DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS

MATH 233 **EXAM 2** **Fall 2009**

NAME:____ Spire ID:______________________

Section Number:_________________________ Instructor's Name: _________________________

In this exam there are six pages, including this one, and five problems . Make sure you have them all before you begin!

1.	(20)	
2.	(20)	
3.	(20)	
4.	(20)	
5.	(20)	
Total	(100)	

Instructions:

- One (single-sided US Letter) page of notes is allowed.
- You may use a calculator, but you must explain how you arrived at your answers, and show your algebraic calculations.
- Simplify your expressions! But please leave fractions and square roots in your answers and do not give decimal expansions.
- All of these expressions are acceptable ways to notate vectors: $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\langle x, y, z\rangle$, $[x, y, z]$, (x, y, z).

1. (a) (10 points) Let $f(x, y) = \sin(x - y) + \cos(x + y)$. Compute an equation for the tangent plane to the graph of f at the point where $x = \frac{\pi}{4}$, $y = \frac{\pi}{4}$.

The formula for the tangent plane to the graph of $f(x, y)$ at the point $(\frac{\pi}{4}, \frac{\pi}{4}, f(\frac{\pi}{4}, \frac{\pi}{4}))$ is

$$z = f(\tfrac{\pi}{4}, \tfrac{\pi}{4}) + f_x(\tfrac{\pi}{4}, \tfrac{\pi}{4})(x - \tfrac{\pi}{4}) + f_y(\tfrac{\pi}{4}, \tfrac{\pi}{4})(y - \tfrac{\pi}{4})$$

So we want to evaluate the functions $f_x(x, y)$, $f_y(x, y)$ and $f(x, y)$ at $(\frac{\pi}{4}, \frac{\pi}{4})$ and plug them into the formula. Compute partial derivatives,

$$\begin{aligned} f_x(x, y) &= \cos(x - y) - \sin(x + y) \\ f_y(x, y) &= -\cos(x - y) - \sin(x + y) \end{aligned}$$

So when you plug in $x = \frac{\pi}{4}$ and $y = \frac{\pi}{4}$ we get

$$\begin{aligned} f(\tfrac{\pi}{4}, \tfrac{\pi}{4}) &= \sin(0) + \cos(\tfrac{\pi}{2}) &&= 0 - 0 = 0 \\ f_x(\tfrac{\pi}{4}, \tfrac{\pi}{4}) &= \cos(0) - \sin(\tfrac{\pi}{2}) &&= 1 - 1 = 0 \\ f_y(\tfrac{\pi}{4}, \tfrac{\pi}{4}) &= -\cos(0) - \sin(\tfrac{\pi}{2}) &&= -1 - 1 = -2 \end{aligned}$$

Putting these values into the formula, we get an equation of the tangent plane

$$z = 0 + (0)(x - \tfrac{\pi}{4}) + (-2)(y - \tfrac{\pi}{4})$$

or more simplified,

$$z = -2(y - \tfrac{\pi}{4}).$$

NOTE: An alternate solution is provided at the end of the exam.

(b) (10 points) Let $g(x, y, z) = x^2y + y^2z + xz^2$. Compute the directional derivative at the point $(1, -1, 1)$ in the direction of the vector $3\mathbf{i} + 4\mathbf{k}$.

The gradient of g is

$$\begin{aligned} \nabla g &= \langle g_x, g_y, g_z \rangle \\ &= \langle 2xy + 0 + z^2, x^2 + 2yz + 0, 0 + y^2 + 2xz \rangle \\ &= \langle 2xy + z^2, x^2 + 2yz, y^2 + 2xz \rangle. \end{aligned}$$

therefore

$$\nabla g(1, -1, 1) = \langle -1, -1, 3 \rangle.$$

Since the magnitude of the vector $\mathbf{v} = 3\mathbf{i} + 4\mathbf{k}$ is $\sqrt{9 + 16} = 5$, a unit vector in the direction of $\mathbf{v}$ is

$$\mathbf{u} = \langle 3/5, 0, 4/5 \rangle.$$

Therefore the directional derivative is

$$D_{\mathbf{u}}g(1, -1, 1) = \nabla g(1, -1, 1) \cdot \mathbf{u} = \langle -1, -1, 3 \rangle \cdot \langle 3/5, 0, 4/5 \rangle = -3/5 + 0 + 12/5 = 9/5$$

2. Suppose $z = e^{x^2+y} + \sin(x + y^2)$, and $x = st$, $y = s/t$. Use the Chain Rule to find $\partial z/\partial t$ and $\partial z/\partial s$ when $s = t = 1$.

