbf{n})(\mathbf{n} \cdot \mathbf{n}) = 0, \text{ as } \mathbf{n} \cdot \mathbf{n} = 1.$$

$\square$

Problem 45(c) - Spring 2009

Given $\mathbf{a} = \langle 3, 6, -2 \rangle, \qquad \mathbf{b} = \langle 1, 2, 3 \rangle.$

Let θ be the angle between $\mathbf{a}$ and $\mathbf{b}$. Find $\cos\theta$.

Solution:

$$\cos(\theta) = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \frac{\langle 3, 6, -2 \rangle \cdot \langle 1, 2, 3 \rangle}{|\langle 3, 6, -2 \rangle||\langle 1, 2, 3 \rangle|} = \frac{9}{\sqrt{49}\sqrt{14}} = \frac{9}{7\sqrt{14}}.$$

$\square$

Problem 46(a) - Spring 2009

Given $A = (-1, 7, 5)$, $B = (3, 2, 2)$ and $C = (1, 2, 3)$.
Let $\mathbf{L}$ be the line which passes through the points $A = (-1, 7, 5)$ and $B = (3, 2, 2)$. Find the parametric equations for $\mathbf{L}$.

Solution:

- To get **parametric equations** for $\mathbf{L}$ you need a point through which the line passes and a vector parallel to the line. For example, take the point to be A and the vector to be $\overrightarrow{AB}$.
- The vector equation of $\mathbf{L}$ is

$$\mathbf{r}(t) = \overrightarrow{OA} + t\overrightarrow{AB} = \langle -1, 7, 5 \rangle + t \langle 4, -5, -3 \rangle = \langle -1 + 4t, 7 - 5t, 5 - 3t \rangle ,$$

where O is the origin.

- The **parametric equations** are:

$$\left| \begin{aligned} x &= -1 + 4t \\ y &= 7 - 5t, \qquad\qquad t \in \mathbb{R}. \\ z &= 5 - 3t \end{aligned} \right.$$

$\square$

Problem 46(b) - Spring 2009

Given $A = (-1, 7, 5)$, $B = (3, 2, 2)$ and $C = (1, 2, 3)$.
A, B and C are three of the four vertices of a parallelogram, while CA and CB are two of the four edges. Find the fourth vertex.

Solution:

Denote the fourth vertex by D. Then

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{CB} = \langle -1, 7, 5 \rangle + \langle 2, 0, -1 \rangle = \langle 1, 7, 4 \rangle \,,$$

where O is the origin. That is,

$$D = (1, 7, 4).$$

$\square$

Problem 47(a) - Spring 2009

Consider the points $P(1,3,5)$, $Q(-2,1,2)$, $R(1,1,1)$ in $\mathbb{R}^3$.
Find an equation for the plane containing P, Q and R.

Solution:

Since a plane is determined by its normal vector $\mathbf{n}$ and a point on it, say the point P, it suffices to find $\mathbf{n}$. Note that:

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & -2 & -3 \\ 0 & -2 & -4 \end{vmatrix} = \langle 2, -12, 6 \rangle = 2\langle 1, -6, 3 \rangle.$$

So the **equation of the plane** is:

$$(x-1) - 6(y-3) + 3(z-5) = 0.$$

□

Problem 47(b) - Spring 2009

Consider the points $P(1,3,5)$, $Q(-2,1,2)$, $R(1,1,1)$ in $\mathbb{R}^3$.
Find the area of the triangle with vertices P, Q, R.

Solution:

The area of the triangle $\boldsymbol{\Delta}$ with vertices P, Q, R can be found by taking the area of the parallelogram spanned by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and dividing it by 2. Thus, using **a)**, we have:

$$\textbf{Area}(\boldsymbol{\Delta}) = \frac{|\overrightarrow{PQ} \times \overrightarrow{PR}|}{2} = \frac{1}{2}\,|2\langle 1, -6, 3\rangle|$$
$$= \sqrt{1+36+9} = \sqrt{46}.$$

$\square$

Problem 48 - Spring 2009

Find parametric equations for the line of intersection of the planes $x + y + 3z = 1$ and $x - y + 2z = 0$.

