Practice problems from old exams for math 233

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October 26, 2012

Disclaimer: Your instructor covers far more materials that we can possibly fit into a four/five questions exams. These practice tests are meant to give you an idea of the kind and varieties of questions that were asked within the time limit of that particular tests. In addition, the scope, length and format of these old exams might change from year to year. Users beware! These are NOT templates upon which future exams are based, so don't expect your exam to contain problems exactly like the ones presented here. Check the course web page for an update on the material to be covered on each exam or ask your instructor.

1 Practice problems for Exam 2.

Fall 2008 Exam

1. (a) For the function $f(x, y) = 2x^2 + xy^2$, calculate f_x, f_y, f_{xy}, f_{xx}:
   - $f_x(x, y) =$
   - $f_y(x, y) =$
   - $f_{xy}(x, y) =$
   - $f_{xx}(x, y) =$

   (b) What is the **gradient** $\nabla f(x, y)$ of f at the point $(1, 2)$? $\nabla f =$

   (c) Calculate the **directional derivative** of f at the point $(1, 2)$ in the direction of the vector $\mathbf{v} = \langle 3, 4 \rangle$?

   (d) Next evaluate $D_{\mathbf{u}} f(1, 2) =$

   (e) What is the **linearization** $L(x, y)$ of f at $(1, 2)$?

   (f) Use the **linearization** $L(x, y)$ in the previous part to estimate $f(0.9, 2.1)$.

2. A hiker is walking on a mountain path. The surface of the mountain is modeled by $z = 100 - 4x^2 - 5y^2$. The positive x-axis points to **East** direction and the positive y-axis points **North**.

   (a) Suppose the hiker is now at the point $P(2, -1, 79)$ heading North, is she **ascending** or **descending**?

   (b) When the hiker is at the point $Q(1, 0, 96)$, in which direction on the map should she initially head to **descend** most rapidly?

   (c) What is her **rate** of **descent** when she travels at a speed of 10 meters per minute in the direction of maximal decent from $Q(1, 0, 96)$?

(d) When the hiker is at the point $Q(1, 0, 96)$, in which two directions on her map can she initially head to **neither** ascend nor descend (to keep traveling at the same height)?
Justify your answers.

3. (a) Let $f(x, y)$ be a differentiable function with the following values of the **partial derivatives** $f_x(x, y)$ and $f_y(x, y)$ at certain points (x, y)

x	y	$f_x(x,y)$	$f_y(x,y)$
1	1	−2	4
−1	2	3	−1
1	2	−1	3

(You are given more values than you will need for this problem.) Suppose that x and y are functions of variable t: $x = t^3$; $y = t^2 + 1$, so that we may regard f as a function of t. Compute the derivative of f with respect to t when $t = 1$.
Use the **Chain Rule** to find $\frac{\partial z}{\partial v}$ when $u = 1$ and $v = 1$, where

$$z = x^3y^2 + y^3x; \quad x = u^2 + v^2, \ y = u - v^2.$$

(b) Use the **Chain Rule** to find $\frac{\partial z}{\partial v}$ when $u = 1$ and $v = 1$, where

$$z = x^3y^2 + y^3x; \quad x = u^2 + v^2, \ y = u - v^2.$$

4. Consider the surface $x^2 + y^2 - 2z^2 = 0$ and the point $P(1, 1, 1)$ which lies on the surface.

(i) Find the equation of the **tangent plane** to the surface at P.

(ii) Find the equation of the **normal line** to the surface at P.

5. Let

$$f(x, y) = 2x^3 + xy^2 + 6x^2 + y^2.$$

Find and classify (as local **maxima**, local **minima** or **saddle** points) all **critical points** of f.

6. A flat circular plate has the shape of the region $x^2 + y^2 \leq 4$. The plate (including the boundary $x^2 + y^2 = 4$) is heated so that the temperature at any point (x, y) on the plate is given by $T(x, y) = x^2 + y^2 - 2x$. Find the temperatures at the **hottest** and the coldest points on the plate, including the boundary $x^2 + y^2 = 4$.

Spring 2008 Exam

7. Consider the equation $x^2 + y^2/9 + z^2/4 = 1$.

(a) Identify this quadric (i.e. quadratic surface), and graph the portion of the surface in the region $x \geq 0, y \geq 0$, and $z \geq 0$. Your graph should include tick marks along the three positive coordinate axes, and must clearly show where the surface intersects any of the three positive coordinate axes.

