Solution for Math 233, Midterm #2

University of Massachusetts Amherst

Department of Mathematics & Statistics. Fall 2019

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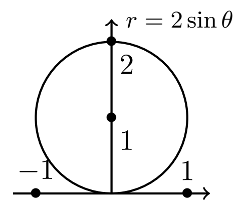
1. (20 points) For each question, please select the best response. Please clearly indicate your choice; ambiguous answers will not receive credit. In this problem, there is *no partial credit* awarded and it is *not necessary to show your work.*

(a) (4 points) The area of the region enclosed by the polar graph $r = 2\sin\theta$ is

(i) $1/\sqrt{2}$ (ii) $\pi/\sqrt{2}$ (iii) 1

(iv) $\pi/2$ (v) $\boxed{\pi}$ (vi) 2π

Solution. By plotting a few points, we readily check that we get the complete graph by letting θ goes from 0 to π. So the area of this polar curve is $\int_0^\pi \int_0^{2\sin\theta} r dr d\theta = \int_0^\pi \frac{r^2}{2}\Big|_0^{2\sin\theta} d\theta = 2\int_0^\pi \sin^2\theta d\theta = 2\int_0^\pi \frac{1-\cos(2\theta)}{2} d\theta = \pi - \int_0^\pi \cos(2\theta)d\theta = \boxed{\pi}$
Alternatively: If you know that it's a circle then you can write down the answer immediately!

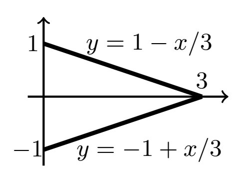


(b) (4 points) Let T be the triangular region in the xy-plane with vertices $(3, 0)$, $(0, -1)$, and $(0, 1)$. Suppose T is a thin plate with constant density 1. Then the x-coordinate of the center of mass of T is

(i) $2\sqrt{3}$ (ii) $\sqrt{3}$ (iii) $\boxed{1}$

(iv) $1/\sqrt{3}$ (v) $1/2$ (vi) $1/3$

Solution. After we sketch the region of integration we can readily setup the double integral for the mass of the thin plate: $\int_0^3 \int_{-1+x/3}^{1-x/3} dydx = \int_0^3 (2 - 2x/3)dx = 3$. Similarly, the moment above the y-axis is $\int_0^3 \int_{-1+x/3}^{1-x/3} xdydx = \int_0^3 (2x - 2x^2/3)dx = 3$. So the x-coordinate of the center of mass is $\boxed{1}$



(c) (4 points) The Jacobian of the transformation $x = u + v$, $y = -u + v$ equals

(i) $2uv$ (ii) $u^2 - v^2$ (iii) $\sqrt{u^2 - v^2}$

(iv) -2 (v) 0 (vi) $\boxed{2}$

Solution. $\dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u} = (1)(1) - (1)(-1) = \boxed{2}$

Continuation of ***1.***

(d) (4 points) Set up the double integral $\iint_R f(x,y)\,dA$ over the shaded region R shown in Figure 1 in the order $dy\,dx$. (The region is bounded by $x = 1$, $y = 1 - x^2$, and $y = e^x$).

(i) $\int_0^1\int_{x^2}^{\ln x} f(x,y)\,dy\,dx$ (ii) $\int_0^1\int_{1-x^2}^{\ln x} f(x,y)\,dy\,dx$

(iii) $\int_1^0\int_{1-x^2}^{e^x} f(x,y)\,dy\,dx$ (iv) $\int_1^0\int_{e^x}^{1-x^2} f(x,y)\,dy\,dx$

(v) $\boxed{\int_0^1\int_{1-x^2}^{e^x} f(x,y)\,dy\,dx}$ (vi) $\int_0^1\int_{e^x}^{1-x^2} f(x,y)\,dy\,dx$

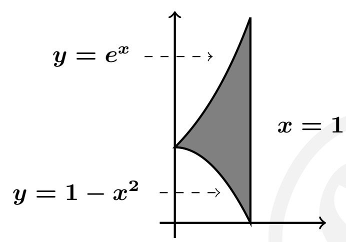


Figure 1: The region R from 1(d)

(e) (4 points) Let E be the solid region bounded by the paraboloid $z = 2 + x^2 + y^2$, the cylinder $x^2 + y^2 = 1$, and the xy-plane (Figure 2). In cylindrical coordinates, when written as an iterated integral the triple integral $\iiint_E e^z\,dV$ becomes

(i) $\int_0^{2\pi}\int_0^1\int_0^{2+r^2} re^z\,dr\,d\theta\,dz$ (ii) $\int_0^{2\pi}\int_0^1\int_0^{2+r^2} re^z\,d\theta\,dz\,dr$