Solution:

- A vector $\mathbf{v}$ parallel to the line is the cross product of the normal vectors of the planes:

$$\mathbf{v} = \langle 1, 1, 3 \rangle \times \langle 1, -1, 2 \rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ 1 & -1 & 2 \end{vmatrix} = \langle 5, 1, -2 \rangle.$$

- A point on $\mathbf{L}$ is any (x_0, y_0, z_0) that satisfies the equations of **both** planes.
- Setting $z = 0$, we obtain the equations $x + y = 1$ and $x - y = 0$ and find such a point $(\frac{1}{2}, \frac{1}{2}, 0)$. Therefore **parametric equations** for $\mathbf{L}$ are:

$$\left| \begin{array}{l} x = \frac{1}{2} + 5t \\ y = \frac{1}{2} + t \\ z = -2t. \end{array} \right.$$

□

Problem 49(a) - Spring 2009

Consider the parametrized curve

$$\mathbf{r}(t) = \left\langle t, t^2, t^3 \right\rangle, \ t \in \mathbb{R}.$$

Set up an integral for the length of the arc between $t = 0$ and $t = 1$. Do **not** attempt to evaluate the integral.

Solution:

- The **velocity field** is:

$$\mathbf{v}(t) = \mathbf{r}'(t) = \left\langle 1, 2t, 3t^2 \right\rangle.$$

- Then the **speed** is

$$\left|\mathbf{r}'(t)\right| = \sqrt{1 + 4t^2 + 9t^4}.$$

- Therefore, the **length** of the arc is:

$$\mathbf{L} = \int_0^1 \sqrt{1 + 4t^2 + 9t^4}\, dt.$$

□

Problem 49(b) - Spring 2009

Consider the parametrized curve

$$\mathbf{r}(t) = \left\langle t, t^2, t^3 \right\rangle, \; t \in \mathbb{R}.$$

Write down the parametric equations of tangent line to $\mathbf{r}(t)$ at $(2, 4, 8)$.

Solution:

- The parametrized curve passes through the point $(2, 4, 8)$ if and only if
$$t = 2, \quad t^2 = 4, \quad t^3 = 8 \Longleftrightarrow t = 2.$$
- The **velocity vector field** to the curve is given by
$$\mathbf{r}'(t) = \left\langle \, 1, 2t, 3t^2 \right\rangle \; \text{hence} \; \mathbf{r}'(2) = \langle 1, 4, 12 \rangle \, .$$
- The **equation of the tangent line** in question is:
$$\left| \begin{array}{l} x = 2 + \tau \\ y = 4 + 4\tau, \quad \tau \in \mathbb{R} \\ z = 8 + 12\tau \end{array} \right.$$

Caution: The parameter along the line, τ, has nothing to do with the parameter along the curve, t. $\square$

Problem 50(a) - Spring 2009

Consider the sphere **S** in $\mathbb{R}^3$ given by the equation

$$x^2 + y^2 + z^2 - 4x - 6z - 3 = 0.$$

Find its center **C** and its radius **R**.

Solution:

- Completing the square we get

$$(x-2)^2 - 4 + y^2 + (z-3)^2 - 9 - 3 = 0$$

$$\iff$$

$$(x-2)^2 + y^2 + (z-3)^2 = 16.$$

- This gives:

$$\mathbf{C} = (2,0,3) \qquad \mathbf{R} = 4$$

$\square$

Problem 50(b) - Spring 2009

What does the equation $x^2 + z^2 = 4$ describe in $\mathbb{R}^3$? Make a sketch.

Solution:

- This a (straight, circular) **cylinder** determined by the circle in the *xz*-plane of radius 2 and center $(0, 0)$ and parallel to the y-axis. $\square$

Problem 51(a) - Spring 2009

Jane throws a basketball at an angle of 45^o to the horizontal at an initial speed of 12 m/s, where m denotes meters. It leaves her hand 2 m above the ground. Assume the acceleration of the ball due to gravity is downward with magnitude 10 m/s^2 and neglect air friction.

(a) Find the velocity function $\mathbf{v}(t)$ and the position function $\mathbf{r}(t)$ of the ball.

Use coordinates in the xy-plane to describe what is happening; assume Jane is standing with her feet at the point $(0,0)$ and y represents the height.

