DEPARTMENT OF MATHEMATICS AND STATISTICS
UNIVERSITY OF MASSACHUSETTS

MATH 233 **EXAM 1** **Fall 2009**

NAME:____ Spire ID:____________________

Section Number:______________________ Instructor's Name: ______________________

In this exam there are six pages, including this one, and five problems . Make sure you have them all before you begin!

1.	(20)	
2.	(20)	
3.	(20)	
4.	(20)	
5.	(20)	
Total	(100)	

Instructions:

- One (single-sided US Letter) page of notes is allowed.
- You may use a calculator, but you must explain how you arrived at your answers, and show your algebraic calculations.
- Simplify your expressions! But please leave fractions and square roots in your answers and do not give decimal expansions.
- All of these expressions are acceptable ways to notate vectors: $x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, $\langle x, y, z\rangle$, $[x, y, z]$, (x, y, z).

1. Let $\mathbf{v} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$ and $\mathbf{w} = 2\mathbf{i} + 6\mathbf{k} + 9\mathbf{j}$.

   (a) (10 points) Find the vector representing the projection of $\mathbf{v}$ onto $\mathbf{w}$.

$$proj_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{w}|^2}\mathbf{w} = \frac{(2)(2)+(-1)(6)+(2)(9)}{(2^2+6^2+9^2)}(2\mathbf{i}+6\mathbf{k}+9\mathbf{j}) = \frac{16}{121}(2\mathbf{i}+6\mathbf{k}+9\mathbf{j})$$

   (b) (10 points) Find $\cos\theta$, where θ is the angle between $\mathbf{v}$ and $\mathbf{w}$.

   By the definition of the dot product, $\mathbf{v}\cdot\mathbf{w} = |\mathbf{v}||\mathbf{w}|\cos\theta$. Therefore

$$\cos\theta = \frac{\mathbf{v}\cdot\mathbf{w}}{|\mathbf{v}||\mathbf{w}|} = \frac{(2)(2)+(-1)(6)+(2)(9)}{\sqrt{2^2+(-1)^2+2^2}\,\sqrt{2^2+6^2+9^2}} = \frac{16}{\sqrt{5}\sqrt{121}} = \frac{16}{11\sqrt{5}}$$

2. Consider the points $P = (0, 3, -3)$, $Q = (-1, 3, 2)$, $R = (-1, 2, -3)$.

(a) (10 points) Find an equation for the plane containing P, Q, R.

The *scalar equation of a plane* is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0,$$

where (x_0, y_0, z_0) is a point on the plane and $\langle a, b, c\rangle$ is a vector normal to the plane.

We may use any of the points P, Q, R as the point (x_0, y_0, z_0) on the plane. For example, we use P, so that

$$(x_0, y_0, z_0) = (0, 3, -3).$$

To obtain a vector normal to the plane, take the cross product of the two vectors $\overrightarrow{PQ} = \langle -1-0, 3-3, 2-(-3)\rangle = \langle -1, 0, 5\rangle$ and $\overrightarrow{RP} = \langle 0-(-1), 3-2, -3-(-3)\rangle = \langle 1, 1, 0\rangle$, which lie flat in the plane. We obtain

$$\begin{aligned}
\langle a, b, c\rangle &= \overrightarrow{PQ} \times \overrightarrow{RP} = \langle -1, 0, 5\rangle \times \langle 1, 1, 0\rangle = \begin{vmatrix} i & j & k \\ -1 & 0 & 5 \\ 1 & 1 & 0 \end{vmatrix} \\
&= \begin{vmatrix} 0 & 5 \\ 1 & 0 \end{vmatrix} \mathbf{i} - \begin{vmatrix} -1 & 5 \\ 1 & 0 \end{vmatrix} \mathbf{j} + \begin{vmatrix} -1 & 0 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\
&= [(0)(0) - (1)(5)]\,\mathbf{i} - [(-1)(0) - (1)(5)]\,\mathbf{j} + [(-1)(1) - (0)(1)]\,\mathbf{k} \\
&= -5\mathbf{i} + 5\mathbf{j} - \mathbf{k} \\
&= \langle -5, 5, -1\rangle.
\end{aligned}$$

