HOMEWORK 8, HONORS CALCULUS II
11/12/19

Problem 1. Calculate the surface area of the funnel formed by revolving $y = e^{-x}$, $x \in [0, b]$, around the x-axis (draw a picture). What happens to the area as $b \to \infty$?

Problem 2. Calculate the area of the spindle formed by revolving $y = \sin x$, $x \in [0, \pi]$, around the x-axis.

Problem 3. Calculate the area of the surface generated by revolving the region between the graphs $y = \sqrt{x}$ and $y = x$ around the y-axis (draw a picture).

Problem 4. Calculate the area of the surface generated by revolving the graph $y = \frac{1}{x}$ around the y-axis for $1 \leq y \leq b$ (draw a picture). What happens to the area when $b \to \infty$?

Problem 5. A "cooling tower" of height $2h > 0$ is obtained by revolving the hyperbola $x^2 - y^2 = 1$ around the y-axis between $-h \leq y \leq h$. Draw a picture. What is its surface area?

Problem 6. Set up and try to calculate the surface area and enclosed volume of an ellipsoid of revolution, that is, the surface obtained by rotating the ellipse

$$(\frac{x}{a})^2 + (\frac{y}{b})^2 = 1$$

around the x-axis. Here $a \geq b > 0$ are the lengths of the major and minor axes of the ellipse.

Problem 7. Let $\gamma\colon [a, b] \to \mathbb{R}^2$ be a non-selfintersecting parameterized curve contained in the right half-plane, that is, $\gamma_1(t) > 0$ for all $t \in [a, b]$. Rotating this curve around the y-axis generates a surface of revolution with profile γ (note, this is a bit more general than taking profiles which are given by graphs of functions). Find a formula—which should only involve terms computable from γ and its derivative γ'—for the volume enclosed by this surface, and also a formula for the surface area. Test your formula agains examples you know the answers for.

Problem 8 (Newton's equation of motion)**.** Newton postulated that the movement of an object of a given mass is governed by the following law, nowadays called *Newton's equation of motion*:

The acceleration experienced by the object is proportional to the sum total of forces acting on the object

where the proportionality factor is given by a mysterious quantity, the *mass* of the object (one could also have the, perhaps not so perverse, viewpoint that Newton's equation of motion defines what we mean by mass).

If we call F the force, $m > 0$ the mass, and a the acceleration, then Newton's law of motion (for any object everywhere in the universe) reads

$$F = m \times a$$

Let's assume we live in 1-dimensional space, so we only need to measure distance from the origin to determine the position p of an object. We did discuss that velocity $v(t) = p'(t)$ is the instantaneous rate of change with respect to time t of the position $p(t)$ of the object (as a function of time t). Likewise, the acceleration

$a(t) = v'(t)$ is the instantaneous rate of change with respect to time t of the velocity $v(t)$ of the object. So we can rewrite Newton's equation of motion as

$$F = mv'$$

Let's do a few examples:

(i) One of the consequences of Newton's law is what is sometimes called *Newton's 1st law of motion:* if no force acts on an object, the object moves with constant velocity. Verify *Newton's 1st law* from Newton's equation of motion.

(ii) Newton also formulated his famous gravitational law: the force of *attraction* between two objects is proportional to the product of their respective masses and inverse proportional to the square of their distance. Since for motion *near the surface* of the earth the distance an object travels is very small as compared to the distance to the center of the earth, this force is almost the same as the constant gravitational force experienced by an object of mass m, namely $F = mg$, where g is the gravitational constant (pull) of the planet we are on (so $9.81\, m/sec^2$ on earth).

Let's climb the famous leaning tower of Pisa (its top platform is 60 meters above ground) and drop a stone of mass m (where we first ignore air drag, which is reasonable for a fairly heavy small round stone). Write down Newton's equation of motion for this scenario. Then find the velocity function $v(t)$ which satisfies your equation. Once you found the velocity function, find the formula for the position $p(t)$ of the object from the equation $p'(t) = v(t)$. What you should get are Galileo's laws of free falling objects without air drag: velocity increases linearly in time, position increases (or rather decreases if the stone is falling) quadratically in time. Draw a picture fo this experiment and choose your vertical axis direction. Also draw the graphs of $p(t)$ and $v(t)$.

(iii) Calculate how long it takes for the object to fall to the ground when released on top of the tower.

(iv) Assume, you hit a baseball vertically upwards (standing at the top of the tower) with initial speed $60\, km/h$. Calculate how far above the ground the baseball flies, how long it takes for it to reach this highest point, and how long it takes the baseball from there to fall to the ground. Also draw the graphs of $p(t)$ and $v(t)$ for this experiment.

(v) **Bonus**: Now write down Newton's equation of motion for the more realistic scenario when the falling object also experiences air drag which you can model as being proportional to the speed of the falling object. The proportionality factor is usually called the *air drag* and will significantly depend on the shape and mass of the material (a feather will have a much higher air drag than a glass marble). Try to solve the equation you obtained, that is, find the velocity $v(t)$ and position $p(t)$ as functions of time.