Solution:

- Acceleration due to gravity is $\mathbf{a} = \langle 0, -g \rangle = \langle 0, -10 \rangle$. Initial velocity is $\mathbf{v}(0) = 12\langle \cos\frac{\pi}{4}, \sin\frac{\pi}{4} \rangle = \langle 6\sqrt{2}, 6\sqrt{2} \rangle$.
  So the velocity function is

  $$\mathbf{v}(t) = \mathbf{v}(0) + \int_0^t \mathbf{a}\, d\tau = \mathbf{v}(0) + \mathbf{a}t = \langle 6\sqrt{2}, 6\sqrt{2} - 10t \rangle.$$

  One can recover the position by integrating the velocity:

  $$\mathbf{r}(t) = \int_0^t \mathbf{v}(\tau)\, d\tau + \mathbf{r}(0).$$

  Notice the initial position is $\mathbf{r}(0) = \langle 0, 2 \rangle$. This integral yields:

  $$\mathbf{r}(t) = \mathbf{r}(0) + \mathbf{v}(0)t + \mathbf{a}\frac{t^2}{2} = \left\langle 6\sqrt{2}t, 2 + 6\sqrt{2}t - 5t^2 \right\rangle.$$

$\square$

Problem 51(b) - Spring 2009

Jane throws a basketball from the ground at an angle of 45^o to the horizontal at an initial speed of 12 m/s, where m denotes meters. It leaves her hand 2 m above the ground. Assume the acceleration of the ball due to gravity is downward with magnitude 10 m/s^2 and neglect air friction.

(b) Find the speed of the ball at its highest point.

Solution:

At the highest point, the vertical component of the velocity is zero, so we only need to calculate the horizontal component which is $6\sqrt{2}$. Thus the speed at the highest point is $6\sqrt{2}$. $\square$

Problem 51(c) - Spring 2009

Jane throws a basketball from the ground at an angle of 45^o to the horizontal at an initial speed of 12 m/s, where m denotes meters. It leaves her hand 2 m above the ground. Assume the acceleration of the ball due to gravity is downward with magnitude 10 m/s^2 and neglect air friction.

(c) At what time **T** does the ball reach its highest point.

Solution:

When the ball reaches its highest point, the vertical component of its velocity is zero. That is,

$$6\sqrt{2} - 10t = 0,$$

so $\mathbf{T} = \frac{3\sqrt{2}}{5}$. $\square$

Problem 57(a) - Spring 2010

Consider the **parallelogram** with vertices A, B, C, D such that B and C are adjacent to A where $A = (1, 2, -1)$, $B = (3, 5, 1)$ and $D = (2, -1, 2)$. Find the **area** of the **parallelogram**.

Solution:

- Recall that:

$$\textbf{area} = \left| \overrightarrow{AB} \times \overrightarrow{BD} \right|.$$

- Since $\overrightarrow{AB} = \langle 2, 3, 2 \rangle$ and $\overrightarrow{BD} = \langle -1, -6, 1 \rangle$,

$$\textbf{area} = \left| \begin{array}{ccc} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 3 & 2 \\ -1 & -6 & 1 \end{array} \right| = |\langle 15, -4, -9 \rangle| = \sqrt{322}.$$

$\square$

Problem 57(b) - Spring 2010

Consider the **parallelogram** with vertices **A**, **B**, **C**, **D** such that **B** and **C** are adjacent to **A** where **A** $= (1, 2, -1)$, **B** $= (3, 5, 1)$ and **D** $= (2, -1, 2)$. Find the coordinates of the point **C**.

Solution:

$$\overrightarrow{O\mathbf{C}} = \overrightarrow{O\mathbf{A}} + \overrightarrow{\mathbf{BD}} = \langle 1, 2, -1 \rangle + \langle -1, -6, 1 \rangle = \langle 0, -4, 0 \rangle,$$

where O is the origin. That is,

$$\mathbf{C} = (0, -4, 0).$$

□

Problem 58(a) - Spring 2010

Consider the points $A = (0, 3, -3)$, $B = (-1, 3, 2)$, $C = (-1, 2, -3)$. Find the **orthogonal projection** $\mathbf{proj}_{\overrightarrow{AB}}(\overrightarrow{AC})$ of the vector $\overrightarrow{AC}$ onto the vector $\overrightarrow{AB}$.

Solution:

- We just plug in the vectors $\mathbf{a} = \overrightarrow{AB} = \langle -1, 0, 5 \rangle$ and $\mathbf{b} = \overrightarrow{AC} = \langle -1, -1, 0 \rangle$ into the formula:

$$\mathbf{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}}\,\mathbf{a}.$$

- Plugging in, we get:

$$\mathbf{proj}_{\overrightarrow{AB}}(\overrightarrow{AC}) = \frac{\langle -1, 0, 5 \rangle \cdot \langle -1, -1, 0 \rangle}{\langle -1, 0, 5 \rangle \cdot \langle -1, 0, 5 \rangle}\,\langle -1, 0, 5 \rangle = \frac{1}{26}\langle -1, 0, 5 \rangle.$$

$\square$

Problem 58(b) - Spring 2010

Consider the points $A = (0, 3, -3)$, $B = (-1, 3, 2)$, $C = (-1, 2, -3)$. Find the distance $\mathbf{d}$ from the point C to the line $\mathbf{L}$ that contains points A and B.

