MATH 233H EXAM I (TAKE HOME)

This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and *any four* of the remaining problems. You must justify your answer to receive credit for a solution; correct answers alone are not necessarily sufficient for credit.

The exam will be submitted on Moodle as individual problems. Please submit exactly five problems (including Problem 1); if you submit more than five (including submitting Problem 1) only the first five (in numerical order) will be graded. If you submit more than four and don't submit Problem 1, then only the first four in numerical order will be graded.

Please make sure your name and student ID are written somewhere in your answers. PDF must be submitted on Moodle; no other formats (such as doc, jpg, tiff, etc) will be accepted. Also please name your PDF files in the form

`StudentID_LastName_FirstName_Exam1_ProblemX.pdf`

where X is the problem number. For example,

`314159_Gunnells_Paul_Exam1_Problem1.pdf`

ADDITIONAL INSTRUCTIONS FOR TAKE-HOME EXAM.

The exam answers must be submitted in PDF. Scans of handwriting are ok, but please be sure that they are at a sufficiently high resolution for me to be able to read them. The following **are allowed**:

- You may use class materials (textbook, your own notes, hw assignments, lecture notes from video lectures, video lectures, and other materials on our course pages) during the exam.
- You may use the Desmos Scientific calculator `https://www.desmos.com/scientific` to assist with numerical computations. Algebraic computations must be done by hand. You may also use your own calculator if you prefer it; Desmos is allowed so that everyone is guaranteed to have access to something.

The following **are not allowed**:

- Discussing the exam with anyone in the class or elsewhere. Exception: you may ask me by email for clarification about a problem, just like in the classroom exam. I will try to check email often but unavoidably there will be delays in replies.
- Using any other sources of information (internet, other books, other notes, tables, Wikipedia, etc.) during the exam. In particular you are allowed to look at your own HW, but not any materials away from WebAssign.
- Using a computer (other than Desmos above or for access to video lectures and our course page). In particular programming is not allowed.

When submitting your exam, you are agreeing to the following statement:

I hereby declare that the work submitted represents my individual effort. I have neither given nor received any help and have not consulted any online resources other than those authorized. I attest that I have followed the instructions of the exam.

Academic honesty is very important to me.

Date: Assigned: Tuesday 22 September 2020; Due: on Moodle by 6pm, Thursday 24 September 2020.

Let me know if you find any mistakes in the answers.

(1) (20 points) Let $\vec{a} = \langle 1,1,1\rangle$, $\vec{b} = \langle 1,2,3\rangle$, $\vec{c} = \langle 1,-2,1\rangle$. Please compute the following. In this problem (and only this problem), there is no partial credit awarded and it is sufficient to just write the answers of the computations.

(a) (4 points) $\vec{a}\cdot\vec{b}$. **Answer:** 6

(b) (4 points) $\vec{b}\times\vec{c}$. **Answer:** $\langle 8,2,-4\rangle$

(c) (4 points) The angle between the vectors $\vec{a}$ and $\vec{c}$. **Answer:** They are orthogonal.

(d) (4 points) The area of the triangle T with vertices at the tips of $\vec{a}$, $\vec{b}$, $\vec{c}$. **Answer:** $1/2|(\vec{b}-\vec{a})\times(\vec{c}-\vec{a})| = 1/2|\langle 0,1,2\rangle\times\langle 0,-3,0\rangle| = 1/2|\langle 6,0,0\rangle| = 3$

(e) (4 points) An equation for the plane containing the triangle T from above. **Answer:** The normal vector is $\langle 6,0,0\rangle$ from the computation above, so the equation looks like $6x = D$. Plugging in the point at the tip of $\vec{a}$ we get $D = 6$, so an equation is $x = 1$.

(2) (20 points) Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS, where the points P, Q, R, S have coordinates

$$P(3,2,1),\quad Q(3,3,0),\quad R(4,2,2),\quad S(4,4,3).$$

Answer: The vectors that run along the edges are $\langle 0,1,-1\rangle$, $\langle 1,0,1\rangle$, $\langle 1,2,2\rangle$. We want the triple scalar product of these, which is given by the determinant of the matrix with these vectors as rows. This gives -3 (putting them in the rows in the order above). Thus we take the absolute value and the volume is 3.

