MATH 233H EXAM 2 (TAKE HOME)

This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and *any four* of the remaining problems. You must justify your answer to receive credit for a solution; correct answers alone are not necessarily sufficient for credit.

The exam will be submitted on Moodle as individual problems. Please submit exactly five problems (including Problem 1); if you submit more than five (including submitting Problem 1) only the first five (in numerical order) will be graded. If you submit more than four and don't submit Problem 1, then only the first four in numerical order will be graded.

Please make sure your name and student ID are written somewhere in your answers. PDF must be submitted on Moodle; no other formats (such as doc, jpg, tiff, etc) will be accepted. Also please name your PDF files in the form

`StudentID_LastName_FirstName_Exam1_ProblemX.pdf`

where X is the problem number. For example,

`314159_Gunnells_Paul_Exam1_Problem1.pdf`

ADDITIONAL INSTRUCTIONS FOR TAKE-HOME EXAM.

The exam answers must be submitted in PDF. Scans of handwriting are ok, but please be sure that they are at a sufficiently high resolution for me to be able to read them. The following **are allowed**:

- You may use class materials (textbook, your own notes, hw assignments, lecture notes from video lectures, video lectures, and other materials on our course pages) during the exam.
- You may use the Desmos Scientific calculator `https://www.desmos.com/scientific` to assist with numerical computations. Algebraic computations must be done by hand. You may also use your own calculator if you prefer it; Desmos is allowed so that everyone is guaranteed to have access to something.

The following **are not allowed**:

- Discussing the exam with anyone in the class or elsewhere. Exception: you may ask me by email for clarification about a problem, just like in the classroom exam. I will try to check email often but unavoidably there will be delays in replies.
- Using any other sources of information (internet, other books, other notes, tables, Wikipedia, etc.) during the exam. In particular you are allowed to look at your own HW, but not any materials away from WebAssign.
- Using a computer (other than Desmos above or for access to video lectures and our course page). In particular programming is not allowed.

When submitting your exam, you are agreeing to the following statement:

I hereby declare that the work submitted represents my individual effort. I have neither given nor received any help and have not consulted any online resources other than those authorized. I attest that I have followed the instructions of the exam.

Academic honesty is very important to me.

Date: Assigned: Tuesday 20 Oct 2020; Due: on Moodle by 6pm, Thursday 22 Oct 2020.

Let me know if you find any mistakes in the answers.

(1) (20 points) Please compute the following. In this problem (and only this problem), there is no partial credit awarded and it is sufficient to just write the answers of the computations.

(a) (4 points) Let $f(x, y) = xy^2 \sin(x^3 + y^2)$. Compute f_x and f_y. **Answer:** $f_x = y^2 \sin(x^3 + y^2) + 3x^3y^2 \cos(x^3 + y^2)$. $f_y = 2xy \sin(x^3 + y^2) + 2xy^3 \cos(x^3 + y^2)$.

(b) (4 points) Let $g(x, y) = x^2y + e^{x^2-y^2}$. Compute g_{xx}, g_{xy}, and g_{yy}. **Answer:** $g_x = 2xy + 2xe^{x^2-y^2}$, so $g_{xx} = 2y + 2e^{x^2-y^2} + 4x^2e^{x^2-y^2}$ and $g_{xy} = 2x - 4xye^{x^2-y^2}$. $g_y = x^2 - 2ye^{x^2-y^2}$ so $g_{yy} = -2e^{x^2-y^2} + 4y^2e^{x^2-y^2}$.

(c) (4 points) Let $h(x, y, z) = x^2z + x^3y + xyz^2$. Compute ∇h. **Answer:** $\nabla h = \langle 2xz + 3x^2y + yz^2, x^3 + xz^2, x^2 + 2xyz\rangle$.

(d) (4 points) Let $t(x, y) = x^3y^2 - xy^3$. Compute the Hessian D of t as a function of x and y. **Answer:** $t_x = 3x^2y^2 - y^3$ so $t_{xx} = 6xy^2$ and $t_{xy} = 6x^2y - 3y^2$. $t_y = 2x^3y - 3xy^2$ so $t_{yy} = 2x^3 - 6xy$. Then $D = t_{xx}t_{yy} - t_{xy}^2 = 6xy^2(2x^3 - 6xy) - (6x^2y - 3y^2)^2 = -24x^4y^2 - 9y^4$.

(e) (4 points) Let $u(x, y, z) = x^2y + xz^2 + yz$. Compute the directional derivative of u at $(1, 2, 3)$ in the direction of the vector $\langle 2, 6, 9\rangle$. **Answer:** The gradient is $\langle 2xy + z^2, x^2 + z, 2xz + y\rangle$ which is $\langle 13, 4, 8\rangle$ at the point $(1, 2, 3)$. The direction vector is $\langle 2, 6, 9\rangle/11$. So the directional derivative is $1/11(26 + 24 + 72) = 122/11$.

(2) (20 points) Let H be the hyperboloid of one sheet given by $x^2 + y^2 - z^2 = 1$. Find all points on H where the tangent plane is parallel to the plane $6x + 7y + 9z = 0$, or show that there are no such points. **Answer:** H is a level surface of $f = x^2 + y^2 - z^2$ so the vector $\nabla f = \langle 2x, 2y, -2z\rangle$ is perpendicular to H. We can get rid of the 2 and work with $\vec{v} = \langle x, y, -z\rangle$ instead. The normal vector to the plane is $\vec{w} = \langle 6, 7, 9\rangle$ so we need to find the points on H where $\vec{v}$ is a multiple of $\vec{w}$. If we put $\vec{v} = \alpha\vec{w}$ and plug into H we get $36\alpha^2 + 49\alpha^2 - 81\alpha^2 = 1$ so $4\alpha^2 = 1$ and $\alpha = \pm 1/2$. So we want $\vec{v} = \pm 1/2\langle 6, 7, 9\rangle$. This gives the points $(3, 7/2, -9/2)$ and $(-3, -7/2, 9/2)$.