Solution (Method 1):

- From the figure drawn on the blackboard, we see that the distance $\mathbf{d}$ from C to $\mathbf{L}$ is the absolute value of the scalar projection of $\overrightarrow{AC}$ in the direction

$$\mathbf{v} = \overrightarrow{AC} - \mathbf{proj}_{\overrightarrow{AB}} \overrightarrow{AC}.$$

- The vector $\mathbf{v}$ lies in the plane containing A, B, C and is perpendicular to $\overrightarrow{AB}$.
- Hence,

$$\mathbf{d} = |\mathbf{v}|.$$

- Next, you the student, do the algebraic calculation of $\mathbf{d}$.

□

Problem 58(b) - Spring 2010

Consider the points $A = (0, 3, -3)$, $B = (-1, 3, 2)$, $C = (-1, 2, -3)$. Find the distance **d** from the point C to the line **L** that contains points A and B.

Solution (Method 1):

- From the figure drawn on the blackboard and by the Pythagorean Theorem, we see that the distance **d** from C to **L** is equal to:

$$\mathbf{d} = \sqrt{|\overrightarrow{AC}|^2 - |\mathbf{proj}_{\overrightarrow{AB}} \overrightarrow{AC}|^2}.$$

- Next, you the student, do the algebraic calculation of **d**. $\square$

Problem 58(b) - Spring 2010

Consider the points $A = (0, 3, -3)$, $B = (-1, 3, 2)$, $C = (-1, 2, -3)$. Find the distance **d** from the point C to the line **L** that contains points A and B.

Solution (Method 2):

- Let P be the point on the line **L** closest to the point C.
- From the figure drawn on the blackboard, we see that the distance **d** from C to **L** is equal to the length of the side PC of the right triangle with legs AP, PC and hypotenuse AC.
- Then, using the rule $|\mathbf{a} \times \mathbf{b}| = |\mathbf{a}||\mathbf{b}|\sin(\theta)$,

$$\mathbf{d} = |\overrightarrow{PC}| = |\overrightarrow{AC}|\sin(\theta) = \frac{|\overrightarrow{AC} \times \overrightarrow{AB}|}{|\overrightarrow{AB}|}.$$

- Next, you the student, do the algebraic calculation of **d**.

$\square$

Problem 59(a) - Spring 2010

Let P_1 be the plane $x + 3y + z = 0$ and P_2 be the plane $2x + y - z = 1$. Find the cosine of the angle between the planes.

Solution:

- Note that the normals of the planes are:

$$\mathbf{n}_1 = \langle 1, 3, 1 \rangle \qquad \mathbf{n}_2 = \langle 2, 1, -1 \rangle.$$

- By the formula for dot products of 2 vectors:

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{\langle 1, 3, 1 \rangle \cdot \langle 2, 1, -1 \rangle}{\sqrt{11}\sqrt{6}} = \frac{4}{\sqrt{11}\sqrt{6}}.$$

$\square$

Problem 59(b) - Spring 2010

Let P_1 be the plane $x + 3y + z = 0$ and P_2 be the plane $2x + y - z = 1$. Find the **parametric equations** of the line of intersection between the 2 planes P_1 and P_2.

Solution:

- The vector part $\mathbf{v}$ of the line $\mathbf{L}$ of intersection is orthogonal to the normal vectors $\langle 1, 3, 1\rangle$ and $\langle 2, 1, -1\rangle$. Hence $\mathbf{v}$ can be taken to be:

$$\mathbf{v} = \langle 1, 3, 1\rangle \times \langle 2, 1, -1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 3 & 1 \\ 2 & 1 & -1 \end{vmatrix} = -4\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}.$$

- Choose $P \in \mathbf{L}$ so the z-coordinate of P is zero. Setting $z = 0$, we obtain:

$$x + 3y = 0$$

$$2x + y = 1.$$

  Solving, we find that $x = \frac{3}{5}$ and $y = -\frac{1}{5}$. Hence, $P = \langle \frac{3}{5}, -\frac{1}{5}, 0\rangle$ lies on the line $\mathbf{L}$.