(3) (20 points) Let $\vec{r}_1 = \langle 4+2t, t^2, -t\rangle$, $\vec{r}_2(s) = \langle 1+s, s, s^2\rangle$ be two space curves, where t, s range over all real numbers.

(a) (6 points) Determine all the points of intersection of these space curves. **Answer:** We set two of the components equal to each other, solve for s and t, and then see if the remaining components are equal. Using the y and z components we get the two equations $t^2 = s, -t = s^2$. The solutions are $(t,s) = (0,0)$ and $(-1,1)$. Only the second pair makes the x-components equal (we need $4+2t = 1+s$). The corresponding point of intersection is $(2,1,1)$.

(b) (7 points) At any point of intersection, find parametric equations for the tangent lines to the space curves. (In other words, for any intersection point P, there is a tangent line to $\vec{r}_1$ at P and a tangent line to $\vec{r}_2$ at P, and you should compute both equations.) **Answer:** The derivatives are $\vec{r}_1'(t) = \langle 2, 2t, -1\rangle$ and $\vec{r}_2'(s) = \langle 1,1,2s\rangle$. The tangent vector $\vec{v}_1$ to $\vec{r}_1$ is $\vec{r}_1'(-1) = \langle 2,-2,-1\rangle$ and the tangent vector $\vec{v}_2$ to $\vec{r}_2$ is $\vec{r}_2'(1) = \langle 1,1,2\rangle$. (Note that we have to plug in the values of t and s that we found above.) Therefore equations of the tangent lines are $\langle 2t+2, -2t+1, -t+1\rangle$ for $\vec{r}_1$ and $\langle t+2, t+1, 2t+1\rangle$ for $\vec{r}_2$.

(c) (7 points) At any point of intersection, determine the angle between the tangent lines at that point. **Answer:** The cosine of the angle is $\vec{v}_1\cdot\vec{v}_2/(|\vec{v}_1||\vec{v}_2|) = -2/(\sqrt{9}\sqrt{6}) = -2/(3\sqrt{6})$. The angle is therefore about 105.8 degrees.

(4) (20 points) Two lines are given by the vector-valued functions

$$\vec{r}_1(t) = \langle 2t+1, t, -3t+2\rangle,\quad \vec{r}_2(s) = \langle -4s, -2s+1, 6s\rangle,$$

where t, s vary over all real numbers.

(a) (8 points) Determine if the lines intersect, are parallel, or are skew. **Answer:** The direction vectors are multiples of each other so the lines either are parallel or intersect. In fact they must coincide if they aren't parallel. Putting $t = 0$ in $\vec{r}_1$ and $s = 1/2$ in $\vec{r}_2$ gives the only points on the lines where the y-coordinate is zero. But the other coordinates don't match, so the lines can't coincide and must be parallel.

(b) (12 points) If the lines intersect, find the point of intersection. Otherwise, find the distance between the two lines. **Answer:** Let $\vec{\boldsymbol{v}} = \langle 2, 1, -3\rangle$ be the direction vector of $\vec{\boldsymbol{r}}_1$. The vector $\vec{\boldsymbol{w}} = \langle 1-0, 0-1, 2-0\rangle = \langle 1, -1, 2\rangle$ runs between the lines. We need to find a vector $\vec{\boldsymbol{n}}$ perpendicular to $\vec{\boldsymbol{v}}$ and going between the lines so that we can compute $|\vec{\boldsymbol{w}} \cdot \hat{\boldsymbol{n}}|$. If one draws a picture, one sees that we can take $\vec{\boldsymbol{n}}$ to be $\vec{\boldsymbol{w}} - \alpha\vec{\boldsymbol{v}}$ for a certain scalar α (in fact α is the length of the component of w in the direction of v). Solving $(\vec{\boldsymbol{w}} - \alpha\vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{v}} = 0$ for α gives $\alpha = -5/14$. Therefore $\vec{\boldsymbol{n}} = \langle 12/7, -9/14, 13/14\rangle$ and $\hat{\boldsymbol{n}} = (\sqrt{14/59})\vec{\boldsymbol{n}}$. Finally the distance is $|\vec{\boldsymbol{w}} \cdot \hat{\boldsymbol{n}}| = \sqrt{59/14}$. (Note that in fact that the distance is the same as $|\vec{\boldsymbol{n}}|$; this happens because of the way we constructed $\vec{\boldsymbol{n}}$.)