   By the chain rule,

$$\begin{aligned}\frac{\partial z}{\partial t} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} \\ &= \left(2xe^{x^2+y} + \cos(x + y^2)\right)(s) + \left(e^{x^2+y} + 2y\cos(x + y^2)\right)\left(-\frac{s}{t^2}\right).\end{aligned}$$

   Notice that when $s = t = 1$, we have $x = 1$ and $y = 1$. Therefore

$$\left.\frac{\partial z}{\partial t}\right|_{s=t=1} = (2e^2 + \cos(2)) - (e^2 + 2\cos(2)) = e^2 - \cos(2)$$

   By the chain rule,

$$\begin{aligned}\frac{\partial z}{\partial s} &= \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} \\ &= \left(2xe^{x^2+y} + \cos(x + y^2)\right)(t) + \left(e^{x^2+y} + 2y\cos(x + y^2)\right)\left(\frac{1}{t}\right).\end{aligned}$$

   Again since $s = t = 1$ implies that $x = 1$ and $y = 1$, we obtain

$$\left.\frac{\partial z}{\partial s}\right|_{s=t=1} = (2e^2 + \cos(2)) + (e^2 + 2\cos(2)) = 3e^2 + 3\cos(2)$$

3. Let $f(x, y) = x^3/3 + xy^2 - 2xy - 3x$.

   (a) (4 points) Compute the gradient of f.

$$\nabla f = \langle f_x, f_y \rangle = \langle x^2 + y^2 - 2y - 3, 2xy - 2x \rangle$$

   (b) (8 points) Find all critical points of f.

   A critical point (x_0, y_0) is a point at which $f_x = f_y = 0$.

   Setting $f_y = 2xy - 2x = 0$, we see that there are only two possibilities: either $x = 0$ or $y = 1$. If $x = 0$ then $f_x = y^2 - 2y - 3 = (y - 3)(y + 1) = 0$ so that either $y = 3$ or $y = -1$. If $y = 1$ then $f_x = x^2 - 4 = 0$ so that either $x = 2$ or $x = -2$. That is, we have obtained the four critical points

$$(0, -1) \quad (0, 3) \quad (2, 0) \quad (-2, 0)$$

   (c) (8 points) For each critical point you found above, classify it as a local maximum, local minimum, or a saddle point.

   To classify the critical points, we will need the determinant of the Hessian matrix

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = \begin{vmatrix} 2x & 2y - 2 \\ 2y - 2 & 2x \end{vmatrix} = (2x)^2 - (2y - 2)^2 = 4(x^2 - y^2 + 2y - 1).$$

   Since $D(0, -1) = -16$, the critical point $(0, -1)$ is a saddle point.
   Since $D(0, 3) = -16$, the critical point $(0, 3)$ is also a saddle point.
   Since $D(2, 0) = 12$ and $f_{xx}(2, 0) = 4 > 0$,
   the critical point $(2, 0)$ is a local minimum.
   Since $D(-2, 0) = 12$ and $f_{xx}(-2, 0) = -4 < 0$,
   the critical point $(-2, 0)$ is a local maximum.

4. Find the absolute maximum and minimum values attained by $f(x, y) = x^2 - 2x + y^2 - 4y + 2$ on the closed square with vertices $(0, 0), (4, 0), (0, 4), (4, 4)$ (in other words, the domain $\{(x, y) \mid 0 \leq x \leq 4,\ 0 \leq y \leq 4\}$).

   The critical points inside the closed square are found by setting $f_x = f_y = 0$. Since $f_x = 2x - 2$ and $f_y = 2y - 4$, the only critical point is $(1, 2)$ and indeed it does lie inside the closed square.