(b) Calculate z_x and z_y at an arbitrary point (x, y, z) on the surface.

(c) Determine the equation of the tangent plane to the surface at the point $(\frac{1}{\sqrt{2}}, \frac{3}{2}, 1)$.

8. Given the function $f(x, y) = x^2y + ye^{xy}$.

   (a) Find the linearization of f at the point $(0, 5)$ and use it to approximate the value of f at the point $(.1, 4.9)$. (An unsupported numerical approximation to $f(.1, 4.9)$ will not receive credit.)

   (b) Suppose that $x(r, \theta) = r\cos\theta$ and $y(r, \theta) = r\sin\theta$. Calculate f_θ at $r = 5$ and $\theta = \frac{\pi}{2}$.

   (c) Suppose a particle travels along a path $(x(t), y(t))$, and that $F(t) = f(x(t), y(t))$ where $f(x, y)$ is the function defined above. Calculate $F'(3)$, assuming that at time $t = 3$ the particle's position is $(x(3), y(3)) = (0, 5)$ and its velocity is $(x'(3), y'(3)) = (3, -2)$.

9. Consider the function $f(x, y) = 2\sqrt{x^2 + 4y}$.

   (a) Find the directional derivative of $f(x, y)$ at $P = (-2, 3)$ in the direction starting from P pointing towards $Q = (0, 4)$.

   (b) Find all unit vectors $\mathbf{u}$ for which the directional derivative $D_{\mathbf{u}}f(-2, 3) = 0$.

   (c) Is there a unit vector $\mathbf{u}$ for which the directional derivative $D_{\mathbf{u}}f(-2, 3) = 4$? Either find the appropriate $\mathbf{u}$ or explain why not.

10. let $f(x, y) = \frac{2}{3}x^3 + \frac{1}{3}y^3 - xy$.

    (a) Find all critical points of $f(x, y)$.

    (b) Classify each critical point as a relative maximum, relative minimum or saddle; you do not need to calculate the function at these points, but your answer must be justified by an appropriate calculation.

11. Use the method of Lagrange multipliers to determine all points (x, y) where the function $f(x, y) = 2x^2 + 4y^2 + 16$ has an extreme value (either a maximum or a minimum) subject to the constraint $\frac{x^2}{4} + y^2 = 4$.

Fall 2007 Exam

12. Find the x and y coordinates of all critical points of the function

$$f(x,y) = 2x^3 - 6x^2 + xy^2 + y^2$$

and use the second derivative test to classify them as local minima, local maxima or saddle points.

13. A hiker is walking on a mountain path. The surface of the mountain is modeled by $z = 1 - 4x^2 - 3y^2$. The positive x-axis points to East direction and the positive y-axis points North. Justify your answers.

    (a) Suppose the hiker is now at the point $P(\frac{1}{4}, -\frac{1}{2}, 0)$ heading North, is she ascending or descending?

    (b) When the hiker is at the point $Q(\frac{1}{4}, 0, \frac{3}{4})$, in which direction should she initially head to ascend most rapidly?

14. Find the volume of the solid bounded by the surface $z = 6 - xy$ and the planes $x = 2$, $x = -2$, $y = 0$, $y = 3$, and $z = 0$.

15. Let $z(x,y) = x^2 + y^2 - xy$ where $x = s - r$ is a known function of r and s and $y = y(r,s)$ is an unknown function of r and s. (Note that z can be considered a function of r and s.) Suppose we know that

$$y(2,3) = 3, \quad \frac{\partial y}{\partial r}(2,3) = 7, \quad \text{and} \quad \frac{\partial y}{\partial s}(2,3) = -5.$$

Calculate $\frac{\partial z}{\partial r}$ when $r = 2$ and $s = 3$.

16. Let $F(x,y,z) = x^2 - 2xy - y^2 + 8x + 4y - z$. This problem continues on the next page.

    (a) Write the equation of the tangent plane to the surface given by $F(x,y,z) = 0$ at the point $(-2, 1, -5)$.

    (b) Find the point (a,b,c) on the surface at which the tangent plane is horizontal, that is, parallel to the $z = 0$ plane.

