MATH 471 EXAM I ANSWERS

This exam is worth 100 points, with each problem worth 20 points. Please complete Problem 1 and then any four of the remaining problems. *There are problems on both sides.* Unless indicated, you must justify your answer to receive credit for a solution.

When submitting your exam, please indicate which problems you want graded by writing them in the upper right corner on the cover of your exam booklet. You must select exactly four problems; any unselected problems will not be graded, and if you select more than four only the first four (in numerical order) will be graded.

(1) Please classify the following statements as *True* or *False*. Write out the word completely; do not simply write T or F. There is no partial credit for this problem, and it is not necessary to show your work for this problem.

Note that for a statement to be *True*, it must be true exactly as written and for all cases. To be *False*, there needs to be only one example showing that the statement is false. A statement that is "true most of the time, except sometimes" is false in mathematics.

(a) $13^{-1} \equiv 20 \mod 37$. **ANS:** True. This notation means the number a mod 37 such that $13a \equiv 1 \mod 37$. Since $13 \cdot 20 = 260 = 7 \cdot 37 + 1$, this is true.

(b) The greatest common divisor of 728, 1365, and 819 is 7. **ANS:** False. The gcd of 728 and 1365 is 91. Since $819/91 = 9$, the gcd is 91.

(c) Given positive integers a, b, there exist unique integers q, r satisfying $a = qb + r$. **ANS:** False. This is almost the statement of the division algorithm, but the condition $0 \leq r < b$ has been omitted. And without this condition q and r aren't unique.

(d) For $n \geq 1$, the integer $n^2 - n + 41$ is always prime. **ANS:** False. From class we don't expect there to be such a simple formula to produce primes (although we might expect that this expression is prime infinitely often, by analogy to $n^2 + 1$, which was discussed in class). Plugging in a few small integers does give primes. But clearly if you plug in 41 you won't get a prime (you'll get 41^2). (Note: this is a famous polynomial. If you plug in $0, \ldots, 40$, you actually do get a prime. 41 is the first positive integer where it fails.)

(e) If $N > 1$ is an odd composite integer, then the technique of Fermat factorization will always find a nontrivial divisor of N. **ANS:** True. Since N is odd composite, there is a nontrivial factorization of the form $a \cdot b$ where both a, b are odd. If we solve $a = (s+t)/2$, $b = (s-t)/2$, we'll

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get integers for s and t (since a and b are odd). These will be the s and t that show up when we run the factorization algorithm.

(2) (a) Compute the prime factorization of 3278600. **ANS:** We can use trial division. You can see that $100 = 2^2 5^2$ divides to give 32786. Then 2 divides to give 16393. We only have to check primes now up to $\lfloor\sqrt{16393}\rfloor = 128$. This shows that $16393 = 13^2 \cdot 97$. Putting it all together we get $3278600 = 2^3 5^2 13^2 97$.
  (b) How many divisors does 3278600 have? **ANS:** If the prime factorization is $p_1^{e_1} \cdots p_k^{e_k}$, then the number of divisors is $(e_1+1)\cdots(e_k+1)$. Thus the number of divisors is $4 \cdot 3 \cdot 3 \cdot 2 = 72$.

(3) Find all integral solutions of each of the following linear equations, or explain why there are no solutions.
  (a) $x + 3y = 7$ **ANS:** 1 and 3 are coprime, so there will be solutions. A particular solution to $x + 3y = 1$ is given by $(x_0, y_0) = (1, 0)$, so all solutions to this equation are given by $x = 1 + 3k, y = -k$, where $k \in \mathbb{Z}$. Thus all solutions to the original are given by $x = 7 + 3k, y = -k$, $k \in \mathbb{Z}$.
  (b) $1302x + 1673y = 9$ **ANS:** The gcd of 1302 and 1673 is 7, which does not divide 9. Thus there are no solutions. (This was a typo: the 9 was supposed to be 7.)
  (c) $26x + 91y = 3$ **ANS:** The gcd of 26 and 91 is 13, which does not divide 3. Thus there are no solutions.

(4) Suppose $a \mid c$ and $b \mid c$ and $a \neq b$.
  (a) Prove that the GCD d of a and b divides c. **ANS:** Since $d \mid a$ and $d \mid b$, it follows that $d \mid c$.
  (b) Prove that the LCM m of a and b divides c. **ANS:** It is sufficient to show that any prime p appearing in the factorizations of a, b doesn't have too large an exponent in the prime factorization of m. Let p appear in the prime factorizations of a and b with exponents e, f, respectively. Then p appears in the prime factorization of m with exponent $E = \max\{e, f\}$. We then have that $p^E \mid c$ (since both p^e and p^f divide c). This proves that m divides c.
  (c) Give an example to show that ab doesn't necessarily divide c. **ANS:** The trick here is to make sure that a, b aren't relatively prime. One example is $a = 3$, $b = 9$, $c = 9$.

(5) Define a *k-set of primes* to be a sequence n, $n+2$, $\ldots$, $n+2(k-1)$ such that each is prime.
  (a) Give five examples of 2-sets of primes. (A famous conjecture says there are infinitely many 2-sets, but you only have to give five.) **ANS:** The first five are $(3,5), (5,7), (11,13), (17,19), (29,31)$.
  (b) Give an example of a 3-set of primes. **ANS:** $(3,5,7)$.

(c) Prove that there are no k-sets of primes for $k > 3$. **ANS:** In fact there are no 3-sets of primes other than the one given, since out of three odd numbers $(k, k+2, k+4)$ at least one is divisible by 3. Since $(3, 5, 7, 9)$ is not a 4-set, there can't be any k-sets where $k > 3$.

(6) Approximate how many prime phone numbers there are in our area code (413). **ANS:** As clarified during the exam, this asks for how many prime numbers p there are satisfying $4130000000 \leq p \leq 4139999999$. We can use the approximation $\pi(x) \sim x/\log(x)$. Thus we want to compute

$$4139999999/\log(4139999999) - 4129999999/\log(4129999999).$$

This turns out to be about 431219. A much better approximation to $\pi(x)$ is given by the logarithmic integral $Li(x) = \int_2^x dt/\log t$. Using this gives about 451615. The actual value is 451532. Note: this is not really the number of prime phone numbers, since not every exchange is used (e.g. there are no phone numbers of the form $413 - 000 - abcd$). The first phone numbers in our area code have the form $413 - 200 - abcd$ (they're in Springfield.)

(7) Let $N = 17!/(3! \cdot 5! \cdot 9!)$. It turns out that N is an integer.

(a) Prove $p \mid N$ for $p = 11, 13, 17$ without computing the prime factorization of N. **ANS:** These primes divide the numerator in the expression for N, but they do not divide the denominator. Therefore they cannot be cancelled when the product in the denominator divides the numerator. Thus they divide N.

(b) Compute the prime factorization of N (thus checking part (a)). **ANS:** One way to do this is to write out 17! and then cancel away the numbers from the denominator. Getting rid of 9!, for instance, we get $17 \cdot 16 \cdots 10$. The rest in the denominator have small prime factors and can easily be divided away. The final answer is $N = 2^4 \cdot 5 \cdot 7 \cdot 11 \cdot 13 \cdot 17$.

(c) Find the number of zeros at the end of N when written in (i) base 10, (ii) binary notation. **ANS:** We can see that 10 divides N and 100 doesn't, so the number of zeros in base 10 is 1. Since 2^4 divides and 2^5 doesn't, the number of zeros in base 2 is 4.