- The **parametric equations** are:

$$x = \frac{3}{5} - 4t$$
$$y = -\frac{1}{5} + 3t$$
$$z = 0 - 5t = -5t.$$

$\square$

Problem 59(c) - Spring 2010

Let P_1 be the plane $x + 3y + z = 0$ and P_2 be the plane $2x + y - z = 1$. Find the **distance** from the plane P_2 to the origin.

Solution:

- Recall the distance formula $\mathbf{D} = \dfrac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$ from a point $P = (x_1, y_1, z_1)$ to a plane $ax + by + cz + d = 0$.
- In order to apply the formula, rewrite the equation of the plane in standard form: $2x + y - z - 1 = 0$.
- So, the **distance** from the origin to the plane is:

$$\mathbf{D} = \frac{|(2 \cdot 0) + (1 \cdot 0) + (-1 \cdot 0) - 1|}{\sqrt{2^2 + 1^2 + (-1)^2}} = \frac{|-1|}{\sqrt{6}} = \frac{1}{\sqrt{6}}$$

□

Problem 60(a) - Spring 2010

Let $\mathbf{r}(t) = \cos(2t)\mathbf{i} + \sin(2t)\mathbf{j} + t\mathbf{k}$. What is the length $\mathbf{L}$ of the curve starting at $t = 0$ and ending at $t = 5$.

Solution:

- Recall that the length of $\mathbf{r}(t)$ on the interval $[0, 5]$ is gotten by integrating the speed $|\mathbf{r}'(t)|$.
- Calculating, we get:

$$\begin{aligned}\mathbf{L} = \int_0^5 |\mathbf{r}'(t)|\,dt &= \int_0^5 |\langle -2\sin 2t, 2\cos 2t, 1\rangle|\ dt \\ = \int_0^5 \sqrt{4\sin^2 2t + 4\cos^2 2t + 1}\ dt &= \int_0^5 \sqrt{5}\,dt \\ = \sqrt{5}t\Big|_0^5 &= 5\sqrt{5}.\end{aligned}$$

- Thus

$$\mathbf{L} = 5\sqrt{5}$$

$\square$

Problem 60(b) - Spring 2010

Let $\mathbf{r}(t) = \cos(2t)\mathbf{i} + \sin(2t)\mathbf{j} + t\mathbf{k}$. Find an **equation for the tangent line** to the graph at the point given by $t = 0$.

Solution:

- The parametrized curve passes through the point $\mathbf{r}(0) = (1, 0, 0)$
- The **velocity vector field** to the curve is given by

$$\mathbf{r}'(t) = \langle\, -2\sin(2t), 2\cos(2t), 1\rangle \text{ hence } \mathbf{r}'(0) = \langle 0, 2, 1\rangle\,.$$

- The **equation of the tangent line** in question is:

$$\begin{aligned} x &= 1 \\ y &= 2\tau, \quad \tau \in \mathbb{R}. \\ z &= \tau \end{aligned}$$

$\square$

Problem 61 - Spring 2010

Show that the limit $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2}$ does not exist.

Solution:

- Let $f(x,y) = \frac{x^2-y^2}{x^2+y^2}$.
- Along the line $\langle t, t\rangle$, $t \neq 0$, $f(x,y)$ has the value $\frac{0}{2t^2} = 0$.
- Along the line $\langle 0, t\rangle$, $t \neq 0$, $f(x,y)$ has the value $\frac{t^2}{t^2} = 1$.
- Since $f(x,y)$ has **2 different limiting values** at $(0,0)$, it does **not** have a **limit** at $(0,0)$.

$\square$

Problem 62(a) - Spring 2010

Consider the sphere **S** in $\mathbb{R}^3$ given by the equation

$$x^2 + y^2 + z^2 - 2x - 8y - 2 = 0.$$

Find the coordinates of its **center** and its **radius**.

Solution:

- **Completing the square** we get

$$x^2 - 2x + y^2 - 8y + z^2 = (x^2 - 2x + 1) - 1 + (y^2 + 8y + 16) - 16 + (z^2)$$
$$= (x-1)^2 - 1 + (y+2)^2 - 16 + z^2 = 2$$
$$\Longleftrightarrow$$
$$(x-2)^2 + (y+2)^2 + z^2 = 19.$$

- This gives:

$$\textbf{Center} = (1, 4, 0) \qquad \textbf{Radius} = \sqrt{19}$$

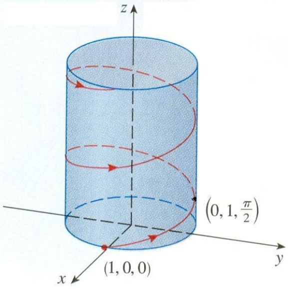
$\square$

Problem 62(b) - Spring 2010

What does the equation $x^2 + y^2 = 64$ describe in $\mathbb{R}^3$. Make a sketch.