   We now find the extrema points along each edge of the square:

   - Along the left vertical edge, $x = 0$ and $0 \leq y \leq 4$ so that the function along this edge is $f(0, y) = y^2 - 4y + 2$. The polynomial $y^2 - 4y + 2$ is at an extreme value when its first derivative $2y - 4$ is zero (so at $y = 2$), or at the endpoints of the interval $0 \leq y \leq 4$ (so at $y = 0$ and $y = 4$). Therefore the extrema points are $(0, 0), (0, 2)$ and $(0, 4)$.
   - Along the right vertical edge, $x = 4$ and $0 \leq y \leq 4$ so that the function along this edge is $f(4, y) = y^2 - 4y + 10$. The polynomial $y^2 - 4y + 10$ is at an extreme value when its first derivative $2y - 4$ is zero (so at $y = 2$), or at the endpoints of the interval $0 \leq y \leq 4$ (so at $y = 0$ and $y = 4$). Therefore the extrema points are $(4, 0), (4, 2)$ and $(4, 4)$.
   - Along the bottom horizontal edge, $y = 0$ and $0 \leq x \leq 4$ so that the function along this edge is $f(x, 0) = x^2 - 2x + 2$. The polynomial $x^2 - 2x + 2$ is at an extreme value when its first derivative $2x - 2$ is zero (so at $x = 1$), or at the endpoints of the interval $0 \leq x \leq 4$ (so at $x = 0$ and $x = 4$). Therefore the extrema points are $(0, 0), (1, 0)$ and $(4, 0)$.
   - Along the top horizontal edge, $y = 4$ and $0 \leq x \leq 4$ so that the function along this edge is $f(x, 4) = x^2 - 2x + 2$. The polynomial $x^2 - 2x + 2$ is at an extreme value when its first derivative $2x - 2$ is zero (so at $x = 1$), or at the endpoints of the interval $0 \leq x \leq 4$ (so at $x = 0$ and $x = 4$). Therefore the extrema points are $(0, 4), (1, 4)$ and $(4, 4)$.

   We now compare the value of the function at all of the critical points inside the square, and at all of the potential extrema points along the edges:

$$
\begin{array}{cc}
f(1,2) = -3 & \\
 & \\
f(0,0) = 2 & f(4,0) = 10 \\
f(0,2) = -2 & f(4,2) = 6 \\
f(0,4) = 2 & f(4,4) = 10 \\
 & \\
f(1,0) = 1 & f(1,4) = 1
\end{array}
$$

   therefore the absolute maximum value is 10 and the absolute minimum value is -3.

5. Use the method of Lagrange multipliers to find the maximum and minimum values attained by the function $f(x, y, z) = x + y + z$ on the ellipsoid $2x^2 + 3y^2 + 6z^2 = 1$.

The gradient of f is $\nabla f = \langle 1, 1, 1 \rangle$ and the gradient of $g = 2x^2 + 3y^2 + 6z^2$ is $\nabla g = \langle 4x, 6y, 12z \rangle$. Setting $\nabla f = \lambda \nabla g$ and also considering the ellipsoid equation we obtain

$$1 = 4\lambda x$$

$$1 = 6\lambda y$$

$$1 = 12\lambda z$$

$$2x^2 + 3y^2 + 6z^2 = 1.$$

First note that none of x, y, z can equal zero since $1 \neq 0$. Solving for λ in the first three equations,

$$\lambda = \frac{1}{4x} = \frac{1}{6y} = \frac{1}{12z}$$

therefore $4x = 6y = 12z$. Writing $y = (2/3)x$ and $z = (1/3)x$ and substituting this into the fourth (ellipsoid constraint) equation,

$$2x^2 + \frac{4}{3}x^2 + \frac{2}{3}x^2 = 1$$

$$4x^2 = 1$$

$$x = \pm\frac{1}{2}.$$

If $x = 1/2$, then $y = 1/3$ and $z = 1/6$. If $x = -1/2$, then $y = -1/3$ and $z = -1/6$. Comparing the value of the function at these two points,

$$f(1/2, 1/3, 1/6) = 1/2 + 1/3 + 1/6 = 1$$

$$f(-1/2, -1/3, -1/6) = -1/2 - 1/3 - 1/6 = -1.$$

Therefore the absolute maximum value is 1 and the absolute minimum value is -1.