17. Find the points on the ellipse $x^2 + 4y^2 = 4$ that are closest to the point $(1,0)$.

Fall 2006 Exam

18. (a) Let $f(x,y)$ be a differentiable function with the following values of the partial derivatives $f_x(x,y)$ and $f_y(x,y)$ at certain points (x,y):

x	y	$f_x(x,y)$	$f_y(x,y)$
1	1	-2	4
-1	2	3	-1
1	2	-1	1

(You are given more values than you will need for this problems.) Suppose that x and y are functions of variable t:

$$x = t^3; \quad y = t^2 + 1,$$

so that we may regard f as a function of t. Compute the derivative of f with respect to t when $t = 1$.

(b) Use the Chain Rule to find $\frac{\partial z}{\partial v}$ when $u = 1$ and $v = 1$, where

$$z = x^3y^2 + y^3x; \quad x = u^2 + v, \quad y = 2u - v.$$

19. (a) Let $f(x, y) = x^2y^3 + y^4$. Find the directional derivative of f at the point $(1, 1)$ in the direction which forms an angle (counterclockwise) of $\pi/6$ with the positive x-axis.

    (b) Find an equation of the tangent line to the curve $x^2y + y^3 - 5 = 0$ at the point $(x, y) = (2, 1)$.

20. Let

$$f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2.$$

Find and classify (as local maxima, local minima or saddle points) all critical points of f.

21. Find the maximum value of $f(x, y) = 2x^2 + y^2$ on the circle $x^2 + y^2 = 1$.

22. Find the volume above the rectangle $-1 \leq x \leq 1, \quad 2 \leq y \leq 5$ and below the surface $z = 5 + x^2 + y$.

23. Evaluate the integral

$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3 + 1}\, dx\, dy$$

by reversing the order of integration.

These problems from older exams

24. Use Chain Rule to find dz/dt or $\partial z/\partial u$, $\partial z/\partial v$.

(1) $z = x^2y + 2y^3, \quad x = 1 + t^2, \quad y = (1 - t)^2.$

(2) $z = x^3 + xy^2 + y^3, \quad x = uv, \quad y = u + v.$

25. If $z = f(x, y)$, where f is differentiable, and $x = 1 + t^2$, $y = 3t$, compute dz/dt at $t = 2$ provided that $f_x(5, 6) = f_y(5, 6) = -1$.

26. For the following functions

    (1). $f(x,y) = x^2y + y^3 - y^2$, (2) $g(x,y) = x/y + xy$, (3) $h(x,y) = \sin(x^2y) + xy^2$.

    (a) Find the gradient.
    (b) Find the directional derivative at the point $(0,1)$ in the direction of $\mathbf{v} = \langle 3,4\rangle$.
    (c) Find the maximum rate of change at the point $(0,1)$.

27. Find an equation of the tangent plane to the surface $x^2 + 2y^2 - z^2 = 5$ at the point $(2,1,1)$.

28. Find parametric equations for the tangent line to the curve of intersection of the surfaces $z^2 = x^2 + y^2$ and $x^2 + 2y^2 + z^2 = 66$ at the point $(3,4,5)$.

29. Find and classify all critical points (as local maxima, local minima, or saddle points) of the following functions.

    (1) $f(x,y) = x^2y^2 + x^2 - 2y^3 + 3y^2$, (2) $g(x,y) = x^3 + y^2 + 2xy - 4x - 3y + 5$.

30. Find the minimum value of $f(x,y) = 3 + xy - x - 2y$ on the closed triangular region with vertices $(0,0)$, $(2,0)$ and $(0,3)$.

31. Use Lagrange multipliers to find the extreme values of the following functions with the given constraint.

    (1) $f(x,y) = xy$ with constraint $x^2 + 2y^2 = 3$;

    (2) $g(x,y,z) = x + 3y - 2z$ with constraint $x^2 + 2y^2 + z^2 = 5$.