Solution:

This equation describes a circular cylinder of radius 8 centered along the z-axis. Here is a sketch of a cylinder of radius 1 centered along the z-axis.



□

Problem 63(a) - Spring 2012

Given $A = (-1, 7, 5)$, $B = (3, 0, 2)$ and $C = (1, 2, 3)$.
Let **L** be the line which passes through the points $A = (-1, 7, 5)$ and $B = (3, 0, 2)$. Find the parametric equations for **L**.

Solution:

- To get **parametric equations** for **L** you need a point through which the line passes and a vector parallel to the line. For example, take the point to be A and the vector to be $\overrightarrow{AB}$.
- The vector equation of **L** is

$$\mathbf{r}(t) = \overrightarrow{OA} + t\overrightarrow{AB} = \langle -1, 7, 5\rangle + t\langle 4, -7, -3\rangle = \langle -1 + 4t, 7 - 7t, 5 - 3t\rangle,$$

where O is the origin.

- The **parametric equations** are:

$$\left| \begin{array}{l} x = -1 + 4t \\ y = 7 - 7t, \\ z = 5 - 3t \end{array} \right. \qquad t \in \mathbb{R}.$$

$\square$

Problem 63(b) - Spring 2012

Given $A = (-1, 7, 5)$, $B = (3, 0, 2)$ and $C = (1, 2, 3)$.
A, B and C are three of the four vertices of a parallelogram, while AB and BC are two of the four edges. Find the fourth vertex.

Solution:

- Denote the fourth vertex by D.
- Then

$$\overrightarrow{OD} = \overrightarrow{OA} + \overrightarrow{BC} = \langle -1, 7, 5 \rangle + \langle -2, 2, 1 \rangle = \langle -3, 9, 6 \rangle ,$$

  where O is the origin.
- That is,

$$D = (-3, 9, 6). \qquad \square$$

Problem 64(a) - Spring 2012

Consider the points $P(1,1,1)$, $Q(-2,1,2)$, $R(1,3,5)$ in $\mathbb{R}^3$.
Find an equation for the plane containing P, Q and R.

Solution:

- Since a plane is determined by its normal vector **n** and a point on it, say the point P, it suffices to find **n**.
- Note that:

$$\mathbf{n} = \overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 0 & 1 \\ 0 & 2 & 4 \end{vmatrix} = \langle -2, 12, -6 \rangle.$$

- So the **equation of the plane** is:

$$-2(x-1) + 12(y-1) - 6(z-1) = 0. \qquad \square$$

Problem 64(b) - Spring 2012

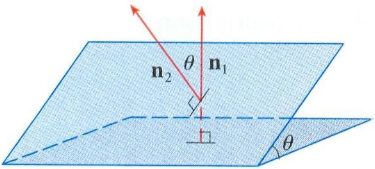
Consider the points $P(1,1,1)$, $Q(-2,1,2)$, $R(1,3,5)$ in $\mathbb{R}^3$.
Find the area of the triangle with vertices P, Q, R.

Solution:

- The area of the triangle $\boldsymbol{\Delta}$ with vertices P, Q, R can be found by taking the area of the parallelogram spanned by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ and dividing it by 2.
- Thus, using part **a)**, we have:

$$\textbf{Area}(\boldsymbol{\Delta}) = \frac{|\overrightarrow{PQ} \times \overrightarrow{PR}|}{2} = \frac{1}{2}\left|\langle -2, 12, -6 \rangle\right|$$
$$= \frac{1}{2}\sqrt{4+144+36} = \frac{1}{2}\sqrt{184}.$$

$\square$



Problem 65 - Spring 2010

Let P_1 be the plane $x - 2y + 2z = 10$ and P_2 be the plane $2x + y + 2z = 0$. Find the cosine of the angle between the planes.

Solution:

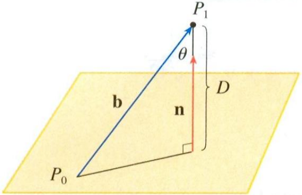
- Note that the normals of the planes are:

$$\mathbf{n}_1 = \langle 1, -2, 2 \rangle \qquad \mathbf{n}_2 = \langle 2, 1, 2 \rangle.$$

- By the formula for dot products of 2 vectors:

$$\cos\theta = \frac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1||\mathbf{n}_2|} = \frac{\langle 1, -2, 2 \rangle \cdot \langle 2, 1, 2 \rangle}{\sqrt{9}\sqrt{9}} = \frac{4}{9}.$$

$\square$



Problem 66(a) - Spring 2012

Find the distance **d** from the point $Q = (1, 6, -1)$ to the plane $2x + y - 2z = 19$.