Therefore the scalar equation of the plane is

$$-5(x - 0) + 5(y - 3) - 1(z - (-3)) =, 0$$

or more simply

$$-5x + 5(y - 3) - (z + 3) = 0.$$

One may also collect the constant terms and (equivalently) write the *linear equation of the plane*

$$-5x + 5y - z - 18 = 0.$$

Note that the equation of the plane is not unique, and could have been written in many ways:

- by using any of the points P, Q, R as the point (x_0, y_0, z_0) on the plane
- by taking the cross product of any two of the vectors $\overrightarrow{PQ}, \overrightarrow{PR}, \overrightarrow{QR}$ or $\overrightarrow{QP}, \overrightarrow{RP}, \overrightarrow{RQ}$ (which all lie flat in the plane) to obtain a vector normal to the plane
- by writing either the scalar or linear equations of the plane.

(b) (10 points) Find the area of the triangle with vertices P, Q, R.

The area of the triangle is equal to half of the area of the parallelogram determined by the vectors $\overrightarrow{PQ}$ and $\overrightarrow{RP}$. That is,

$$Area\Delta = \frac{|\overrightarrow{PQ} \times \overrightarrow{RP}|}{2}.$$

Using our calculation of the cross product from part (a),

$$Area\Delta = \frac{|\langle -5, 5, -1\rangle|}{2} = \frac{\sqrt{(-5)^2 + 5^2 + (-1)^2}}{2} = \frac{\sqrt{51}}{2}$$

Note that we would have obtained the same area had we used the two vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$, or alternatively $\overrightarrow{RP}$ and $\overrightarrow{RQ}$.

3. Let P_1 be the plane $x + y - z = 0$ and P_2 be the plane $x - 2y + z = 1$.

(a) (10 points) Find parametric equations for the line of intersection of P_1 and P_2.

The parametric equations of a line are

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

where (x_0, y_0, z_0) is a point on the line and $\langle a, b, c\rangle$ is a vector parallel to the line.

To find a point on the line, we must find a point that lies on *both* planes. Setting $x = 0$ in both plane equations we obtain

$$y - z = 0, \qquad -2y + z = 1.$$

Solving the first equation for y, we obtain $y = z$. Substituting this into the second equation we obtain $-2z + z = 1$, or rather $-z = 1$ so that $z = -1$. Therefore

$$(x_0, y_0, z_0) = (0, -1, -1)$$

is a point on the line (you may check this, by verifying that $x = 0, y = -1$ and $z = -1$ satisfy both plane equations).

To find a vector parallel to the line, take the cross product of the normal vectors of the two planes. That is,

$$\begin{aligned} \langle a, b, c\rangle &= \langle 1, -2, 1\rangle \times \langle 1, 1, -1\rangle \\ &= \begin{vmatrix} i & j & k \\ 1 & -2 & 1 \\ 1 & 1 & -1 \end{vmatrix} \\ &= \begin{vmatrix} -2 & 1 \\ 1 & -1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & -2 \\ 1 & 1 \end{vmatrix} \mathbf{k} \\ &= [(-2)(-1) - (1)(1)]\,\mathbf{i} - [(1)(-1) - (1)(1)]\,\mathbf{j} + [(1)(1) - (1)(-2)]\,\mathbf{k} \\ &= \langle 1, 2, 3\rangle \end{aligned}$$

Therefore the equations of the line are

$$\begin{aligned} x &= t \\ y &= -1 + 2t \\ z &= -1 + 3t \end{aligned}$$

(b) (10 points) Find the distance from the origin to the plane P_2.

The distance between a point (x_0, y_0, z_0) and a plane $ax + by + cz + d = 0$ is given by

$$d = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

therefore the distance between $(0, 0, 0)$ and the plane $x - 2y + z - 1 = 0$ is

$$d = \frac{|(1)(0) + (-2)(0) + (1)(0) - 1|}{\sqrt{1^2 + (-2)^2 + 1^2}} = \frac{1}{\sqrt{6}}.$$

Another way to do this problem is to take a vector from the origin to the plane, and then compute its component in the direction of a vector perpendicular to the plane.