32. Find the following iterated integrals.

    (1) $\int_1^4 \int_0^2 (x + \sqrt{y})\, dx\, dy$

    (2) $\int_1^2 \int_0^1 (2x + 3y)^2\, dy\, dx$

    (3) $\int_0^1 \int_x^{2-x} (x^2 - y)\, dy\, dx$

    (4) $\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx$ (hint: reverse the order of integration)

33. Evaluate the following double integrals.

    (1) $\int\int_R \cos(x + 2y)\, dA$, $R = \{(x,y) \mid 0 \le x \le \pi,\ 0 \le y \le \pi/2\}$

    (2) $\int\int_R e^{y^2}\, dA$, $R = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$

    (3) $\int\int_R x\sqrt{y^2 - x^2}\, dA$, $R = \{(x,y) \mid 0 \le y \le 1,\ 0 \le x \le y\}$

34. Find the volume.

(1) The solid under the surface $z = 4 + x^2 - y^2$ and above the rectangle

$$R = \{(x, y) \mid -1 \le x \le 1,\ 0 \le y \le 2\}$$

(2) The solid under the surface $z = 2x + y^2$ and above the region bounded by curves $x - y^2 = 0$ and $x - y^3 = 0$.

Spring 2009 Exam

35. Let $f(x, y) = x^2y - y^2 - 2y - x^2$.

(a) Find all of the critical points of f and classify them as either local maximum, local minimum, or saddle points.

(b) Find the linearization $L(x, y)$ of f at the point $(1, 2)$ and use it to approximate $f(0.9, 2.1)$.

36. Consider the function $f(x, y) = x^2 - 2xy + 3y + y^2$.

(a) Find the gradient $\nabla f(x, y)$.

(b) Find the directional derivative of f at the point $(1, 0)$ in the direction $\langle 3, 4\rangle$.

(c) Compute all second partial derivatives of f.

(d) Suppose $x = st^2$ and $y = e^{s-t}$. Find $\frac{\partial f}{\partial s}$ and $\frac{\partial f}{\partial t}$ at $s = 2$ and $t = 1$.

37. Consider the function $f(x, y) = e^{xy}$ over the region D given by $x^2 + 4y^2 \le 2$.

(a) Find the critical points of f.

(b) Find the extreme values on the boundary of D.

(c) What is the absolute maximum value and absolute minimum value of $f(x, y)$ on D?

38. (a) Evaluate the following iterated integral.

$$\int_{-1}^{2}\int_{0}^{1} (x^2y - xy)\, dy\, dx$$

(b) Find the volume of the region below $z = x^2 - 2xy + 3$ and above the rectangle $R = [0, 1] \times [-1, 1]$.

39. Consider the surface S given by the equation $x^2 + y^3 + z^2 = 0$.

(a) Give an equation for the tangent plane of S at the point $(2, -2, 2)$.

(b) Give an equation for the normal line to S at the point $(2, -2, 2)$.

Fall 2009 Exam

40. (a) Let $f(x,y) = \sin(x-y) + \cos(x+y)$. Compute an equation for the tangent plane to the graph of f at the point where $x = \pi/4$, $y = \pi/4$.

    (b) Let $g(x,y,z) = x^2y + y^2z + xz^2$. Compute the directional derivative at the point $(1,-1,1)$ in the direction of the vector $3\mathbf{i} + 4\mathbf{k}$.

41. Suppose $z = e^{x^2+y} + \sin(x+y^2)$, and $x = st$, $y = s/t$. Use the Chain Rule to find $\partial z/\partial t$ and $\partial z/\partial s$ when $s = t = 1$.

    (a) Use the chain rule to write expressions for $\partial z/\partial t$ and $\partial z/\partial s$, but do not evaluate the partial derivatives.

    (b) Compute all the partial derivatives you wrote in (a). Your answers may involve x, y as well as s, t.

    (c) Now use the partial derivatives you computed in (b) together with the formulas in (a) to compute $\partial z/\partial t$ and $\partial z/\partial s$. Your answer should only involve the variables s, t.

42. Let $f(x,y) = x^3/3 + xy^2 - 2xy - 3x$.

    (a) Compute the gradient of f.

    (b) Find all critical points of f.

    (c) For each critical point you found above, classify it as a local maximum, local minimum, or a saddle point.

43. Find the absolute maximum and minimum values attained by $f(x,y) = x^2 - 2x + y^2 - 4y + 2$ on the closed square with vertices $(0,0), (4,0), (0,4), (4,4)$ (in other words, the domain $\{(x,y) \mid 0 \le x \le 4,\ 0 \le y \le 4\}$).

44. Use the method of Lagrange multipliers to find the maximum and minimum values attained by the function $f(x,y,z) = x+y+z$ on the ellipsoid $2x^2 + 3y^2 + 6z^2 = 1$.