HOMEWORK 8, ADVANCED CALCULUS
DUE 4/21/17

Please hand in your home work before class, have it neatly written, organized (the grader will not decipher your notes), stapled, with your name and student ID on top.

Problem 1. In class we considered the predator-prey model

$$x' = F(x)$$

for $x = (x_1, x_2)$ and the vector field $F(x_1, x_2) = (ax_1 - bx_1x_2, -cx_2 + dx_1x_2)$ with $a, b, c, d > 0$ positive real numbers.

(i) Find all the equilibrium points of F, i.e. all the points $\mathring{x}$ such that $F(\mathring{x}) = 0$.
(ii) Linearize the vector field at the equilibrium point $\mathring{x} = (0, 0)$ and write down the corresponding linear 2-dimensional ODE.
(iii) Characterize whether $(0, 0)$ is a stable, unstable, saddle, or spiraling (inward or outward) fixed point of the linear ODE.
(iv) How would you answer the question: if the predator-prey population initially is small, what will happen? Will the population die out, or stay kind of small, or grow?

Problem 2. A pendulum of rod length l (and pendant of a certain mass) satisfies the 2nd order ODE

$$\omega'' + c\,\omega' + \tfrac{g}{l}\sin(\omega) = 0$$

where g is the gravitational constant, $\omega(t)$ is the elongation angle at time t (i.e. the angle measured from the pendulums resting position) and c is a friction coefficient (which will depend among other things on the mass). If $c = 0$ then there is no friction in the pendulum (e.g., pendulum in vacuum with almost friction free pivoting devise etc.).

(i) Rewrite the 2nd order ODE as a first order ODE $x' = F(x)$ for $x = (\omega, v)$ where $v = \omega'$ (angular velocity). Write down the formula for the vector field $F(x)$.
(ii) Find all the equilibrium points for F, i.e. all the zeros of F. Notice that since ω is an angle, it suffices to consider values of ω and v between $-\pi$ and π, where $\omega = 0$ corresponds to the pendulum vertically down.
(iii) Linearize F at the equilibrium points and write down the corresponding linear 2-dimensional ODEs $x' = Ax$ with $A = DF(\mathring{x})$.
(iv) Characterize the origin of the linear ODE at each equilibrium point as a stable, unstable, saddle, or spiraling (inward or outward) fixed point of the linear ODE.
(v) Specialize to the case of no dampening, i..e $c = 0$. Does this effect anything?
(vi) Now characterize the equilibrium points $\mathring{x}$ of F as stable, unstable, saddle, or spiraling (inward or outward). Does this agree with what you think should be the case for a pendulum?

Problem 3. Let $U \subset \mathbb{R}^n$ be open and $f\colon U \to \mathbb{R}$ a smooth function. Consider the gradient vector field $F = \operatorname{grad} f\colon U \to \mathbb{R}^n$.

(i) If $\gamma\colon I \to U$ is an integral curve of F, that is
$$\gamma'(t) = \operatorname{grad}_{\gamma(t)} f$$
verify that the function $f \circ \gamma\colon I \to \mathbb{R}$ is increasing. In other words, the function f increases along the integral curves of its gradient vector field. Also check that if you "flow in the negative gradient direction", then f decreases. *Hint*: 1st derivative test for increasing/decreasing...

(ii) Show that the equilibrium points of the vector field $F = \operatorname{grad} f$ are the critical points of f.

(iii) What is the linearization of $F = \operatorname{grad} f$ at an equilibrium point, i.e. in the linearized ODE $x' = Ax$ what is A (have we encountered this object before?) in terms of f.

(iv) Explain why the linearized ODE at each equilibrium point has real eigenvalues only, and thus the equilibrium points are either stable, unstable, or saddles.

(v) Explain what all of this has to do with min/max theory of f. Think of the mountain landscape the graph of f makes and then interpret the integral curves in this picture. Draw the curves on the graph ($n = 2$ for that) or on your hiking map (assume a landscape which has at least a mountain top, a mountain hole, and a saddle)

Problem 4. Let $U \subset \mathbb{R}^n$ be open and let $F\colon U \to \mathbb{R}^n$ be a vector field. For any point $x \in U$ consider the $n \times n$ Jacobi matrix $DF(x)$.

(i) First show that any $n \times n$ matrix B can be uniquely written as a sum $B = S + A$ where S is symmetric ($S^T = S$) and A is anti-symmetric $A^T = -A$. What are S, A in terms of B? Check that the trace of B equals the trace of S, where the trace of a square matrix is the sum of its diagonal entries.

(ii) Decompose $DF(x) = S + A$ and identify the entries of S and A in terms of the partial derivatives of $F = (F_1, \ldots, F_n)$.

(iii) In the case $n = 3$ have a closer look at A. Do you recognize the entries of the anti-symmetric part of $DF(x)$ in terms of something you have seen in Calc III? Next have a closer look at $\operatorname{trace} DF(x)$. Do you recognize this expression in terms of something you have seen in Calc III?

(iv) Calculate (for general n) the anti-symmetric part of $DF(x)$ in case $F = \operatorname{grad} f$ is a gradient vector field. What is the symmetric part, i.e have you encountered it before? And what is $\operatorname{trace} DF(x)$?

Problem 5. Consider the vector fields

$F(x) = \frac{x}{||x||^3}$ on $\mathbb{R}^n \setminus \{0\}$;

$F(x) = (x_2x_3, x_1x_3, x_1x_2)$ on $\mathbb{R}^3$;

$F(x) = (-x_2, x_1)$ on $\mathbb{R}^2$.

(i) Calculate $DF(x)$ for each of the vector fields.

(ii) Calculate the symmetric and anti-symmetric parts of $DF(x)$ for each of the vector fields.

(iii) Calculate $\operatorname{trace} DF(x)$ for each of the vector fields.