(5) (20 points) Let S be the graph of $x^2 + z^2 = 4$ and T the graph of $y = x^2 - 2z^2$. Determine a vector-valued function for the curve given by the intersection of S and T. **Answer:** The curve lies on the cylinder $x^2 + z^2 = 4$ so its projection down to the xz-plane is the circle $\langle 2\cos t, 0, 2\sin t\rangle$. Above the circle, at every value of x, z we get a the y-coordinate $x^2 - 2z^2$. So the vector-valued function is $\langle 2\cos t, 4\cos^2 t - 8\sin^2 t, 2\sin t\rangle$. We only need $0 \le t \le 2\pi$ to get the intersection, but all real t is fine too.

(6) (20 points) A particle moves along the graph of $\vec{\boldsymbol{r}}(t) = \langle \cos(t^3), \sin(t^3), t^3\rangle$ for all real numbers t.

(a) (4 points) Compute the velocity of the particle. **Answer:** $\vec{\boldsymbol{v}} = \langle -3t^2\sin(t^3), 3t^2\cos(t^3), 3t^2\rangle$

(b) (2 points) Compute the speed of the particle. **Answer:** $|\vec{\boldsymbol{v}}| = \sqrt{18t^4} = 3t^2\sqrt{2}$ since $t^4 \ge 0$ for all t.

(c) (6 points) Compute the acceleration of the particle. **Answer:** $\langle -9t^4\cos(t^3) - 6t\sin(t^3), -9t^4\sin(t^3) + 6t\cos(t^3), 6t\rangle$

(d) (8 points) Compute the distance traveled by the particle from $t = -2\pi$ to $t = 2\pi$. **Answer:** The particle never goes backwards (the speed is always ≥ 0), so we can just find the arc length. $\sqrt{2}\int_{-2\pi}^{2\pi} 3t^2\,dt = 16\pi^3\sqrt{2}$. Note that we have to make sure that this integral is the right one to compute; if the particle goes backwards we have to be more careful.

(7) (20 points) Let $\vec{\boldsymbol{a}} = \langle a_1, a_2, a_3\rangle$, $\vec{\boldsymbol{b}} = \langle b_1, b_2, b_3\rangle$, $\vec{\boldsymbol{c}} = \langle c_1, c_2, c_3\rangle$. Show that $\vec{\boldsymbol{a}} \times (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) = \alpha\vec{\boldsymbol{b}} - \beta\vec{\boldsymbol{c}}$, where $\alpha = \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{c}}$ and $\beta = \vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}$. (*Hint:* Just compute everything.) **Answer:** One simply has to chug through the computation (there are other ways to do it that go beyond the scope of the course, but computing will definitely work). For the LHS,

$$\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}} = \langle c_3b_2 - c_2b_3, -c_3b_1 + c_1b_3, c_2b_1 - c_1b_2\rangle,$$

so

$$\vec{\boldsymbol{a}} \times (\vec{\boldsymbol{b}} \times \vec{\boldsymbol{c}}) =$$
$$\langle (c_2b_1 - c_1b_2)a_2 + (c_3b_1 - c_1b_3)a_3, (-c_2b_1 + c_1b_2)a_1 + (c_3b_2 - c_2b_3)a_3, (-c_3b_1 + c_1b_3)a_1 + (-c_3b_2 + c_2b_3)a_2\rangle.$$

For the RHS, $\alpha = a_1c_1 + a_2c_2 + a_3c_3$ and $\beta = a_1b_1 + a_2b_2 + a_3b_3$, so

$$\alpha\vec{\boldsymbol{b}} - \beta\vec{\boldsymbol{c}} =$$
$$\langle (c_2b_1 - c_1b_2)a_2 + (c_3b_1 - c_1b_3)a_3, (-c_2b_1 + c_1b_2)a_1 + (c_3b_2 - c_2b_3)a_3, (-c_3b_1 + c_1b_3)a_1 + (-c_3b_2 + c_2b_3)a_2\rangle.$$

The two sides match.