(3) (20 points) Let D be the closed triangle in the xy-plane with vertices $(-2, 0)$, $(2, 0)$, and $(0, 2)$. Find the absolute maximum and the absolute minimum of $f(x, y) = 2x^3 + xy^2 + 5x^2 + y^2$ on D. **Answer:** We have to look for critical points from 3 sources: (1) $\nabla f = 0$, (2) the edges of the domain, and (3) the corners of the domain. From (3) we get the three candidates $C_1 = (0, 2)$, $C_2 = (2, 0)$, and $C_3 = (-2, 0)$. From (1) we get four solutions to $\nabla f = 0$, but two of them $(-1, -2)$, $(-1, 2)$ are not in the domain. The other two are $C_4 = (0, 0)$ and $C_5 = (-5/3, 0)$. Finally we consider the edges (A) $y = 0$ (B) $y = 2 - x$ (C) $y = 2 + x$. For each of these we plug the equation for y into f and then look for critical points in the style of 1-variable calculus. (A) doesn't give a new candidate. (B) gives $x = 0$, $x = -4/9$. The first is a candidate we already have, the second is not on the edge of D. (C) gives two new points. They are roots of a quadratic $9x^2 + 20x + 8$ that doesn't factor, unfortunately, but we can use the QF to find them. These points (as floating point numbers) are $C_6 = (-1.69906\cdots, 0.30094\cdots)$ and $C_7 = (-0.52317\cdots, 1.4768\cdots)$, and both are on the edge of D. Thus we have 7 candidates altogether. When we plug them into f, we see the max occurs at $C_2 = (2, 0)$ and is 36, and the min occurs at $(0, 0)$ and is 0.

(4) (20 points) Find and classify all the critical points of $f(x, y) = x^3 - 3x^2 + 3xy^2 + 3y^2$. **Answer:** We set $\nabla f = 0$. We have (i) $f_x = 3x^2 - 6x + 3y^2$ and (ii) $f_y = 6xy + 6y$. There are no points in the domain of f where ∇f doesn't exist, so we don't get any critical points that way. To solve the system $f_x = 0$, $f_y = 0$, first look at (ii). This equation says either $y = 0$ or $x = -1$. If $y = 0$ then in (i) we get either $x = 2$ or $x = 0$. So we have at least two critical points $P_1 = (0, 0)$ and $P_2 = (2, 0)$. If $y = -1$ then in (i) we have no real solutions.

Thus the only critical points are P_1 and P_2. The hessian is the determinant D of

$$\begin{pmatrix} 6x-6 & 6y \\ 6y & 6x+6 \end{pmatrix}.$$

At P_1 we have $D < 0$, so it's a saddle point. At P_2 we have $D > 0$ and $f_{xx} > 0$, so this is a relative minimum.

(5) (20 points) Suppose that $0 < a < b < c$ are three fixed numbers satisfying $ab + ac + bc = 1$. Find the maximum and minumum value of $2x + 2y + 2z$ on the surface $ax^2 + by^2 + cz^2 = 1$. **Answer:** We use Lagrange multipliers. If we put $g = 2x+2y+2z$ and $f = ax^2+by^2+cz^2$ and put $\lambda\nabla g = \nabla f$, we get the system of equations $\lambda = ax, \lambda = by, \lambda = cz, ax^2 + by^2 + cz^2 = 1$. It's not possible that $\lambda = 0$, since that forces $x = y = z = 0$ which is not on the surface (an ellipsoid, by the way). We can solve for x, y, z in terms of λ and plug into $f = 1$ to get $\lambda = \pm\sqrt{abc}$ (this is where the condition $ab + ac + bc = 1$ gets used). We then get $(x, y, z) = \pm\sqrt{abc}(1/a, 1/b, 1/c)$. The maximum value is $2\sqrt{abc}(1/a + 1/b + 1/c)$ and the minimum value is $-2\sqrt{abc}(1/a + 1/b + 1/c)$.

(6) (20 points) Compute

$$\iint_R e^{y^3}\, dA$$

where R is the region in the xy-plane bounded by the y-axis, $y = 1$, and $y = \sqrt{x}$. **Answer:** We use the order of integration $dx\,dy$. Then the region is $0 \le y \le 1$ and $0 \le x \le y^2$ and the integral is

$$\int_0^1\int_0^{y^2} e^{y^3}\, dx\, dy.$$

The answer is $(e - 1)/3$.

(7) (20 points) Let R_1 be the region $x^2 + y^2 \le 1$ and let R_2 be the region bounded by the graph of $r = 1 + \cos\theta$. Compute the area of the intersection of R_1 and R_2. **Answer:** R_1 is the unit disk. The circle and the cardioid intersect at $\theta = 3\pi/2$ and $\theta = \pi/2$. The radius goes from 0 the circle for $\theta = -\pi$ to π, and from 0 to the cardioid when θ goes from $\pi/2$ to $3\pi/2$. So we do the computation as two integrals:

$$\int_{-\pi}^{\pi}\int_0^1 r\, dr\, d\theta \quad \text{and} \quad \int_{\pi}^{3\pi/2}\int_0^{1+\cos\theta} r\, dr\, d\theta.$$

The first is just $\pi/2$ (1/2 the area of a circle of radius 1). The second is $3\pi/4 - 2$. The total area is $5\pi/4 - 2$.