(iii) $\boxed{\int_0^{2\pi}\int_0^1\int_0^{2+r^2} re^z\,dz\,dr\,d\theta}$ (iv) $\int_0^{2\pi}\int_0^2\int_0^{2+r^2} r^2e^z\,dr\,d\theta\,dz$

(v) $\int_0^{2\pi}\int_0^2\int_0^{2+r^2} r^2e^z\,d\theta\,dz\,dr$ (vi) $\int_0^{2\pi}\int_0^2\int_0^{2+r^2} r^2e^z\,dz\,dr\,d\theta$

3dfigure.pdf

Figure 2: The region E from 1(e)

2. (20 points) Let $f(x, y) = x + y$ and let E be the ellipse

$$x^2 + \frac{y^2}{8} = 1.$$

Find the minimum and maximum value of f on E.

Solution. We want to maximize/minimize $f(x, y) = x + y$ subject to the constrain $g(x, y) = 1$ where $g(x, y) := x^2 + y^2/8$. By Lagrange multiplier, we have

$$\nabla f = \lambda \nabla g \text{ for some } \lambda.$$

In terms of coordinates, that mean

$$\langle 1, 1 \rangle = \lambda \langle 2x, y/4 \rangle;$$

Note that this implies in particular

$$\lambda \neq 0,$$

whence

$$y = 8x. \tag{1}$$

Substitute this back into the constrain $g(x, y) = 1$, we get $9x^2 = 1$, whence

$$x = \pm 1/3.$$

Recall (1) and we find that

$$(x, y) = (1/3, 8/3) \quad \text{or} \quad (-1/3, -8/3).$$

We readily check that the maximal and minimal of f are, respectively,

$$\boxed{3 \text{ and } -3.}$$

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3. (20 points) Let R be the triangular region in the xy-plane with vertices $(0, 0)$, $(1, 1)$, and $(1, 0)$. Find the volume over R and under the paraboloid $z = 2 - x^2 - y^2$.

Solution. The volume of this solid is given by

$$\begin{aligned}
&\int_0^1 \int_0^x (2 - x^2 - y^2)\, dy\, dx \\
=&\int_0^1 (2y - x^2 y - y^3/3)\Big|_0^x dx \\
=&\int_0^1 (2x - 4x^3/3) dx \\
=&(x^2 - x^4/3)\Big|_0^1 \\
=&\boxed{2/3}
\end{aligned}$$

4. (20 points) Find the surface area of the part of the graph of $z = 3 + 2y + x^4/4$ that lies over the region R in the xy-plane bounded by $y = x^5$, $x = 1$, and the x-axis.

Solution. The surface area is given by

$$\begin{aligned}\int_0^1 \int_0^{x^5} \sqrt{1 + (z_x)^2 + (z_y)^2}\, dy\, dx &= \int_0^1 \int_0^{x^5} \sqrt{1 + (x^3)^2 + (2)^2}\, dy\, dx \\ &= \int_0^1 \int_0^{x^5} \sqrt{5 + x^6}\, dy\, dx \\ &= \int_0^1 x^5 \sqrt{5 + x^6}\, dx \\ &= \int_5^6 \sqrt{u}(du/6) \quad = \boxed{\frac{6^{3/2} - 5^{3/2}}{9}}\end{aligned}$$

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5. (20 points) Let E be the solid region bounded by the unit sphere $x^2 + y^2 + z^2 = 1$ and inside the cone $z = \sqrt{x^2 + y^2}$. Evaluate $\iiint_E z\, dV$.

Solution. (#1) First we compute the intersection the of the sphere and the cone:

$$1 - (x^2 + y^2) = z^2 = x^2 + y^2,$$

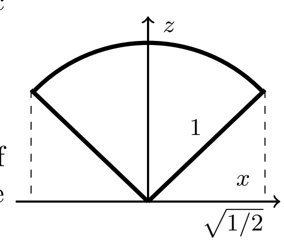
so the intersection is

$$x^2 + y^2 = 1/2,\ z = \sqrt{1/2}.$$

The spherical top suggests that compute this triple integrals using *spherical coordinates*. Since the solid is symmetric around the z-axis,

$$\theta \text{ goes from } 0 \text{ to } 2\pi.$$

To determine the range of ϕ, consider the xz-cross section of the solid as depicted on the right. From this picture we see that



$$\phi \text{ goes from } 0 \text{ to } \pi/4.$$

Put everything together and we see that this triple integral in spherical coordinates is

$$\begin{aligned}&\int_0^{2\pi} \int_0^{\pi/4} \int_0^1 (\rho \cos\phi)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/4} \frac{1}{4} \cos\phi \sin\phi\, d\phi\, d\theta \\ &= \int_0^{2\pi} \frac{1}{16}\, d\theta \\ &= \boxed{\frac{\pi}{8}}\end{aligned}$$