4. Let $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j} + t^6\mathbf{k}$.

(a) (6 points) Find an equation for the tangent line to the graph at the point given by $t = 1$.

The parametric equations of a line are

$$\begin{aligned} x &= x_0 + at \\ y &= y_0 + bt \\ z &= z_0 + ct \end{aligned}$$

where (x_0, y_0, z_0) is a point on the line and $\langle a, b, c\rangle$ is a vector parallel to the line.

By plugging $t = 1$ into the position vector we obtain the point $(x_0, y_0, z_0) = (1, 1, 1)$, which lies on both the curve and the tangent line.

By taking the derivative of the position vector we obtain the velocity vector $\mathbf{r}'(t) = \langle 2t, 3t^2, 6t^5\rangle$. At $t = 1$, $\mathbf{r}'(1) = \langle 2, 3, 6\rangle$ gives a vector parallel to the tangent line.

Therefore the equations of the tangent line are

$$\begin{aligned} x &= 1 + 2t \\ y &= 1 + 3t \\ z &= 1 + 6t. \end{aligned}$$

(b) (6 points) Find the unit tangent vector $\mathbf{T}$ to the graph at the point given by $t = 1$.

Above we computed the tangent vector $\langle 2, 3, 6\rangle$ at $t = 1$. This vector has length $\sqrt{2^2 + 3^2 + 6^2} = \sqrt{49} = 7$. Therefore the unit tangent vector is

$$\mathbf{T}(1) = \langle 2/7, 3/7, 6/7\rangle.$$

(c) (8 points) Write a definite integral that computes the length of the graph of $\mathbf{r}(t)$ from $t = 1$ to $t = 2$, but do **not** attempt to evaluate it.

The length of the curve $\mathbf{r}(t)$ between $t = \alpha$ and $t = \beta$ is given by $L = \int_\alpha^\beta |\mathbf{r}'(t)|dt$. Since $\mathbf{r}'(t) = \langle 2t, 3t^2, 6t^5\rangle$,

$$\begin{aligned} L &= \int_1^2 \sqrt{(2t)^2 + (3t^2)^2 + (6t^5)^2}dt \\ &= \int_1^2 \sqrt{4t^2 + 9t^4 + 36t^{10}}dt \end{aligned}$$

5. Consider a particle moving with acceleration $\mathbf{a}(t) = \langle t, e^t, -\sin(t) \rangle$.

(a) (10 points) Find the velocity vector $\mathbf{v}(t)$ of the particle, assuming that $\mathbf{v}(0) = \mathbf{0}$.

The velocity vector is

$$\begin{aligned} \mathbf{v}(t) &= \int \mathbf{a}(t) dt \\ &= \langle \frac{1}{2} t^2 + c_1, e^t + c_2, \cos(t) + c_3 \rangle \end{aligned}$$

for integration constants c_1, c_2, c_3. Using the initial condition to find the constants,

$$\mathbf{v}(0) = \langle 0 + c_1, 1 + c_2, 1 + c_3 \rangle = \langle 0, 0, 0 \rangle.$$

Therefore $c_1 = 0, c_2 = -1$ and $c_3 = -1$ and the velocity vector is

$$\mathbf{v}(t) = \langle \frac{1}{2} t^2, e^t - 1, \cos(t) - 1 \rangle$$

(b) (10 points) Find the position vector $\mathbf{r}(t)$ of the particle, assuming that $\mathbf{r}(0) = \mathbf{0}$.

The position vector is

$$\begin{aligned} \mathbf{r}(t) &= \int \mathbf{v}(t) dt \\ &= \langle \frac{1}{6} t^3 + d_1, e^t - t + d_2, \sin(t) - t + d_3 \rangle \end{aligned}$$

for integration constants d_1, d_2, d_3. Using the initial condition to find the constants,

$$\mathbf{r}(0) = \langle 0 + d_1, 1 - 0 + d_2, 0 - 0 + d_3 \rangle = \langle 0, 0, 0 \rangle.$$

Therefore $d_1 = 0, d_2 = -1$ and $d_3 = 0$ and the position vector is

$$\mathbf{r}(t) = \langle \frac{1}{6} t^3, e^t - t - 1, \sin(t) - t \rangle$$