Homework 9, Honors Calculus II
11/19/19

Problem 1. Last week we have obtained, via an infinite series which converges, a new function:

$$\exp\colon \mathbb{R} \to (0, \infty)\,, \qquad \exp(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

We now want to claim that this function $\exp(x)$ is the *only* function satisfying the differential equation $Q'(x) = Q(x)$ with initial condition $Q(0) = 1$ (which was around shortly after the Big Bang 12 billion or so years ago...). There could be some mystery function $f(x)$ which cannot be expressed in terms of an infinite series satisfying the differential equation $f' = f$.

(i) Show that a differentiable function $f(x)$ which satisfies the condition $f'(x) = f(x)$ must be of the form $f(x) = c \cdot \exp(x)$ for some constant $c \in \mathbb{R}$.
Hint: show that $f(x)\exp(-x)$ is constant, by showing that its derivative $[f(x)\exp(-x)]' = 0$ is zero for all values of x. To accomplish this show that exp and f satisfy the same differential equation.

(ii) Now show, that if f from (i) satisfies in addition the initial condition $f(0) = 1$, then $f = \exp$.

These considerations essentially say the following: any solution of $Q' = Q$ has to be a constant multiple of exp and the initial condition determines the constant.

Problem 2. Find the unique solution of the differential equation $Q'(x) = k \cdot Q(x)$ with initial condition $Q(0) = Q_0$ and $k \in \mathbb{R}$ a real constant. *Hint*: modify $\exp(x)$ appropriately and use a similar argument as in Problem 1 to argue that the solution you write down is the only one. You never need to use infinite series.

Problem 3. Show that the infinite series

$$\sum_{k=1}^{\infty} \frac{1}{k^2}$$

has a finite limit (even though you will not be able to show what value the series sums up to). *Hint*: $\frac{1}{k^2} \le \frac{1}{k(k-1)} =?$ (why is this estimate true) for $k \ge 2$, then think of partial fraction decomposition to calculate ?, and then think of telescoping terms in the partial sums $s_n = \sum_{k=2}^{n} \frac{1}{k(k-1)} =??$.

Problem 4. Investigate the series

$$\sum_{k=1}^{\infty} \frac{1}{k^\alpha}$$

for their convergency depending on the value of the real parameter $\alpha \in \mathbb{R}$, that is, show:

(i) If $\alpha > 1$ the series converges. *Hint*: Riemann sum for which function (draw a picture)?

(ii) If $\alpha \le 1$ the series does not converge, that is, the sequence of partial sums is unbounded. *Hint*: Riemann sum for which function (draw a picture)?

Problem 5. Provide a value (preferably small) for $n \in \mathbb{N}$, so that the *finite* sum $\sum_{k=0}^{n} \frac{1}{k!}$ agrees with the value of $e = \exp(1) = \sum_{k=0}^{\infty} \frac{1}{k!}$ to five decimals. At this stage we don't know the exact value of e, and the question is how we can find the value of e up to any prescribed accuracy, in this case 10^{-5}. As a check of your answer carry out the finite sum (up to your predicted length n) to see if this number agrees with e (which you read off your calculator) up to five decimals.

Problem 6 (For the die hard types)**.** Here a purely algebraic way to verify the functional equation

$$\exp(x+y) = \exp(x)\exp(y)$$

of the exponential function

$$\exp(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Hints: you will have to use the binomial formula (which you probably know from high school)

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\cdots(n-k+1)}{k!}.$$

Then you will have to stare at the summands a bit, perhaps interchange sums, rearrange terms, rename summation indices, and combine binomial coefficients—this is all pretty hard to do (some algebra), but you will understand formal manipulations of series much better after you solved this problem.

Those who have done the above problem (and came back alive from the binomial debauch) may appreciate the much more elegant proof using Calculus and Problem 2:

Problem 7 (Functional equation of exp via Calculus)**.** To show $\exp(x+y) = \exp(x)\exp(y)$ for all $x, y \in \mathbb{R}$ can be done as follows: treat x as the variable and y as a constant. Consider the two functions $f(x) = \exp(x+y)$ and $g(x) = \exp(x)\exp(y)$.

(i) Show that f and g both satisfy the *same* differential equation $Q' = kQ$ (what is k?) with the same initial condition.
(ii) Use Problem 2 to deduce form (i) that $f = g$.

Problem 8. One version of Zeno's Paradox is the following: suppose I wish to cross a room of a certain length. First, of course, I must cover half the distance. Then, I must cover half the remaining distance. Then, I must cover half the remaining distance. Then I must cover half the remaining distance...and so on forever. The consequence is that I can never get to the other side of the room, that is, it would take infinitely long. Since everyday experience shows that we can actually cross a room in finite time, something must be wrong with Zeno's reasoning. Can you resolve the paradox?

Problem 9. You compound an initial investment of 1 with annual interest rate x (e.g. $x = 0.03$ would be a 3% rate) n-times through the year.

(i) Show that your initial investment of 1 grew to $(1+\frac{x}{n})^n$ after one year.
(ii) Following Euler, use the binomial formula to expand the above expression and calculate the limit $n \to \infty$ of $(1+\frac{x}{n})^n$. What do you get? Interpret your result in terms of compounding.