Solution:

Method 1

- The distance from the point (x_1, y_1, z_1) to the plane $Ax + By + Cz + D = 0$ is:

$$\mathbf{d} = \frac{|Ax_1 + By_1 + Cz_1 + D|}{\sqrt{A^2 + B^2 + C^2}}.$$

- In our case this gives

$$\mathbf{d} = \frac{|2 + 6 + 2 - 19|}{\sqrt{9}} = \frac{9}{\sqrt{9}} = 3$$

□

Problem 66(a) - Spring 2012

Find the distance **d** from the point $Q = (1, 6, -1)$ to the plane $2x + y - 2z = 19$.

Solution:

Method 2

- Note that $P = (0, 19, 0)$ is on the plane.
- The distance between Q and the plane equals the length of the component of $\mathbf{b} = \overrightarrow{PQ}$ which is **orthogonal** to the plane, i.e., which is *parallel* to the normal vector to the plane.
- If the normal vector of the plane is $\mathbf{n}$, then this is distance is the absolute value of the scalar projection: $\mathbf{d} = \left|\mathbf{b} \cdot \frac{\mathbf{n}}{|\mathbf{n}|}\right|$.
- Here $\mathbf{b} = \overrightarrow{PQ} = \langle 1, -13, -1\rangle$ and $\mathbf{n} = \langle 2, 1, -2\rangle$ with $|\mathbf{n}| = \sqrt{2^2 + 1 + (-2)^2} = 3$.
- Then

$$\mathbf{d} = \frac{|\langle 1, -13, -1\rangle \cdot \langle 2, 1, -2\rangle|}{\sqrt{9}} = \frac{|2 - 13 + 2|}{3} = 3.$$

□

Problem 66(a) - Spring 2012

Find the distance **d** from the point $(1, 6, -1)$ to the plane $2x + y - 2z = 19$.

Solution:

- **Method 3** Write the equation of the line through $P(1, 6, -1)$ which is orthogonal to the plane. Find the point Q where it intersects the plane. Find the distance **d** between P and Q from the distance formula.
- The line in question passes through $(1, 6, -1)$ and is parallel to $\mathbf{n} = \langle 2, 1, -2 \rangle$, so it has **parametric equations**

$$\left| \begin{array}{l} x = 1 + 2t \\ y = 6 + t, \qquad t \in \mathbb{P}. \\ z = -1 - 2t \end{array} \right.$$

- It intersects the plane $2x + y - 2z = 19$ if and only if

$$2(1 + 2t) + (6 + t) - 2(-1 - 2t) = 19 \Leftrightarrow 9t = 9 \Leftrightarrow t = 1.$$

- Substituting $t = 1$ above gives the point $Q(3, 7, -3)$.
- So the distance between P and Q is $\mathbf{d} = \sqrt{(3-1)^2 + (7-6)^2 + (-3+1)^2} = \sqrt{4+1+4} = 3.$ $\square$

Problem 66(b) - Spring 2012

Write the parametric equations of the line **L** containing the point $T(1,2,3)$ and perpendicular to the plane $2x + y - 2z = 19$.

Solution:

- The line **L** passes through $(1,2,3)$ and is parallel to $\mathbf{n} = \langle 2, 1, -2 \rangle$.
- So, **L** has **parametric equations**:

$$\begin{aligned} x &= 1 + 2t \\ y &= 2 + t, \qquad t \in \mathbb{R}. \\ z &= 3 - 2t \end{aligned}$$

$\square$

Problem 66(c) - Spring 2012

Find the point of intersection of the line **L** in part **b)** with the plane $2x + y - 2z = 19$.

Solution:

- The line **L** has **parametric equations**:

$$\begin{aligned} x &= 1 + 2t \\ y &= 2 + t, \qquad t \in \mathbb{R}. \\ z &= 3 - 2t \end{aligned}$$

- So, **L** intersects the plane $2x + y - 2z = 19$ if and only if

$$2(1 + 2t) + (2 + t) - 2(3 - 2t) = 19 \Longleftrightarrow 9t = 21 \Longleftrightarrow t = \frac{7}{3}.$$

- Substituting $t = \frac{7}{3}$ in the **parametric equations** of **L** gives the point $Q = (\frac{17}{3}, \frac{13}{3}, -\frac{5}{3})$. $\square$

Problem 67 - Spring 2012

Find the equation of the sphere with center the point $(1, 2, 3)$ and which contains the point $(3, 1, 5)$.