Alternate solution to problem 1. (a).

1. (a) (10 points) Let $f(x,y) = \sin(x-y) + \cos(x+y)$. Compute an equation for the tangent plane to the graph of f at the point where $x = \frac{\pi}{4}$, $y = \frac{\pi}{4}$.

The graph of $f(x,y)$ is the set of all points (x,y,z) in $\mathbf{R}^3$ which satisfy the equation $z = f(x,y)$. Equivalently, the graph is the set of points (x,y,z) in $\mathbf{R}^3$ which satisfy the equation $f(x,y) - z = 0$. Therefore, the graph of the 2-variable function f is a level set of a (new) 3-variable function which we'll call F. Specifically,

$$F(x,y,z) = f(x,y) - z = 0.$$

Therefore, the gradient vector $\nabla F(\frac{\pi}{4}, \frac{\pi}{4}, f(\frac{\pi}{4}, \frac{\pi}{4}))$ is normal to the surface $\{F = 0\}$ at the point

$$(x,y,z) = (\tfrac{\pi}{4}, \tfrac{\pi}{4}, f(\tfrac{\pi}{4}, \tfrac{\pi}{4})).$$

In particular, the gradient vector above is a normal vector for the tangent plane we're looking for. In order to write the equation of the tangent plane, we need a point on the plane, which is $(\frac{\pi}{4}, \frac{\pi}{4}, f(\frac{\pi}{4}, \frac{\pi}{4}))$, and a normal vector for this plane which is $\nabla F(\frac{\pi}{4}, \frac{\pi}{4}, f(\frac{\pi}{4}, \frac{\pi}{4}))$. Compute

$$\begin{aligned}\nabla F &= \langle \tfrac{\partial F}{\partial x}, \tfrac{\partial F}{\partial y}, \tfrac{\partial F}{\partial z} \rangle \\ &= \langle \tfrac{\partial}{\partial x}(f-z), \tfrac{\partial}{\partial y}(f-z), \tfrac{\partial}{\partial z}(f-z) \rangle \\ &= \langle f_x, f_y, -1 \rangle\end{aligned}$$

Therefore, an equation of the plane tangent to the graph $z = f(x,y)$ at the point $(\frac{\pi}{4}, \frac{\pi}{4}, f(\frac{\pi}{4}, \frac{\pi}{4}))$ is

$$0 = f_x(\tfrac{\pi}{4}, \tfrac{\pi}{4})(x - \tfrac{\pi}{4}) + f_y(\tfrac{\pi}{4}, \tfrac{\pi}{4})(y - \tfrac{\pi}{4}) + (-1)(z - f(\tfrac{\pi}{4}, \tfrac{\pi}{4}))$$

The numbers $f(\frac{\pi}{4}, \frac{\pi}{4}) = 0$, $f_x(\frac{\pi}{4}, \frac{\pi}{4}) = 0$ and $f_y(\frac{\pi}{4}, \frac{\pi}{4}) = -2$ are computed in exactly the same way as in the first solution (see the first page). Putting these values into the equation above gives an equation for the tangent plane

$$0 = (0)(x - \tfrac{\pi}{4}) + (-2)(y - \tfrac{\pi}{4}) + (-1)(z - 0)$$

which simplifies to

$$z = -2(y - \tfrac{\pi}{4}).$$

<u>REMARK:</u> A common mistake when attempting to construct this solution is to say that ∇f is the normal vector (incorrect) instead of saying ∇F is the normal vector (correct). If you make this mistake, your final answer will be

$$0 = -2(y - \tfrac{\pi}{4}).$$

That simplifies to $y = \frac{\pi}{2}$ and is obviously wrong because this plane is vertical, so it couldn't possibly be tangent to the graph of a continuous function.