Solution:

- The distance from $P = (1, 2, 3)$ to $Q = (3, 1, 5)$ is the length of the vector $\overrightarrow{PQ}$ which is 3.
- So the radius of the sphere is $r = 3$.
- Hence the equation of the sphere is:

$$(x-1)^2 + (y-2)^2 + (z-3)^2 = 3^2 = 9$$

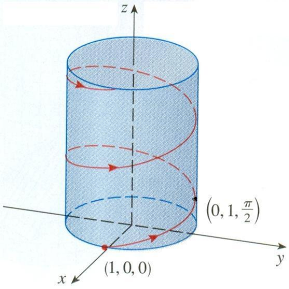
□

Problem 68 - Spring 2012

Make a sketch of the surface in $\mathbb{R}^3$ described by the equation $y^2 + z^2 = 36$. In your sketch of the surface, include the labeled coordinate axes and the trace curves on the surface for the planes $x = 0$ and $x = 4$.

Solution:

This equation describes a circular cylinder of radius 6 centered along the x-axis. Below is a sketch of a cylinder of radius 1 centered along the z-axis. Th red curve should be removed.



□

Problem 69 - Spring 2012

Find the equation of the plane which contains the points $A(1,2,3)$ and $B(1,0,4)$ and which is also perpendicular to the plane $4x - 2y + z = 8$.

Solution:

- Since a plane is determined by its normal vector $\mathbf{n}$ and a point on it, say the point A, it suffices to find $\mathbf{n}$.
- Note that the normal $\langle 4, -2, 1\rangle$ to the plane $4x - 2y + z = 8$ and the vector $\overrightarrow{AB} = \langle 0, -2, 1\rangle$ are both perpendicular to the the normal vector $\mathbf{n}$ that we want.
- It follows that:

$$\mathbf{n} = \overrightarrow{AB} \times \langle 4, -2, 1\rangle = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -2 & 1 \\ 4 & -2 & 1 \end{vmatrix} = \langle 0, 4, 8\rangle.$$

- So the **equation of the plane** is:

$$0(x-1) + 4(y-2) + 8(z-3) = 4(y-2) + 8(z-3) = 0. \quad \square$$

Problem 70(a) - Spring 2012

Suppose that $\mathbf{a} = \langle 2, 1, 2 \rangle$ and $\mathbf{b} = \langle 8, 2, 0 \rangle$. Find the vector projection, call it $\mathbf{c}$, of $\mathbf{b}$ in the direction of $\overrightarrow{a}$.

Solution:

- We just plug in the vectors $\mathbf{a} = \langle 2, 1, 2 \rangle$ and $\mathbf{b} = \langle 8, 2, 0 \rangle$ into the formula:

$$\mathbf{proj_a b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a}.$$

- Plugging in, we get:

$$\mathbf{proj_a b} = \frac{\langle 2, 1, 2 \rangle \cdot \langle 8, 2, 0 \rangle}{\langle 2, 1, 2 \rangle \cdot \langle 2, 1, 2 \rangle} \langle 2, 1, 2 \rangle = \langle 4, 2, 4 \rangle.$$

□

Problem 70(b) - Spring 2012

Suppose that $\mathbf{a} = \langle 2, 1, 2 \rangle$ and $\mathbf{b} = \langle 8, 2, 0 \rangle$ and $\mathbf{c} = \mathbf{proj_a b} = \frac{\mathbf{a} \cdot \mathbf{b}}{\mathbf{a} \cdot \mathbf{a}} \mathbf{a}$. Calculate the vector $\mathbf{b} - \mathbf{c}$ and then show that it is orthogonal to $\mathbf{a}$.

Solution:

- By the previous problem, $\mathbf{c} = \langle 4, 2, 4 \rangle$.
- Calculating, we get

$$\mathbf{b} - \mathbf{c} = \langle 8, 2, 0 \rangle - \langle 4, 2, 4 \rangle = \langle 4, 0, -4 \rangle.$$

- Since

$$\langle 4, 0, -4 \rangle \cdot \langle 2, 1, 2 \rangle = 0,$$

the vector $\mathbf{b} - \mathbf{c}$ is orthogonal to $\mathbf{a}$. $\square$